The regular topology on $C(X)$

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Abstract. Hewitt [*Rings of real-valued continuous functions. I.*, Trans. Amer. Math. Soc. **64** (1948), 45–99] defined the m-topology on $C(X)$, denoted $C_m(X)$, and demonstrated that certain topological properties of X could be characterized by certain topological properties of $C_m(X)$. For example, he showed that X is pseudocompact if and only if $C_m(X)$ is a metrizable space; in this case the m-topology is precisely the topology of uniform convergence. What is interesting with regards to the m-topology is that it is possible, with the right kind of space X, for $C_m(X)$ to be highly non-metrizable. E. van Douwen [*Nonnormality of spaces of real functions*, Topology Appl. **39** (1991), 3–32] defined the class of DRS-spaces and showed that if X was such a space, then $C_m(X)$ satisfied the property that all countable subsets of $C_m(X)$ are closed. In J. Gomez-Perez and W.Wm. McGovern, *The m-topology on $C_m(X)$ revisited*, Topology Appl. **153**, (2006), no. 11, 1838–1848, the authors demonstrated the converse, completing the characterization. In this article we define a finer topology on $C(X)$ based on positive regular elements. It is the authors' opinion that the new topology is a more well-behaved topology with regards to passing from $C(X)$ to $C^*(X)$. In the first section we compute some common cardinal invariants of the preceding space $C_r(X)$. In Section 2, we characterize when $C_r(X)$ satisfies the property that all countable subsets are closed. We call such a space for which this happens a weak DRS-space and demonstrate that X is a weak DRS-space if and only if βX is a weak DRS-space. This is somewhat surprising as a DRS-space cannot be compact. In the third section we give an internal characterization of separable weak DRS-spaces and use this to show that a metrizable space is a weak DRS-space precisely when it is nowhere separable.

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1. Introduction

Given a topological space X we let $C(X)$ denote the set of real-valued continuous functions defined on X. It is well-known that $C(X)$ is an $\mathbb{R}$-algebra under pointwise operations of addition, multiplication, and scalar multiplication and that there are several topologies on $C(X)$ that one may consider. The topology of pointwise convergence, the topology of uniform convergence, and the m-topology are but three examples. In this article we are interested in a ring topology which

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is in the same vein as the m-topology but finer. Our goal will be to show that the two topologies are not only different in general but also have different algebraic properties. We call this topology the r-topology since it is based on regular elements (i.e. non zero-divisors). Recall that each of the uniform topology and m-topology takes as a base the collection of sets of the form

$$B(f,e) = \{g \in C(X) : |f(x) - g(x)| < e(x), \forall\, x \in X\}$$

where $f \in C(X)$ and e is from a pre-defined set. In particular to obtain the topology of uniform convergence we allow e to be any strictly positive constant function. To get the m-topology, e is allowed to be any positive multiplicative unit. Furthermore, the r-topology is obtained by allowing e to be any positive regular element of $C(X)$. For more information on the m-topology the reader is urged to read [10] and problems 2N and 7Q of [6].

For the ease of the reader we recall some basic definition from the theory of $C(X)$. Our standard references for rings of continuous functions and topological spaces are [6] and [3].

Definition 1.1. Let $f \in C(X)$. Set $Z(f) = \{x \in X : f(x) = 0\}$ and let $\mathrm{coz}(f)$ be its set-theoretic complement. We call $Z(f)$ the *zeroset* of f and $\mathrm{coz}(f)$ the *cozeroset* of f, respectively. By a *zeroset* (*cozeroset*) of X we mean a set of the form $Z(f)$ ($\mathrm{coz}(f)$) for some $f \in C(X)$.

Units and regular elements of $C(X)$ are characterized topologically in the following way.

(1) For $f \in C(X)$, f is a unit of $C(X)$ if and only if $Z(f) = \emptyset$ if and only if $\mathrm{coz}(f) = X$.
(2) For $f \in C(X)$, f is a regular element of $C(X)$ if and only if $\int_X Z(f) = \emptyset$ if and only if $\mathrm{coz}(f)$ is a dense subset of X.

We let $C(X)^+ = \{f \in C(X) : f(x) \geq 0 \text{ for all } x \in X\}$ and call this the set of *positive elements* of $C(X)$. When $f(x) > 0$ for all $x \in X$, we will say f is *strictly positive* or is a *positive unit*. Set $U(X)^+ = \{f \in C(X) : f(x) > 0 \text{ for all } x \in X\}$, the set of positive multiplicative units of $C(X)$. Define

$$r(X)^+ = \{f \in C(X)^+ : f \ \text{ is a regular element of } \ C(X)\},$$

the set of positive regular elements of $C(X)$. It is straightforward to check that if $r, s \in r^+(X)$, then so is $r \wedge s$.

All topological spaces considered in this article shall be assumed to be Tychonoff, that is, Hausdorff and completely regular. For such a space X, we shall denote its Stone-Čech compactification by βX.

Formally, the r-topology on $C(X)$ is the one obtained by taking sets of the form

$$R(f,r) = \{g \in C(X) : |f(x) - g(x)| < r(x), \forall\, x \in \mathrm{coz}(r)\}$$

for $f \in C(X)$ and $r \in r^+(X)$ as a base for the topology. (Note that if $g \in R(f,r)$, then by the continuity of r and density of $\operatorname{coz}(r)$ we have $g(x) = f(x)$ for all $x \in Z(r)$.)

Proposition 1.2. *For any space X, the collection $\{R(f,r) : f \in C(X), r \in r(X)^+\}$ is a neighborhood system. Consequently, the rule $R(f,r)$ defines a base for a topology on $C(X)$.*

Proof: We supply a sketch. For $f \in C(X)$, let $R(f) = \{R(f,r)\}_{r\in r(X)^+}$. Observe the following. 1) for each $f \in C(X)$, $R(f) \neq \emptyset$. 2) If $r, s \in r(X)^+$, then so is $r \wedge s \in r(X)^+$ and that $R(f, r\wedge s) \subseteq R(f,r) \cap R(f,s)$. 3) Suppose $g \in R(f,r)$. Set $s = r - |f-g|$ and observe that $\operatorname{coz}(r) = \operatorname{coz}(s)$, whence $s \in r(X)^+$. Next, it is straightforward to check that $R(g,s) \subseteq R(f,r)$.

We have demonstrated that the collection $\{R(f,r) : f \in C(X), r \in r(X)^+\}$ satisfies the conditions (BP1)–(BP3) of [3, Section 1.1]. Consequently, said collection is a neighborhood system. □

Since $U(X)^+ \subseteq r^+(X)$ we conclude that the r-topology is finer than the m-topology. We leave it to the interested reader to check that the r-topology makes $C(X)$ into a topological ring, i.e., $+, \cdot$ are continuous operations, though, in general and not unlike the m-topology, the r-topology does not make $C(X)$ into a topological algebra. (For more information on this fact the reader is encouraged to read [11].)

We will use the notation $C_r(X)$ to denote $C(X)$ equipped with the r-topology.

We conclude this section with a few theorems answering the questions of coincidence of the three topologies defined above. First we give a few topological definitions.

Definition 1.3. Recall that a space X is called *pseudocompact* if every element of $C(X)$ is bounded, that is, for each $f \in C(X)$ there is a natural number M for which $|f(x)| < M$ for all $x \in X$. The collection of bounded continuous functions on X will be denoted by $C^*(X)$. (Pseudocompactness is the same as saying $C^*(X) = C(X)$.) Obviously compact spaces are pseudocompact. The standard example of a noncompact pseudocompact space is the collection of countable ordinals under the order topology. (See Chapter 5 of [6].)

Definition 1.4. Recall that the space X is called a *Frechèt-Urysohn space* if whenever $p \in \operatorname{cl}_X A$ then there exists a sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq A$ such that $\lim\limits_{n\to\infty} a_n = p$. A more general concept is that of a countably tight space. For a point $p \in X$, the *tightness of* p is defined to be the least cardinal κ such that whenever $p \in \operatorname{cl}_X A \smallsetminus A$ there is an $S \subseteq A$ of cardinality κ for which $p \in \operatorname{cl}_X S$. The space X is called *countably tight* if the tightness at each point is $\aleph_0$.

The space Σ from [6] is an example of a countable space, and hence countably tight, that is not a Frechèt-Urysohn space.

Theorem 1.5 ([2, Corollary 2.5]). *For a space X the following are equivalent:*

(1) *X is pseudocompact;*
(2) *$C_m(X) = C_u(X)$;*
(3) *$C_m(X)$ is metrizable;*
(4) *$C_m(X)$ is first countable;*
(5) *$C_m(X)$ is a Frechèt-Urysohn space;*
(6) *$C_m(X)$ is a countably tight space.*

Definition 1.6. In [12] the author defined a space X to be an *almost P-space* if every nonempty G_δ-set of X has nonempty interior. We presently recall some equivalent conditions for a space X to be an almost P-space.

Proposition 1.7 ([12, Proposition 1.1]). *For a Tychonoff space X, the following statements are equivalent.*

(1) *X is an almost P-space.*
(2) *Each non-empty zeroset of X has non-empty interior.*
(3) *Each zeroset of X is a regular closed subset of X.*

Moving from global to local we say $p \in X$ is an *almost P-point* if every G_δ-set of X containing p has nonempty interior. We shall have cause to use the set of almost P-points of X; denote the set of almost P-points of X by $\mathfrak{a}(X)$.

As for examples of P-spaces it is the case that if X is a (non-compact) locally compact and realcompact space, then $\beta X \setminus X$ is a compact almost P-space (see Lemma 3.1 of [4]).

In terms of the elements of $C(X)$, X is an almost P-space if and only if every regular element of $C(X)$ is a unit. This yields one direction of our next theorem.

Theorem 1.8. *Let X be a Tychonoff space. The following are equivalent.*

(i) *$C_r(X) = C_m(X)$.*
(ii) *X is an almost P-space.*
(iii) *$r(X)^+ = U(X)^+$.*

Proof: The proof that (ii) and (iii) are equivalent is straightforward. Since $r(X)^+$ and $U(X)^+$ are the sets used to create the r-topology and m-topology, respectively, we have that (iii) implies (i).

Next, if X is not an almost P-space then there is a nonempty zeroset, say $Z(f)$, whose interior is empty. Now, $Z(f) = Z(|f|)$ and so without loss of generality we assume that $f \geq 0$ and hence $f \in r(X)^+$. Consider $R(\mathbf{0}, f)$. If (i) holds then there is some $g \in U(X)^+$ such that $R(\mathbf{0}, g) \subseteq R(\mathbf{0}, f)$. Let $p \in Z(f)$. Then

$$0 < \frac{g(p)}{2} < f(p) = 0,$$

a contradiction. Therefore, (i) implies (ii). □

We are now able to prove our main result of this section.

Theorem 1.9. *For any Tychonoff space X, the following are equivalent.*

(i) $C_r(X)$ *is first countable.*
(ii) $C_r(X)$ *is a Frechèt-Urysohn space.*
(iii) $C_r(X)$ *is countably tight.*
(iv) $C_r(X) = C_u(X)$.
(v) X *is a pseudocompact, almost P-space.*
(vi) βX *is an almost P-space.*
(vii) $C_r(\beta X) = C_u(\beta X)$.

Proof: We start by showing that (i), (ii), and (iii) are all equivalent. It suffices to show that (iii) $\Rightarrow$ (i). Notice that $\mathbf{0} \in \operatorname{cl} r(X)^+$ so that by (iii) we can find a countable sequence, say $\{r_n\}_{n\in\mathbb{N}}$ for which $\mathbf{0} \in \operatorname{cl}\{r_n\}_{n\in\mathbb{N}}$. With not too much effort we can suppose that $r_n \geq r_{n+1}$ for all natural n. We claim that the collection $\{R(\mathbf{0}, r_n)\}_{n\in\mathbb{N}}$ is a base of neighborhoods for $\mathbf{0}$. To see this let $r \in r(X)^+$, then there is some $r_n \in R(\mathbf{0}, r)$. By design it follows that $R(\mathbf{0}, r_n) \subseteq R(\mathbf{0}, r)$. Therefore, the claim is true and so by translation $C_r(X)$ is first countable.

From Theorems 1.5 and 1.8 we gather that (iv) and (v) are equivalent. That (iv) $\Rightarrow$ (i) is patent.

Next we prove that (i) implies (iv). So suppose that $C_r(X)$ is first countable. Now, if we can show that X is an almost P-space, then it will follow from the facts that $C_r(X) = C_m(X)$, and then by Theorem 1.5 X is pseudocompact. By means of contradiction suppose that $p \in X$ is not an almost P-point and let $r \in r(X)^+$ for which $r(p) = 0$. Next, let $\{r_n\} \subseteq r(X)^+$ be a sequence which generates a countable base of neighborhoods for $\mathbf{0}$. We might as well assume that for all natural n,

$$0 \leq r_{n+1} \leq r_n \leq r \leq \mathbf{1}.$$

Notice that $p \in Z(r) \subseteq Z(r_n)$ for each n. For each n let $O_n = r^{-1}((0, \frac{1}{n}))$ and observe that each of these sets is nonempty. Otherwise, it would follow that $Z(r)$ is clopen contradicting that r is regular. Furthermore, the regularity of r_n implies that $\operatorname{coz}(r_n)$ is a dense open set. Therefore, we may choose a sequence $\{x_n\}_{n\in\mathbb{N}}$ so that $x_n \in \operatorname{coz}(r_n) \cap O_n$. Moreover, we can select the sequence so that $r(x_n) > r(x_{n+1})$. Thus (i) implies (v) and so (i) through (v) are equivalent.

Next, choose a sequence of positive real numbers $\{\delta_n\}_{n\in\mathbb{N}}$ such that $\delta_{n+1} < \delta_n$ and that

$$\delta_n < \frac{r_n(x_n)}{2}.$$

Let $h \in C([0,1])$ such that $\mathbf{0} \leq h \leq \mathbf{1}$, $h(r(x_n)) = \delta_n$, and $Z(h) = \{0\}$. Set $f = h \circ r$ and observe that $f \in C(X)^+$ and $Z(f) = Z(r)$, hence $f \in r(X)^+$. Finally, for each n we have that

$$0 < f(x_n) = (h \circ r)(x_n) = \delta_n < r_n(x_n)$$

whence $R(\mathbf{0}, r_n) \nsubseteq R(\mathbf{0}, f)$ for all n, contradicting that the collection of $R(\mathbf{0}, r_n)$ is a base around $\mathbf{0}$.

Finally, from what we have just proved it follows that (vi) and (vii) are equivalent. Proposition 2.2 of [12] states that (v) and (vi) are equivalent. $\square$

Definition 1.10. For a topological space X and a point $p \in X$ recall that the character of the point p is

$$\chi(p, X) = \aleph_0 + \min\{|\mathcal{U}| : \mathcal{U} \text{ is a base of neighborhoods for } p\}.$$

The character of X is defined as

$$\chi(X) = \sup\{\chi(p, X) : p \in X\}.$$

In [2] the authors determined the character of the space $C_m(X)$. There they utilized the dominating number of a space, which is defined as follows. A subset $\mathfrak{F}$ of $C(X)$ is called *dominating* if for every $g \in C(X)$ there exists an $f \in \mathfrak{F}$ such that $g \leq f$. Then the dominating number of X is

$$\mathrm{dn}(X) = \aleph_0 + \min\{|\mathfrak{F}| : \mathfrak{F} \text{ is a dominating subset of } C(X)\}.$$

When $X = \mathbb{N}$, then we write $d = \mathrm{dn}(\mathbb{N})$. It is known that $\aleph_1 \leq d \leq \mathfrak{c}$.

Theorem 1.11 ([2, Theorem 2.3])**.** *Let X be any space. Then $\chi(C_m(X)) = \mathrm{dn}(X)$.*

Corollary 1.12. *Suppose X is an almost P-space. Then $\chi(C_r(X)) = \mathrm{dn}(X)$. In particular, $\chi(C_r(\mathbb{N})) = d$.*

To determine the character of $C_r(X)$ we need to recall the definition of the collection of almost real-valued continuous functions defined on the space X. The space $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is the two-point compactification of the real numbers. The collection of almost real-valued continuous functions on X is defined as

$$D(X) = \{f : X \to \overline{\mathbb{R}} : f^{-1}(\mathbb{R}) \text{ is a dense subset of } X\}.$$

In general, $D(X)$ is not closed under sums or products but it is always a lattice. (For a more detailed discussion of $D(X)$ see [1].) We call a subset $\mathfrak{F}$ of $D(X)$ *D-dominating* if for every $g \in D(X)$ there exists an $f \in \mathfrak{F}$ such that $g \leq f$. We define the D-dominating number of X as follows:

$$\mathrm{dn}_D(X) = \aleph_0 + \min\{|\mathfrak{F}| : \mathfrak{F} \text{ is a } D\text{-dominating subset of } D(X)\}.$$

Proposition 1.13. *Let X be a any Tychonoff space. Then*

$$\chi(C_r(X)) = \mathrm{dn}_D(X).$$

Proof: The proof is similar to that of [2, Theorem 2.3]. The only thing one needs to check is that $r \in r(X)^+$ if and only if $\frac{1}{r} \in D(X)^+$. $\square$

We would like a more internal characterization (relative to X) of the character of $C_r(X)$. We will not be able to do this exactly but we shall be able to give an appropriate upper and lower bound for the character. Denote the collection of nowhere dense zerosets of X by $Z_{\mathrm{nd}}[X]$, that is, let $Z_{\mathrm{nd}}[X] = \{Z(f) : f \in r(X)^+\}$. Observe that $Z_{\mathrm{nd}}[X]$ is an ideal of the lattice of all zerosets of X and that $\bigcup\{Z \in Z_{\mathrm{nd}}[X]\}$ is precisely the set of non-almost P-points. Let

$$z(X) = \min\{\kappa : \kappa \text{ is the cardinality of a generating set for } Z_{\mathrm{nd}}[X]\}.$$

A subcollection $\mathcal{Z}$ is a generating set for $Z_{\mathrm{nd}}[X]$ if for every $Z \in Z_{\mathrm{nd}}[X]$ there is some $Z' \in \mathcal{Z}$ such that $Z \subseteq Z'$. A generating set of minimal cardinality will be called a *minimal generating set*. It is straightforward to check that all minimal generating sets have the same cardinality. Next, let $\mathcal{F}$ be any collection of dense cozerosets of X and define

$$\mathrm{dn}_X(\mathcal{F}) = \aleph_0 + \sup\{\mathrm{dn}(C) : C \in \mathcal{F}\}.$$

When $\mathcal{Z} \subseteq Z_{\mathrm{nd}}[X]$ we define $\mathcal{F}_{\mathcal{Z}} = \{X \smallsetminus Z : Z \in \mathcal{Z}\}$. A generalization of Corollary 1.12 is given by the following proposition.

Proposition 1.14. *Suppose X has the property that $Z_{\mathrm{nd}}[X]$ has a maximum element, say Z. Then $Z = X \smallsetminus \mathfrak{a}(X)$ is the collection of non almost P-points of X. Furthermore,*

$$\chi(C_r(X)) = \mathrm{dn}(\mathfrak{a}(X)).$$

Proof: By hypothesis, there is a $\varphi \in r(X)^+$ (with $\mathbf{0} \le \varphi \le \mathbf{1}$) such that $Z(\varphi)$ such that $Z(\varphi)$ is the largest element of $Z_{\mathrm{nd}}[X]$. Since $Z(\varphi)$ is nowhere dense and by the definition of almost P-point it follows that $Z(\varphi) \subseteq X \smallsetminus \mathfrak{a}(X)$. If x is not an almost P-point, then there is some $Z \in Z_{\mathrm{nd}}[X]$ such that $x \in Z$. Now, $Z \cup Z(\varphi) \in Z_{\mathrm{nd}}[X]$ and therefore $Z \subseteq Z(\varphi)$. Therefore, $Z(\varphi) = X \smallsetminus \mathfrak{a}(X)$ which demonstrates the first statement.

Let $\mathcal{C} = \{R(\mathbf{0}, r_i)\}_{i \in I}$ be a base of neighborhoods around $\mathbf{0}$. Set $T = \{r \in r(X)^+ : R(\mathbf{0}, r) \in \mathcal{C}\}$ and then $T' = \{r \wedge \varphi : r \in T\}$. Observe that the cardinality of T' is no greater than that of T. Next, let $\mathcal{C}' = \{R(\mathbf{0}, r)\}_{r \in T'}$. $\mathcal{C}'$ is also a base of neighborhoods of $\mathbf{0}$ and its cardinality is no greater than that of the original base. Also, for every $r \in T'$ we have $Z(r) = Z(\varphi)$.

It is obvious that the collection $\mathfrak{F} = \{\frac{1}{r} : r \in T'\}$ is a subset of $C(\mathfrak{a}(X))$ of cardinality equal to that of T'. We claim that $\mathfrak{F}$ is in fact a dominating set for $C(\mathfrak{a}(X))$. Let $g \in C(\mathfrak{a}(X))$ and without loss of generality we assume that $g \ge \frac{1}{\varphi}$. It follows then that the element $\frac{1}{g} \in C(a(X))$ can be extended to all of X; namely define $\frac{1}{g}(x) = 0$ for all $x \in X \smallsetminus \mathfrak{a}(X)$. Moreover, $\frac{1}{g} \in r(X)^+$ and so there is some $r \in T'$ such that $r \le \frac{1}{g}$. Hence $g \le \frac{1}{r}$ and so $\mathfrak{F}$ is a dominating set for $C(\mathfrak{a}(X))$. We conclude that $\chi(C_r(X)) \ge \mathrm{dn}(\mathfrak{a}(X))$.

The reverse inequality is obtained in the reverse manner used above. Begin with a dominating set $\mathfrak{F}$ for $C(\mathfrak{a}(X))$ and assume that each member of $\mathfrak{F}$ is greater than or equal to $\frac{1}{\varphi}$. We leave it to the interested reader to show that the collection $\{R(\mathbf{0}, \frac{1}{f})\}_{f \in \mathfrak{F}}$ is a base of neighborhoods around $\mathbf{0}$. $\square$

Example 1.15. Let X be an almost discrete space, that is, X has exactly one non-isolated point, say $X = D \cup \{\alpha\}$. If α is not an almost P-point of X, then $Z_{\mathrm{nd}}[X] = \{\emptyset, \{\alpha\}\}$. By the previous proposition it follows that $\chi(C_r(X)) = \mathrm{dn}(D)$. In particular,

$$\chi(C_r(\alpha\mathbb{N})) = \mathrm{dn}(\mathbb{N}) = d = \chi(C_r(\mathbb{N})).$$

We are now in position to give an upper and lower bound for the character of $C_r(X)$ using our new cardinal function.

Theorem 1.16. *Let $\mathcal{Z}$ be any minimal generating set for $Z_{\mathrm{nd}}[X]$. Then*

$$z(X) \leq \chi(C_r(X)) \leq z(X) \cdot \mathrm{dn}_X(\mathcal{F}_{\mathcal{Z}}).$$

Proof: Let $\mathcal{C}$ be a base around $\mathbf{0}$ and suppose that $\kappa = |\mathcal{C}| = \chi(C_r(X)$. Without loss of generality, we can assume that every element of $\mathcal{C}$ has the form $R(\mathbf{0}, r)$ for some $r \in R(X)^+$. Furthermore, we can enumerate $\mathcal{C}$ as a κ-sequence, say $\mathcal{C} = \{R(\mathbf{0}, r_\sigma)\}_{\sigma<\kappa}$. Let

$$\mathcal{Z} = \{Z(r_\sigma) : \sigma < \kappa\}$$

and note that $\mathcal{Z} \subseteq Z_{\mathrm{nd}}[X]$. Let $Z(r) \in Z_{\mathrm{nd}}[X]$. Then there exists a $\sigma < \kappa$ such that $R(\mathbf{0}, r_\sigma) \subseteq R(\mathbf{0}, r)$ and so therefore $Z(r) \subseteq Z(r_\sigma)$. It follows that the collection $\mathcal{Z}$ is a generating set for $Z_{\mathrm{nd}}[X]$ and so $z(X) \leq \kappa = \chi(C_r(X))$.

Next, let $\mathcal{Z}$ be as in the hypothesis of the theorem. If we close up $\mathcal{Z}$ under finite unions then the cardinality will not change. Therefore, we assume that $\mathcal{Z}$ is closed under finite unions. For each $Z \in \mathcal{Z}$ let $Z = Z(\phi_Z)$ where $\phi_Z \in r(X)^+$ and $\mathbf{0} \leq \phi_Z \leq \mathbf{1}$. Let $\mathfrak{F}_Z$ be a minimal dominating set for $C(\mathrm{coz}(\phi_Z))$. Without loss of generality we assume that every $f \in \mathfrak{F}$ satisfies $f \geq \frac{1}{\phi_Z}$. It is straightforward to check that $\frac{1}{f} \in C(X)$ and that $Z(\frac{1}{f}) = Z(\phi_Z)$, hence $\frac{1}{f} \in r(X)^+$. Next, define $B_Z = \{R(\mathbf{0}, \frac{1}{f})\}_{f \in \mathfrak{F}_Z}$.

We claim that

$$\mathcal{B} = \bigcup_{Z \in \mathcal{Z}} B_Z$$

is a base of neighborhoods of $\mathbf{0}$ in $C_r(X)$. To see this let $r \in r(X)^+$ and consider $R(\mathbf{0}, r)$. Now since $\mathcal{Z}$ is a generating set for $Z_{\mathrm{nd}}[X]$ it follows that there are $Z_1, \cdots, Z_n \in \mathcal{Z}$ such that $Z(r) \subseteq Z_1 \cup \cdots \cup Z_n$. Since $\mathcal{Z}$ is closed under unions it follows that $Z_1 \cup \cdots \cup Z_n = Z \in \mathcal{Z}$. At this point we know that $R(\mathbf{0}, r \wedge \phi_Z) \subseteq R(\mathbf{0}, r)$ and so without loss of generality we suppose that $Z(r) = Z$. Therefore, since $\frac{1}{r} \in C(\mathrm{coz}(\phi_z))$ we can find an element $f \in \mathfrak{F}_Z$ such that $\frac{1}{r} \leq f$ and thus, $\frac{1}{f} \leq r$ with $Z(\frac{1}{f}) = Z(\phi_Z) = Z(r)$. It follows then that $R(\mathbf{0}, \frac{1}{f}) \subseteq R(\mathbf{0}, r)$. Therefore, $\mathcal{B}$ is a base of neighborhoods of $\mathbf{0}$ in $C_r(X)$, whence $\chi(C_r(X)) \leq z(X) \cdot \mathrm{dn}_X(\mathcal{F}_{\mathcal{Z}})$. □

In Example 1.15 we computed $\chi(C_r(X))$ for some specific spaces. We would consider more examples. In order to calculate $\chi(C_r([0,1])$ we find it useful to remind the reader of Martin's Axiom.

Definition 1.17. A space X is said to satisfy the *countable chain condition* (or *ccc*) if there is no uncountable family of pairwise disjoint non-empty open subsets of X.

Recall that Martin's Axiom states that if X is a compact Hausdorff which satisfies the ccc, then X is not the union of κ or fewer nowhere dense subsets for any $\kappa < \mathfrak{c}$.

The proof of the next lemma is straightforward and left to the interested reader.

Lemma 1.18. *X has no almost P-points if and only if $Z_{\mathrm{nd}}[X]$ covers X.*

Proposition 1.19 (MA)**.** *Let X be a compact Hausdorff space with no almost P-points and satisfying ccc. Then $z(X) = \chi(C_r(X))$. In particular, if X is either the Cantor set or the unit interval, then $\chi(C_r(X)) = \mathfrak{c}$.*

Proof: By assumption X has no almost P-points and so by Lemma 1.18 the collection of nowhere dense zerosets covers X. Let $\mathcal{Z}$ be a minimal generating set for $Z_{\mathrm{nd}}[X]$ and notice that $\mathcal{Z}$ is a cover of X. Combining together X satisfying ccc and Martin's Axiom we conclude that $\mathcal{Z}$ has cardinality no smaller than $\mathfrak{c}$, whence $z(X) \geq \mathfrak{c}$.

Now any cozeroset of any space is an F_σ-set and so in this case, since X is compact, every cozeroset of X is locally compact and σ-compact. By [2, Proposition 2.2] we have that $\mathrm{dn}(C) = d$ for every proper dense cozeroset C. Thus, $\mathrm{dn}_X(\mathcal{F}_{\mathcal{Z}}) = d$ for any nontrivial collection of nowhere dense zero sets $\mathcal{Z}$. Since $d \leq \mathfrak{c}$ it follows from Theorem 1.16 that $z(X) = \chi(C_r(X))$. □

2. When $C_r(X)$ is a weak P-space

We now turn our attention to determining when $C_r(X)$ is a weak P-space. Recall that a *weak P-space* is a space for which every countable subset is closed. It follows that a metrizable weak P-space is discrete.

Recall from [15] that a space X is called a *discrete refining sequence space* or *DRS-space* for short, if for every sequence of nonempty open sets, say $\{O_n\}_{n\in\mathbb{N}}$, there is a discrete sequence of nonempty open sets, say $\{V_n\}_{n\in\mathbb{N}}$, such that $V_n \subseteq O_n$ for each $n \in \mathbb{N}$. (Note that we do not require that $O_n \neq O_m$ for $n \neq m$. By a discrete sequence $\{V_n\}_{n\in\mathbb{N}}$ we mean that each point of x has a neighborhood which intersects at most one of the V_n.)

Van Douwen was interested in constructing spaces X for which $C_m(X)$ is a weak P-space. This notion led him to the definition of a DRS-space. Van Douwen was able to prove that if X is a DRS-space, then $C_m(X)$ is a weak P-space. In [7] the authors prove the converse. Our aim is to modify the definition of DRS-space to obtain a similar characterization of when $C_r(X)$ is a weak P-space.

Some facts about DRS-spaces (Proposition 5.5 of [15]) include that they are never pseudocompact, they do not contain isolated points, a dense subspace of a DRS-space is a DRS-space, and if X is a DRS-space, then so is $X \times Y$ for any space Y.

Definition 2.1. We call X a *weak DRS-space* if for every sequence of nonempty open sets, say $\{O_n\}_{n\in\mathbb{N}}$, there is a sequence $\{V_n\}_{n\in\mathbb{N}}$ of nonempty open sets such that for each $n \in \mathbb{N}$ $V_n \subset O_n$, and $\{V_n\}_{n\in\mathbb{N}}$ is a discrete family of non-empty subsets when restricted to the complement of some nowhere dense zeroset of X. Note that a weak DRS-space contains no isolated points.

Remark 2.2. Clearly a DRS-space is a weak DRS-space. We will show later that there exist compact weak DRS-spaces. Since a DRS-space is never pseudocompact it follows that there are weak DRS-spaces that are not DRS-spaces. It will follow from our main theorem that an almost P-space is a DRS-space if and only if it is a weak DRS-space.

Lemma 2.3. *X is a weak DRS-space if and only if for every sequence of nonempty open sets $\{O_n\}_{n\in\mathbb{N}}$ there exists a sequence of distinct points $\{x_n\}$ with $x_n \in O_n$ for each $n \in \mathbb{N}$, and an $r \in r(X)^+$, such that $r(x_n) = \frac{1}{n}$ for all $n \in \mathbb{N}$.*

Proof: If X is a weak DRS-space, then given a sequence of nonempty sets $\{O_n\}_{n\in\mathbb{N}}$ there is a refinement $\{V_n\}$ which is discrete in $\mathrm{coz}(v)$ for some $v \in r(X)^+$. We further assume that $\mathbf{0} \leq v \leq \mathbf{1}$. Choose a distinct sequence of points, say $\{x_n\}_{n\in\mathbb{N}}$, such that $x_n \in V_n \cap \mathrm{coz}(v)$ for each natural number n. Since X is Tychonoff, for each $n \in \mathbb{N}$, there exists an $f_n \in C(\mathrm{coz}(v))$ such that $f_n(x_n) = \frac{n}{v(x_n)}$ and $f(y) = 0$ for all $y \in (X \smallsetminus V_n) \cap \mathrm{coz}(v)$. Since the sequence $\{V_n \cap \mathrm{coz}(v)\}$ is a discrete sequence of open subsets of $\mathrm{coz}(v)$, it is straightforward to check that the function $f = \sum f_n$ belongs to $C(\mathrm{coz}(v))$. Next, let $r = v(f \bigvee \mathbf{1})^{-1}$. Observe that $r \in C(X)^+$ and $\mathrm{coz}(r) = \mathrm{coz}(v)$ so that $r \in r(X)^+$. Moreover, $r(x_n) = \frac{1}{n}$ for each $n \in \mathbb{N}$.

The converse is clear. □

Definition 2.4. Recall that a π-base for X is a collection of nonempty open sets, say $\mathcal{U}$, such that for any open subset O of X there is some $U \in \mathcal{U}$ such that $U \subseteq O$. The *π-weight* of a space is defined as

$$\pi\omega(X) = \aleph_0 + \min\{|\mathcal{U}| : \mathcal{U} \text{ is a } \pi\text{-base for } X\}.$$

Since every base for X is a π-base for X it follows that the weight of X exceeds its π-weight.

Proposition 2.5. *If X is a weak DRS-space, then $\pi\omega(X) > \aleph_0$. In particular, X is not second countable.*

Proof: Let X be a weak DRS-space. Suppose, on the contrary that $\{B_n\}$ is a countable π-base of nonempty open sets. By Lemma 2.3 there is an $r \in r(X)^+$ and a sequence of distinct points, say $S = \{x_n\}_{n\in\mathbb{N}}$, such that $x_n \in B_n$ and $r(x_n) = \frac{1}{n}$ for each natural number n. Since $\{B_n\}_{n\in\mathbb{N}}$ is a π-base it is straightforward to check that S is a countable dense subset of X. Since we know that X has no isolated points it follows that there exists a $y \in X \smallsetminus S$ such that $r(y) > 0$. But by the density of S and continuity of r, $r(y) = 0$, a contradiction. □

The proofs of the following two lemmas are similar to the proofs for a DRS-space.

Lemma 2.6. *A nonempty open subset of a weak DRS-space is a weak DRS-space.*

Lemma 2.7. *A nonempty dense subset of a weak DRS-space is a weak DRS-space.*

Lemma 2.8. *Suppose X is a weak DRS-space and Y is any space. Then $X \times Y$ is a weak DRS-space.*

Lemma 2.9. *Suppose X is a space containing a dense cozeroset, say U, for which U is a weak DRS-space. Then X is a weak DRS-space.*

Proof: Let $\{O_n\}_{n\in\mathbb{N}}$ be a sequence of nonempty open subsets of X. The sequence $\{O_n \cap U\}_{n\in\mathbb{N}}$ is a sequence of nonempty open subsets of U. Therefore, by hypothesis, there is a sequence $\{V_n\}_{n\in\mathbb{N}}$ of cozerosets of U with $V_n \subseteq O_n \cap U$ for each n and $\{V_n\}_{n\in\mathbb{N}}$ is discrete when restricted to a dense cozeroset of U. Since a cozeroset of a cozeroset is a cozeroset (and a dense subspace of a dense subspace is dense) we have that in X, the sequence $\{V_n\}_{n\in}$ is discrete when restricted to a dense cozeroset of X. □

Corollary 2.10. *A space X is a weak DRS-space if and only if each dense cozeroset of X is a weak DRS-space.*

Proof: If X is a weak DRS-space, then by Lemma 2.7 every dense cozeroset of X is a weak DRS-space. Conversely, let U be a dense cozeroset of X and so by Lemma 2.9 X is as well. □

Proposition 2.11. *If $\{X_\alpha\}$ is an uncountable collection of nontrivial spaces, then $\prod X_\alpha$ is a weak DRS-space.*

Proof: Let $\{O_n\}$ be a sequence of nonempty open sets in $\prod X_\alpha$. We may assume, without loss of generality, that the O_n's are basic open sets in the cartesian product topology. Altogether, there are at most a countable number of coordinates where the full space does not occur in the product expression of the O_n's. Since α is an uncountable index we can find a countably infinite subset $\{X_{\alpha_i}\}$ of $\{X_\alpha\}$ such that $\pi(O_n) = \prod X_{\alpha_i}$ for all n. Here π is the projection of $\prod X_\alpha$ onto $\prod X_{\alpha_i}$. Now, since these are all nontrivial Tychonoff spaces, there exist continuous real-valued functions s_i on X_{α_i} with minimum and maximum values 0 and $1/i$ respectively. Define s on $\prod X_{\alpha_i}$ by $s((x_{\alpha_i})) = \sup\{s_i(x_{\alpha_i})\}$. To see that s is continuous note that if a and b are real numbers, $b > 0$, and π_k the projection of $\prod X_{\alpha_i}$ onto X_{α_k}, then $s^{-1}((a,1]) = \bigcup \pi_i^{-1}(s_i^{-1}((a,1/i]))$ and $s^{-1}([0,b)) = \prod s_i^{-1}([0,b))$ are open sets, so $s^{-1}((a,b))$ is open. Moreover, s is a regular element in $C(\prod X_{\alpha_i})$ whose image contains $1/n$ for all $n \in \mathbb{N}$. Thus $r = s\pi$ is also a regular element and one can find a sequence of points $x_n \in O_n$ with $r(x_n) = 1/n$. This proves the proposition. □

The previous proposition provides examples of weak DRS-spaces where no point is first countable; also examples of compact weak DRS-spaces. Here is an example with a point that satisfies first countability.

Example 2.12. Let $\{X_n\}_{n\in\mathbb{N}}$ be a denumerable collection of DRS-spaces and let Y be the topological sum of these spaces together with an extra point, say p. Define a neighborhood of p as any open subset O such that $O \cap X_n = X_n$ for all but a finite number of n. Then p has a countable base of neighborhoods and hence Y is not a DRS-space. But by Lemma 2.9 Y is a weak DRS-space.

We now characterize when $C_r(X)$ is a weak P-space.

Theorem 2.13. *X is a weak DRS-space if and only if $C_r(X)$ is a weak P-space.*

Proof: We first prove the necessity. Since $C_r(X)$ is a homogeneous space it is enough to show that $\mathbf{0}$ is not in the closure of any sequence of non-zero elements in $C_r(X)$. Moreover, it is enough to show that $\mathbf{0}$ is not in the closure of any nonzero nonnegative sequence, say $\{f_n\}_{n\in\mathbb{N}}$. Given the sequence of nonempty open sets $\{\mathrm{coz}(f_n)\}_{n\in\mathbb{N}}$, by Lemma 2.3, there is a distinct sequence $\{x_n\}_{n\in\mathbb{N}}$, and a positive regular element $r \in r(X)^+$, such that for every $n \in \mathbb{N}$, $x_n \in \mathrm{coz}(f_n)$ and $r(x_n) = \frac{1}{n}$. Next, choose a decreasing sequence of positive real numbers, say $\{s_n\}_{n\in\mathbb{N}}$, such that $s_n < \min\{\frac{1}{n}, f_n(x_n)\}$ for each natural number n. Let $h \in C(R)^+$ such that $h(\frac{1}{n}) = s_n$ and $Z(h) = \{0\}$. Then $Z(r) = Z(h \circ r)$ and $(h \circ r)(x_n) < f_n(x_n)$ for each $n \in \mathbb{N}$. The former implies that $h \circ r \in r(X)^+$ and the latter forces $f_n \notin R(\mathbf{0}, h \circ r)$. Therefore, $\mathbf{0} \notin \mathrm{cl}\{f_n\}$.

Conversely, first observe that X has no isolated points. Next, let $\{O_n\}_{n\in\mathbb{N}}$ be a sequence of nonempty open sets. We assume, without loss of generality, that $O_n = \mathrm{coz}(f_n)$ with $\mathbf{0} \le f_n \le \frac{\mathbf{1}}{\mathbf{n}}$. Since $C_r(X)$ is a weak P-space there is an $r \in r(X)^+$ with $f_n \notin R(\mathbf{0}, r)$ for all n. Thus one can find an $x_n \in O_n$ with $0 < r(x_n) < f_n(x_n) \le \frac{1}{n}$. Moreover, since X has no isolated points we can choose the sequence $\{x_n\}_{n\in\mathbb{N}}$ to be distinct. It follows by Lemma 2.3 that X is a weak DRS-space. ☐

Remark 2.14. Recall that there is a natural (ring) isomorphism between $C^*(X)$ and $C(\beta X)$. Namely, for any $f \in C^*(X)$, the unique extension of f to all of βX is denoted by f^β. A natural question is whether the subspace topology on $C^*(X)$ inherited from $C_r(X)$ coincides with the r-topology on $C(\beta X)$. It is known that for the m-topology, the analogous question is answered in the negative. This is because it is possible for $u \in U(X)^+ \cap C^*(X)$ but $u^\beta \notin U(\beta X)^+$. We now answer the question for the r-topology. It is because of the next result that it is our opinion that the r-topology is a much more well-behaved topology than the m-topology.

Proposition 2.15. *The subspace topology on $C^*(X)$ inherited from $C_r(X)$ is homeomorphic to the r-topology on $C(\beta X)$. Moreover, the two topologies on $C^*(X)$ inherited from $C_m(X)$ and $C_r(X)$ are equal.*

Proof: Observe that $r \in r(X)^+ \cap C^*(X)$ if and only if $r^\beta \in r(\beta X)^+$. Since the collection $\{R(\mathbf{0}, r \wedge \mathbf{1})\}_{r\in r(X)^+}$ forms a base around $\mathbf{0} \in C^*(X)$ with respect to the subspace topology inherited from $C_r(X)$ and this collection corresponds exactly to the base around $\mathbf{0} \in C(\beta X)$, the result follows. ☐

Proposition 2.16. *$C_r(X)$ is a weak P-space if and only if $C_r(\beta X)$ is a weak P-space.*

Proof: Since a subspace of a weak P-space is again a weak P-space it follows then that if $C_r(X)$ is weak P-space, then so is $C^*(X)$ with respect to the subspace topology. But by Proposition 2.15 we conclude that $C_r(\beta X)$ is a weak P-space.

Next, suppose that $C_r(X)$ is not a weak P-space. This implies that there is a sequence of continuous functions, say $\{f_n\}$, which is not closed. By translation, we can assume that $\mathbf{0} \in \mathrm{cl}\{f_n\}_{n\in\mathbb{N}} \smallsetminus \{f_n\}_{n\in\mathbb{N}}$. It is straightforward to check that $0 \in \mathrm{cl}\{f_n \wedge \mathbf{1}\}_{n\in\mathbb{N}} \smallsetminus \{f_n \wedge \mathbf{1}\}_{n\in\mathbb{N}}$. But this implies that $C^*(X)$ is not a weak P-space, i.e., $C_r(\beta X)$ is not a weak P-space. □

Corollary 2.17. *X is a weak DRS-space if and only if βX is a weak DRS-space. In particular, the Stone-Čech compactification of a DRS-space is a weak DRS-space.*

3. A topological characterization of separable weak DRS-spaces

The motivating example for this section is the following:

Example 3.1. $\mathbb{R}$ is not a weak DRS-space.

In fact we shall prove more.

Definition 3.2. For a given $x \in X$, a *π-base of neighborhoods of x* is a collection of nonempty open subsets of X, say $\mathcal{U}$, such that for any neighborhood O of x there is a $U \in \mathcal{U}$ such that $U \subseteq O$. We define the π-character of x as

$$\pi\chi(x, X) = \aleph_0 + \min\{|\mathcal{U}| : \mathcal{U} \text{ is a } \pi\text{-base of neighborhoods of } x\},$$

and the *π-character of X* as

$$\pi\chi(X) = \sup\{\pi\chi(x, X) : x \in X\}.$$

As with the weight we always have $\chi(X) \geq \pi\chi(X)$.

In [15] the author showed that a DRS-space cannot have any points of countable π-character. He then showed that for a countable space that this was also sufficient. Formally, we have:

Theorem 3.3 ([15])**.** *Let X be a countable space. X is a DRS-space if and only if $\pi\chi(x, X) > \aleph_0$ for all $x \in X$.*

In a weak DRS-space you can have points of countable π-character. However, a weak DRS-space cannot have a countable π-base, as we presently show. Later, we will generalize Theorem 3.3 to separable spaces.

Lemma 3.4. *Let X be a separable space with a dense sequence of distinct points, say $\{x_n\}_{n\in\mathbb{N}}$, such that $\pi\chi(x_n, X) > \aleph_0$ for all n. Given a sequence of nonempty open sets $\{U_{2i}\}_{i\in\mathbb{N}}$, there is a cover of $\{x_n\}_{n\in\mathbb{N}}$ consisting of a sequence of disjoint nonempty open sets $\{V_n\}_{n\in\mathbb{N}}$ such that $V_{2i} \subset U_{2i}$ for all $i \in \mathbb{N}$. Moreover, $\bigcup V_i$ is a dense cozeroset of X.*

Proof: First of all we note that by hypothesis none of the x_n are isolated points. Let $U_{2i-1} = X$ for all $i \in \mathbb{N}$ and let $v_1 = x_1$. Since $\pi_\chi(v_1, X) > \aleph_0$ there is a cozeroset O_1 with $v_1 \in O_1 \subset U_1$ such that $U_i \not\subset O_1$ for all $i > 1$. Choose $f_1 \in C(X)^+$ such that $f(v_1) = 1$ and $f(y) = 0$ for all $y \in X \setminus O_1$. Since $\{x_n\}_{n\in\mathbb{N}}$ is a countable set, there is an $r_1 \in (0,1)$ such that $f_1^{-1}(r_1) \cap \{x_n\}_{n\in\mathbb{N}} = \emptyset$. Let

$$W_1 = f_1^{-1}[0, r_1) \qquad\qquad V_1 = f_1^{-1}(r_1, \infty]$$

and observe that W_1 and V_1 are disjoint cozerosets of X with $V_1 \subseteq O_1 \subseteq U_1$. Note that $U_i \cap W_1 \neq \emptyset$ for all $i > 1$. We use recursion now.

Suppose we have a pair of collections of cozerosets, say $\{W_i\}_{i=1}^n$ and $\{V_i\}_{i=1}^n$, and a sequence $\{v_i\}_{i=1}^n \subseteq \{x_i\}_{i\in\mathbb{N}}$ such that 1) $W_i \cap V_i = \emptyset$ for each $i = 1, \cdots, n$, 2) $v_i \in V_i$ and $v_i \neq v_j$ for each $1 \leq i < j \leq n$, 3) $U_i \cap (\bigcap_{j=1}^n W_j) \neq \emptyset$ for each $i > n$, 4) for each $i = 1, \cdots, n$, $V_i \subseteq U_i$, and 5) $\{V_i\}_{i=1}^n$ is pairwise disjoint. Now let

$$i_{n+1} = \min\left\{ i \in \mathbb{N} : x_i \in U_{n+1} \cap \Big(\bigcap_{j=1}^{n} W_j \Big) \right\}$$

and set $v_{n+1} = x_{i_{n+1}}$. Since for each $i = 1, \cdots, n$ we have $v_i \in V_i$ and hence $v_i \notin W_i$ it follows that $v_i \neq v_{n+1}$. Choose a cozeroset neighborhood of v_{n+1}, say O_{n+1} such that $v_{n+1} \in O_{n+1} \subseteq U_{n+1} \cap (\bigcap_{j=1}^n W_j)$ and $U_k \cap (\bigcap_{j=1}^n W_j) \nsubseteq O_{n+1}$ for all $k > n$. We can do this because $\pi\chi(v_{n+1}, X) > \aleph_0$. Next, choose a function $f \in C(X)^+$ such that $f(v_{n+1}) = 1$ and $f(y) = 0$ for all $y \in X \setminus O_{n+1}$. There is an $0 < r_{n+1} < 1$ such that $f(x_k) \neq r$ for all $k \in \mathbb{N}$. Let $W_{n+1} = f^{-1}([0, r_{n+1}))$ and $V_{n+1} = f^{-1}((r_{n+1}, \infty))$. Since $V_{n+1} \subseteq O_{n+1} \subseteq U_{n+1} \cap (\bigcap_{j=1}^n W_j)$ it follows that $V_{n+1} \subseteq U_{n+1}$ and that $V_{n+1} \cap V_i = \emptyset$ for all $i = 1, \cdots, n$. Thus, our new pair of collections of cozerosets $\{W_i\}_{i=1}^{n+1}$ and $\{V_i\}_{i=1}^{n+1}$ satisfies the properties 1) through 5) from above.

By induction there is a sequence, $\{V_n\}_{n\in\mathbb{N}}$, of pairwise disjoint cozeroset with $V_i \subseteq U_i$ for all $i \in \mathbb{N}$. Letting $V = \bigcup_{i\in\mathbb{N}} V_n$ we get that since V is a countable union of cozerosets it is a cozeroset. Furthermore, it is straightforward to check that $\{x_i\}_{i\in\mathbb{N}} \subseteq \bigcup_{i\in\mathbb{N}} V_i$ so that V is a dense subset of X. This concludes the proof of the lemma. □

We prove the main result of this section:

Theorem 3.5. *Let X be a separable space. X is a weak DRS-space if and only if there exists a countable dense subset of X, say $\{x_j\}_{j\in\mathbb{N}}$, such that $\pi\chi(x_j, X) > \aleph_0$ for all $j \in \mathbb{N}$.*

Proof: Suppose X is a weak DRS-space. Note that there cannot exist a countable dense subset of X, say $\{x_j\}_{j\in\mathbb{N}}$, such that $\pi\chi(x_j, X) = \aleph_0$ for every $j \in \mathbb{N}$. If so then it would follow that X has countable π-weight, contradicting Theorem 2.5. Thus, let S be a countable dense subset of X and split S into two disjoint sets, S_0 and $S \setminus S_0$, where $x \in S$ belongs to S_0 if and only $\pi\chi(x, X) = \aleph_0$. If $S \setminus S_0$ is a dense subset of X, we are done. Otherwise let $O = X \setminus \mathrm{cl}_X(S \setminus S_0)$ and

$T = S_0 \cap O$. Then T is a dense subset of the nonempty set O. But since O is an open subset of X it follows that $\pi\chi(x, O) = \aleph_0$ for each $x \in T$ and so O cannot be a weak DRS-space, contradicting Lemma 2.6.

As for the sufficiency suppose $\{x_j\}_{j\in\mathbb{N}}$ is a countable dense subset (of distinct points) of X with $\pi\chi(x_j, X) > \aleph_0$ for each $j \in \mathbb{N}$. Let $\{U_n\}_{n\in\mathbb{N}}$ be a sequence of nonempty open sets of X. By Lemma 3.4 we can find a sequence of pairwise disjoint cozerosets, say $\{V_n\}_{n\in\mathbb{N}}$, whose union is a dense cozeroset. It follows that when restricted to the dense cozeroset the collection is discrete. Therefore, X is a weak DRS-space. □

Theorem 3.6. *Suppose X satisfies the property that every dense open set contains a dense cozeroset, e.g. a perfectly normal space. Furthermore, suppose that $\pi\chi(x, X) = \aleph_0$ for all $x \in X$. Then X is a weak DRS-space if and only if X is nowhere separable.*

Proof: To prove the necessity observe that any weak DRS-space satisfying $\pi\chi(x, X) = \aleph_0$ for all $x \in X$ will be nowhere separable. This follows from 2.6 that if O is any open subset of X, then O is a weak DRS-space. It is straightforward to check that $\pi\chi(x, O) = \aleph_0$ for all $x \in O$. But this contradicts Theorem 3.5.

Conversely, suppose X is nowhere separable and let $\{f_i\}_{i\in\mathbb{N}}$ be a sequence of continuous functions all of which are different than $\mathbf{0}$. Without loss of generality we assume that $f_i > \mathbf{0}$. Choose a sequence $\{x_i\}_{i\in\mathbb{N}}$ of distinct points with $x_i \in \operatorname{coz}(f_i)$. Let $T = \operatorname{cl}\{x_i\}_{i\in\mathbb{N}}$. We claim that T is a nowhere dense subset of X. If it is not, then $\operatorname{int} T \neq \emptyset$ is an open subset of a separable set, hence separable. This contradicts that X is nowhere separable. Next, since $X \smallsetminus T$ is a dense open subset of X we can apply the hypothesis and conclude that $X \smallsetminus T$ densely contains a cozeroset, say $\operatorname{coz}(r)$. Observe that $\operatorname{coz}(r)$ is a dense subset of X. Therefore, $r \in r(X)^+$. Finally, $0 = r(x_i) < f_i(x_i)$ so that $\mathbf{0} \notin \operatorname{cl}\{f_i\}_{i\in\mathbb{N}}$; whence X is a weak DRS-space. □

Corollary 3.7. *Suppose X is a metric space. X is a weak DRS-space if and only if X is nowhere separable.*

Example 3.8. Let E be the (Iliadis) absolute of the space $[0, 1]$. It is known that E has countable π-weight (see [14]). In Example 4.6 of [7] it is shown that E is not a DRS-space. Since E has countable π-weight it follows that $\pi\chi(x, E) = \aleph_0$ for all $x \in E$ and so, by Theorem 3.5, E is not a weak DRS-space.

For the purpose of this example (and throughout the rest of the article) by a *crowded* space we mean a space without isolated points.

Not every crowded basically disconnected space is a weak DRS-space even though every crowded P-space is a DRS-space, and so every basically disconnected space without isolated points which is of the form βX for X a P-space is a weak DRS-space.

We finish this article by showing that even though $C_r(X)$ might not be a weak P-space for a crowded basically disconnected space, it does share a property with weak P-spaces. Recall from [9] that a space X is called a *cozero complemented space* if for every cozeroset $C \subseteq X$ there is a cozeroset C' such that $C \cap C' = \emptyset$ and $C \cup C'$ is a dense subset of X. Basically disconnected spaces and perfectly normal spaces are cozero complemented.

Remark 3.9. Observe that if $x \in X$ is an isolated point and we let $f = \chi_{\{x\}}$ denote the characteristic function on $\{x\}$, then the sequence $\{\frac{1}{n}f\}_{n\in\mathbb{N}}$ converges to $\mathbf{0}$ in $C_r(X)$.

Proposition 3.10. *Suppose X is a crowded cozero complemented space. Then there are no nontrivial convergent sequences in $C_r(X)$.*

Proof: Note that if such a sequence exists then there is one converging to $\mathbf{0}$. Let $\{\mathrm{coz}(f_j)\}_{j\in\mathbb{N}}$ be a sequence of nonempty cozerosets of X with $f_j \in C(X)^+$. Since X has no isolated points we can find a subsequence $S \subseteq N$ and a discrete sequence of distinct points, say $\{x_n\}_{j\in S}$, such that $x_n \in \mathrm{coz}(f_n)$. This means that there is a discrete sequence of cozerosets, say $\{V_n\}_{n\in S}$, which is pairwise disjoint and so that $x_n \in V_n \subseteq \mathrm{coz}(f_n)$ for each $n \in S$. Now, the union C of these cozerosets is again a cozero set, say $C = \mathrm{coz}(f)$. Furthermore, we can assume that $0 < f(x_n) < f_n(x_n)$ for each $n \in S$. By hypothesis, there is a cozeroset C' so that $C \cap C' = \emptyset$ and $C \cup C'$ is a dense subset of X. Let $g \in C(X)^+$ satisfy $C' = \mathrm{coz}(g)$. Consider the function $f + g \in C(X)$. Since $\mathrm{coz}(f + g) = C \cup C'$ it follows that $f + g \in r(X)^+$. Therefore, $\mathbf{0} \notin \mathrm{cl}_{\{}\, f_n\}_{n\in S}$, whence $\mathbf{0}$ is not the limit of the sequence $\{f_j\}_{j\in\mathbb{N}}$. □

Remark 3.11. It follows from Corollary 3.7 and Proposition 3.10 that $C_r(\mathbb{R})$ is not a weak P-space yet $\mathbf{0}$ is not a limit of a non-trivial convergent sequence of functions. We conclude with the following question. Does there exist a basically disconnected space X for which $\pi\chi(x, X) = \aleph_0$ for all $x \in X$ which is nowhere separable yet X is not a weak DRS-space?

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