Closed discrete subsets of separable spaces and relative versions of normality, countable paracompactness and property (a)

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Abstract. In this paper we show that a separable space cannot include closed discrete subsets which have the cardinality of the continuum and satisfy relative versions of any of the following topological properties: normality, countable paracompactness and property (a). It follows that it is consistent that closed discrete subsets of a separable space X which are also relatively normal (relatively countably paracompact, relatively (a)) in X are necessarily countable. There are, however, consistent examples of separable spaces with uncountable closed discrete subsets under the described relative topological requirements, and therefore the existence of such uncountable sets is undecidable within ZFC. We also investigate what are the outcomes of considering the set-theoretical hypothesis "$2^\omega < 2^{\omega_1}$" within our discussion and conclude by giving some notes and posing some questions.

Keywords: relative normality, relative countable paracompactness, relative property (a), closed discrete subsets, separable spaces

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1. Preliminaries and introduction

Throughout this paper, all spaces are assumed to be T_1 topological spaces.

It is well-known that separable spaces which satisfy $\mathcal{P}$, for any property $\mathcal{P} \in$ {normality, countable paracompactness, property (a)}, cannot include closed discrete subsets of size $\mathfrak{c}$ (resp. [10], [7], [12]) and, moreover, $2^\omega < 2^{\omega_1}$ suffices to show that separable normal spaces cannot include uncountable closed discrete subsets. By previous results due to Watson and the author (resp. [21], [18]), for separable spaces which are either (i) countably paracompact; or (ii) locally compact (a)-spaces, the existence of uncountable closed discrete subsets implies the existence of small dominating families in the families of functions from ω_1 into ω (and therefore the existence of such subsets is related to large cardinals, as we will recall later). The questions whether $2^\omega < 2^{\omega_1}$ alone implies countable extent

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for separable countably paracompact spaces, or for separable (a)-spaces, are still open (resp. [16], [18])[1].

By writing the papers [20] and [17], the author initiated a search for "relative versions" of some of these results and questions, and for many others related. The research on *relative topological properties* was introduced by Arhangel'skii in the late 80's (see [1], [2]), and since then this programme has been widely detached (see e.g. [3], [6], [8], [11], [13] and [22]).

In the author's quoted papers, it is shown that: (i) the existence of uncountable closed discrete subsets which are also relatively countably paracompact in a separable space implies the existence of small dominating families in ${}^{\omega_1}\omega$ ([20]); and (ii) the analogous result for uncountable closed discrete subsets which are also relatively (a) and relatively locally compact in a separable space ([17]).

These results provide set-theoretical restrictions on the existence of such subsets, because of the well-known relationships between small dominating families and large cardinals. We recall here these relationships briefly.

The *mod countable order* in the family of functions from ω_1 into ω is defined as follows: for $f, g \in {}^{\omega_1}\omega$ we have $f \leqslant^* g$ if the set $\{\alpha < \omega_1 : g(\alpha) < f(\alpha)\}$ is countable.

$D \subseteq {}^{\omega_1}\omega$ is a *dominating family* in the mod countable order if it is cofinal, meaning that $(\forall f \in {}^{\omega_1}\omega)(\exists g \in D)[f \leqslant^* g]$. It is well-known that $\mathrm{cf}(\langle {}^{\omega_1}\omega, \leqslant^* \rangle) = \mathrm{cf}(\langle {}^{\omega_1}\omega, \leqslant \rangle)$, where $f \leqslant g$ means $f(\alpha) \leqslant g(\alpha)$ for every $\alpha < \omega_1$ (see [5]).

Jech and Prikry [9], using Dodd and Jensen's results on the *core model*, showed that "$2^\omega < 2^{\omega_1}$" + "2^ω regular" + "There is a dominating family in $\langle {}^{\omega_1}\omega, \leqslant^* \rangle$ of cardinality 2^ω" implies that "There is an inner model with a measurable cardinal". They also showed that there can be no dominating family of size less than 2^{ω_1} in $\langle {}^{\omega_1}\omega, \leqslant^* \rangle$ if either 2^ω is a real-valued measurable cardinal or if $2^\omega < \min\{2^{\omega_1}, \aleph_{\omega_1}\}$.

In this paper, the expression "small dominating family" is always an abbreviation for "dominating family of functions in $\langle {}^{\omega_1}\omega, \leqslant \rangle$ with cardinality not larger than the continuum".

With the results of Jech and Prikry in mind, the referred theorems from [17], [18], [20] and [21] provide the following set-theoretical restrictions on the existence of certain uncountable subsets of separable spaces:

Proposition 1.1. *Suppose* "$\mathrm{cf}(2^\omega) = 2^\omega < 2^{\omega_1}$" *and* "There are no inner models with measurable cardinals". *Then, the following statements hold:*

(i) *separable countably paracompact spaces have countable extent;*
(ii) *locally compact separable (a)-spaces have countable extent;*
(iii) *closed discrete subsets which are also relatively countably paracompact in separable spaces are countable sets;*
(iv) *closed discrete subsets which are also relatively (a) in locally compact separable spaces are countable sets;*

[1]Recall that the *extent* of a topological space X, $e(X)$, is the supremum of the cardinalities of all closed discrete subsets of X, provided this is an infinite cardinal, or is ω otherwise. So, "countable extent" is a short for "non-existence of uncountable closed discrete subsets".

(v) *closed discrete subsets which are also relatively* (a) *and relatively locally compact in separable spaces are countable sets.* □

We keep on investigating restrictions on the existence of uncountable closed discrete subsets of separable spaces satisfying relative versions of the three properties of our interest by showing that, for those with cardinalities not smaller than the continuum, there are absolute restrictions.

These are the *relative topological properties* we are considering in this paper:

Definition 1.2. Let X be a topological space and $Y \subseteq X$.

(i) ([1]) Y is *normal in* X (or *is relatively normal in* X) if for every pair F, G of closed disjoint subsets of X there is pair U, V of open disjoint subsets of X such that $F \cap Y \subseteq U$ and $G \cap Y \subseteq V$.

(ii) ([1], [20]) Y is (*countably*) *paracompact in* X (or *is relatively* (*countably*) *paracompact in* X) if for every (countable) open cover $\mathcal{U}$ of X there is a family $\mathcal{V}$ of open subsets of X such that $\mathcal{V}$ is locally finite at each point of Y (that is, for every $y \in Y$ there is a set U_y such that $y \in U_y$, U_y is an open subset of X and $\{V \in \mathcal{V} : V \cap U_y \neq \emptyset\}$ is a finite set), $\mathcal{V}$ refines $\mathcal{U}$ (that is, for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subseteq U$) and $Y \subseteq \bigcup \mathcal{V}$.

(iii) ([13]) Y has *property* (a) *in* X (or *is relatively* (a) *in* X) if for every open cover $\mathcal{U}$ of X and every dense set $D \subseteq X$ there is $C \subseteq D$ such that C is a closed and discrete subset of X and $Y \subseteq \mathrm{St}(C,\mathcal{U}) := \bigcup\{U \in \mathcal{U} : U \cap C \neq \emptyset\}$.

(iv) ([1], [22]) Y is *compact in* X (or *is relatively compact in* X) if every open cover $\mathcal{U}$ of X has a finite subfamily $\mathcal{V}$ such that $Y \subseteq \bigcup \mathcal{V}$.

(v) ([1], [22]) Y is *locally compact in* X (or *is relatively locally compact in* X) if for every $y \in Y$ there is a set U_y such that U_y is a neighbourhood of y in X and U_y is compact in X.

Clearly, Y is normal in X if and only if for every pair F, G of closed disjoint subsets of X there is an open set U such that $F \cap Y \subseteq U$ and $G \cap Y \subseteq X \setminus \overline{U}$.

Let us describe the organization of this paper. In Sections 2, 3 and 4 we prove the central theorems, those which declare that closed discrete subsets of size $\mathfrak{c}$ of separable spaces cannot satisfy relative versions of, respectively, normality, countable paracompactness and property (a). We also establish, in each section, the independency (with respect to ZFC) of the existence of uncountable subsets os separable spaces which satisfy the desired relative topological requirements. In Section 5, we give some notes and questions.

We close this preliminary discussion by showing that the existence of uncountable sets which are closed discrete subsets of separable spaces and satisfy relative versions of all of our three topological properties is consistent with ZFC. For this, we will use classical examples from Set Theoretic Topology: spaces from almost disjoint families, the so-called Ψ*-spaces*.

A family $\mathcal{A}$ of infinite subsets of ω is said to be an *almost disjoint family* (or *a.d. family*) if every pair of distinct elements of $\mathcal{A}$ has finite intersection. For every almost disjoint family $\mathcal{A}$ we may consider a topological space $\Psi(\mathcal{A})$, whose underlying set is given by $\mathcal{A} \cup \omega$. The points in ω are declared isolated and the basic neighbourhoods of a point $A \in \mathcal{A}$ are given by the sets of the form $\{A\} \cup (A \setminus F)$, for F varying over the finite subsets of ω. One easily checks that ω is a dense set of isolated points and $\mathcal{A}$ is a closed and discrete subset of $\Psi(\mathcal{A})$. The space $\Psi(\mathcal{A})$ is a Hausdorff zero-dimensional (thus, completely regular) first-countable locally compact separable space, and, in fact, it is well-known that if X is a Hausdorff first-countable locally compact separable space such that the derived set X' is non-empty and discrete, then there is an a.d. family $\mathcal{A}$ such that X and $\Psi(\mathcal{A})$ are homeomorphic (see [19, Proposition 1.1]).

In [19], the author surveyed and presented a number of results related to the presence of normality, countable paracompactness and property (a) in spaces from almost disjoint families. The reader may find in the referred paper all the references for the original works (due to Bing, Heath, Tall, Szeptycki, Vaughan, among others) that ensure the validity of the following statement:

Proposition 1.3. *If $|\mathcal{A}| < \mathfrak{p}$, then $\Psi(\mathcal{A})$ satisfies $\mathcal{P}$ for any property $\mathcal{P} \in$ {normality, countable paracompactness, property (a)}.* □

In the preceding proposition, $\mathfrak{p}$ denotes the minimal cardinality of a family $\mathcal{F}$ of infinite subsets of ω which satisfies the *strong finite intersection property* (meaning that every non-empty finite subfamily has infinite intersection) and *has no infinite pseudo-intersection* (meaning that there is no infinite $A \subseteq \omega$ such that $A \setminus F$ is finite for all $F \in \mathcal{F}$). It is well-known that $\mathfrak{p} = \mathfrak{m}_{\sigma\text{-centered}}$, i.e., $\mathfrak{p}$ is the least cardinal for which the Martin's Axiom restricted to σ-centered p.o.'s fails ([4]). So, the consistent statement "$\omega_1 < \mathfrak{p} = \mathfrak{c}$" is, in fact, equivalent to $\mathrm{MA}_{\sigma\text{-centered}} + \neg\mathrm{CH}$.

Ψ-spaces will be very useful for our intents because it is straightforward to write down a proof for the following

Theorem 1.4. *Let $\mathcal{A}$ be an a.d. family of subsets of ω and let $\Psi(\mathcal{A})$ be the corresponding Ψ-space. Then we have*

$$\Psi(\mathcal{A}) \text{ satisfies } \mathcal{P} \iff \mathcal{A} \text{ satisfies relative } \mathcal{P} \text{ in } \Psi(\mathcal{A})$$

for any property $\mathcal{P} \in$ {normality, countable paracompactness, property (a)}. □

From the two preceding results, we deduce that models of $\omega_1 < \mathfrak{p} = \mathfrak{c}$ give us the following:

Proposition 1.5. *The statement*

"for every $\omega_1 \leqslant \kappa < \mathfrak{c}$, there is a separable space with an uncountable closed discrete subset of size κ which satisfies relative $\mathcal{P}$ for any $\mathcal{P} \in$ {normality, countable paracompactness, property (a)}"

is consistent with ZFC + ¬CH. □

2. On relative normality

We proceed as in the original Jones' Lemma ([10]).

Theorem 2.1 (Relative version of Jones' Lemma). *If X is a topological space, $D \subseteq X$ is a dense set and $H \subseteq X$ is a closed discrete subset which is also relatively normal in X, then $2^{|H|} \leqslant 2^{|D|}$.*

Proof: Subsets of a closed discrete subset of X are closed (and discrete) subsets of X. As the closed discrete subset H is supposed to be relatively normal, for every $A \subseteq H$ we can fix an open set U_A such that $A \subseteq U_A$ and $H \setminus A \subseteq X \setminus \overline{U_A}$. Exactly as in the proof of the original Jones' Lemma, we can define a function $\varphi : \mathcal{P}(H) \to \mathcal{P}(D)$ by putting $\varphi(A) = U_A \cap D$ for all $A \subseteq H$, and it is easy to check that φ is an injective function. □

In the separable case, we have that if $H \subseteq X$ is closed discrete and relatively normal then $|H| < 2^{|H|} \leqslant 2^{\omega}$, so the following corollary holds:

Corollary 2.2. *Separable spaces cannot include closed discrete subsets which are also relatively normal in X and have the cardinality of the continuum.*

It follows that in models of CH relatively normal closed discrete subsets of separable spaces are necessarily countable. Together with Proposition 1.5, this ensures that the existence of uncountable closed discrete subsets of separable spaces satisfying relative normality is undecidable within ZFC.

And, because of the inequality $2^{|H|} \leqslant 2^{|D|}$, we are able to say a little more. Recall that the *density* of X, $d(X)$, is the smallest cardinality of a dense subset of X, provided this is an infinite cardinal, or is ω otherwise.

Corollary 2.3. *Let X be a topological space and $\kappa = d(X)$. If $2^{\kappa} < 2^{\kappa^+}$ then X cannot include closed and discrete subsets which are relatively normal in X and have cardinality κ^+.* □

In particular, if $2^{\omega} < 2^{\omega_1}$ then separable spaces cannot include uncountable closed discrete subsets which are also relatively normal in them.

3. On relative countable paracompactness

The following is an adaptation of arguments from [7], [20] and [21]. It may be seen, also, as a diagonal argument.

Theorem 3.1. *Separable spaces cannot include closed and discrete subsets which are also relatively countably paracompact in X and have the cardinality of the continuum.*

Proof: Let H be a closed discrete subset of X with the cardinality of the continuum and suppose D is a countable dense subset of X. $(2^{\omega})^{\omega} = 2^{\omega}$, so we can use H as an index set for the family of all the sequences of subsets of D which are locally finite at each point of H. Let $\{G_x : x \in H\}$ be such a family, and for every $x \in H$ let $G_x = \langle G_{x,n} : n < \omega \rangle$.

Define a function $f : H \to \omega$ such that, for every $x \in H$,

$$f(x) = \min\{n : x \notin \overline{G_{x,n}}\}.$$

By the local finiteness of the G_x's, f is well defined. For every $n < \omega$ let $H_n = f^{-1}(n)$. Then $\{H_n : n < \omega\}$ is a partition of H. Consider the countable open cover of X given by

$$\mathcal{U} = \{X \setminus (H \setminus H_n) : n < \omega\}.$$

We claim that for any family of open sets $\mathcal{V}$ which refines $\mathcal{U}$ and is locally finite at each point of H we have $H \nsubseteq \bigcup \mathcal{V}$, and this clearly suffices for us.

Let $\mathcal{V}$ be as in the preceding paragraph. For every $n < \omega$ let $S_n = \mathrm{St}(H_n, \mathcal{V}) \cap D$. Then $\mathcal{S} = \langle S_n : n < \omega \rangle$ is a sequence of subsets of D which is locally finite at each point of H, and therefore there is $z \in H$ such that $\mathcal{S} = G_z$.

Suppose for a contradiction that $z \in \bigcup \mathcal{V}$. If $m = f(z)$ then we have $z \in H_m$ and therefore

$$z \in \mathrm{St}(H_m, \mathcal{V}) \subseteq \overline{\mathrm{St}(H_m, \mathcal{V})} = \overline{\mathrm{St}(H_m, \mathcal{V}) \cap D} = \overline{G_{z,m}}$$

but this is an absurd, because $x \notin \overline{G_{x,f(x)}}$ for every $x \in H$.

Thus $H \nsubseteq \bigcup \mathcal{V}$, as desired. $\square$

It follows that in models of CH relatively countably paracompact closed discrete subsets of separable spaces are necessarily countable. Together with Proposition 1.5, this ensures that the existence of uncountable closed discrete subsets of separable spaces satisfying relative countable paracompactness is undecidable within ZFC.

4. On relative property (a)

The following is an adaptation of arguments from [12] and [18]. We have a kind of diagonal argument again.

Theorem 4.1 (Relative version of Matveev's (a)-Jones' Lemma)**.** *Separable spaces cannot include closed and discrete subsets which are also relatively (a) in X and have the cardinality of the continuum.*

Proof: Let H be a closed discrete subset of X with the cardinality of the continuum and let D be a countable dense subset of X. As $|H| > |D|$ we may suppose without loss of generality that H and D are disjoint sets. We are allowed to use H to index the family of all closed discrete subsets of D, so let $\{G_x : x \in H\}$ be such a family.

For every $x \in X$ let U_x be the open neighbourhood of x given by $U_x = X \setminus ((H \setminus \{x\}) \cup G_x)$ and consider the open cover of X given by

$$\mathcal{U} = \{X \setminus H\} \cup \{U_x : x \in H\}.$$

Notice that, for all $x \in X$, we have

(1) $U_x \cap H = \{x\}$ and
(2) $U_x \cap G_x = \emptyset$,

and notice that for every $x \in H$ the open set U_x is the only element of $\mathcal{U}$ which contains x.

We claim that $\mathcal{U}$ witnesses that H is not relatively (a) in X. Indeed: let $C \subseteq D$ be an arbitrary closed discrete subset of D. There is $z \in H$ such that $C = G_z$, and therefore $U_z \cap C = \emptyset$, by (2). By the uniqueness property already remarked, we have $z \notin \mathrm{St}(C,\mathcal{U})$ and it follows that $H \not\subseteq \mathrm{St}(C,\mathcal{U})$. As C was chosen arbitrarily, H is not relatively (a) in X. □

It follows that in models of CH, relatively (a) closed discrete subsets of separable spaces are necessarily countable. Together with Proposition 1.5, this ensures that the existence of uncountable closed discrete subsets of separable spaces satisfying relative property (a) is undecidable within ZFC.

5. Notes and questions

With the background presented within this paper and in all referred previous ones, it is natural to formulate "relative versions" of several questions formerly posed in the literature.

Towards to this aim, we first ask the reader to notice that, with easy adaptations of the proofs of Theorems 2.1 and 4.1, one has the following general result:

Proposition 5.1. *If X is a topological space and $\kappa = d(X)$, then X cannot include closed discrete subsets which have cardinality 2^κ and satisfy relative versions of any among normality and property (a).* □

However, a result analogous to Corollary 2.3 for relatively (a), closed discrete subsets, was never established. So, it is very natural to present the following question, which could be seen as a "relative version" of Question 3.1 of [18].

Question 5.2. *Let X be a topological space and $\kappa = d(X)$. Does $2^\kappa < 2^{\kappa^+}$ imply that X cannot include closed discrete subsets which are relatively (a) in X and have cardinality κ^+ ?*

In the separable case, what we are asking is if $2^\omega < 2^{\omega_1}$ suffices to show that relatively (a) closed discrete subsets of separable spaces are countable sets. We recall that, for locally compact separable spaces, this question is related to large cardinals (as we already remarked in Proposition 1.1).

Before asking some analogous question for relative countable paracompactness, we have to point out that the situation in this case is much more subtle. We have remarked that Watson's result of [21] ensures that the existence of a separable countably paracompact space with an uncountable closed discrete subset implies the existence of small dominating families. However, Watson has shown more: these statements are, in fact, equivalent ([21, Theorem 2, p. 840]). Therefore, the existence of such spaces is directly related to large cardinal axioms.

And as, obviously, any subset of a countably paracompact space X is countably paracompact in X, it follows from Watson's result and from the author's Theorem 5.4 of [20] that the following interesting statement holds, bringing a "large cardinal related situation" to the realm of relative topological properties.

Theorem 5.3. *The existence of small dominating families is equivalent to the existence of a separable space X with an uncountable closed discrete subset which is also relatively countably paracompact in X.* □

In [16], the author and Morgan asked if $2^\omega < 2^{\omega_1}$ alone is sufficient to prove that there are no small dominating families. Notice that this is the same as asking if $2^\omega < 2^{\omega_1}$ implies countable extent for separable countably paracompact spaces, and it is also the same as asking the following question on relative countable paracompactness:

Question 5.4. *Does $2^\omega < 2^{\omega_1}$ imply that closed discrete subsets of separable spaces which are also relatively countably paracompact in them are, necessarily, countable ?*

Related to small dominating families, and as a relative version of Question 5.2 of [18], we pose the following

Question 5.5. *Does the existence of small dominating families imply the existence of separable spaces with uncountable closed discrete subsets which are also relatively (a) in them ?*

We also present the following slight variations of the preceding question:

Question 5.6. *The one obtained by adding "Assume $2^\omega < 2^{\omega_1}$" at the beginning of Question 5.5.*

Question 5.7. *The same as Question 5.5, but with "separable spaces" replaced by "locally compact separable spaces".*

Question 5.8. *The same as Question 5.5, but with "relatively (a)" replaced by "relatively (a) and relatively locally compact".*

Finally, we remark that there are some *weak parametrized diamond principles* which imply restrictions on the validity of relative versions of property (a) and countable paracompactness for uncountable closed discrete subsets of separable spaces. The class of such combinatorial "guessing" principles were introduced by Moore, Hrušák and Džamonja in [14].

The weak parametrized diamond principle $\Phi(\omega, <)$ corresponds to the following combinatorial statement:

(∗) *For every function F with values in ω, defined in the binary tree of height ω_1, there is a function $g : \omega_1 \to \omega$ such that g "guesses" every branch of the tree, meaning that for all $f \in {}^{\omega_1}2$ the set given by $\{\alpha < \omega_1 : F(f \restriction \alpha) < g(\alpha)\}$ is stationary.* [2]

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[2]We assume the reader is familiar with the notions of clubs and stationary subsets of ω_1. In any case, definitions for these notions may be found in your favourite textbook of Set Theory.

As any other of the similar weak parametrized diamond principles defined in [14], $\Phi(\omega,<)$ implies $2^\omega < 2^{\omega_1}$.

If we restrict the validity of $(*)$ to functions F that are Borel, we obtain the *Borel version* of the principle, denoted by $\diamondsuit(\omega,<)$.

With obvious adaptations of the proofs of Propositions 2.1 and 3.1 of [16], we have the following results:

Theorem 5.9. $\Phi(\omega,<)$ *implies that closed discrete subsets of separable spaces which are also relatively countably paracompact in them are, necessarily, countable sets.* $\square$

In particular, $\Phi(\omega,<)$ implies the non-existence of small dominating families (see also Proposition 4.3 of [16]).

Theorem 5.10. $\diamondsuit(\omega,<)$ *implies that closed discrete subsets of separable spaces which are also relatively* (a) *and relatively locally compact in them are, necessarily, countable sets.* $\square$

We remark that the Borel version $\diamondsuit(\omega,<)$ is consistent with $2^\omega = 2^{\omega_1}$ ([15]). It follows that $\Phi(\omega,<)$ cannot be replaced by its Borel version in Theorem 5.9, because in models of $2^\omega = 2^{\omega_1}$ there are, obviously, small dominating families.

Finally, we point out that, because of Theorem 1.4, answers obtained by using Ψ-spaces would take care of both kinds of questions, the "relative ones" (presented in this paper) and the "absolute ones" (formerly presented).

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