Uncountably many solutions of a system of third order nonlinear differential equations

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Abstract. In this paper, we aim to study the global solvability of the following system of third order nonlinear neutral delay differential equations

$$\begin{aligned}&\frac{d}{dt}\Big\{r_i(t)\frac{d}{dt}\Big[\lambda_i(t)\frac{d}{dt}\Big(x_i(t)-f_i(t,x_1(t-\sigma_{i1}),x_2(t-\sigma_{i2}),x_3(t-\sigma_{i3}))\Big)\Big]\Big\}\\&\quad+\frac{d}{dt}\Big[r_i(t)\frac{d}{dt}g_i(t,x_1(p_{i1}(t)),x_2(p_{i2}(t)),x_3(p_{i3}(t)))\Big]\\&\quad+\frac{d}{dt}h_i(t,x_1(q_{i1}(t)),x_2(q_{i2}(t)),x_3(q_{i3}(t)))\\&=l_i(t,x_1(\eta_{i1}(t)),x_2(\eta_{i2}(t)),x_3(\eta_{i3}(t))),\quad t\ge t_0,\quad i\in\{1,2,3\}\end{aligned}$$

in the following bounded closed and convex set

$$\begin{aligned}\Omega(a,b)=\Big\{x(t)=\big(x_1(t),x_2(t),x_3(t)\big)\in C([t_0,+\infty),\mathbb{R}^3):a(t)\le x_i(t)\le b(t),\\ \forall\, t\ge t_0, i\in\{1,2,3\}\Big\},\end{aligned}$$

where $\sigma_{ij}>0$, $r_i,\lambda_i,a,b\in C([t_0,+\infty),\mathbb{R}^+)$, $f_i,g_i,h_i,l_i\in C([t_0,+\infty)\times\mathbb{R}^3,\mathbb{R})$, $p_{ij},q_{ij},\eta_{ij}\in C([t_0,+\infty),\mathbb{R})$ for $i,j\in\{1,2,3\}$. By applying the Krasnoselskii fixed point theorem, the Schauder fixed point theorem, the Sadovskii fixed point theorem and the Banach contraction principle, four existence results of uncountably many bounded positive solutions of the system are established.

Keywords: system of third order nonlinear neutral delay differential equations, contraction mapping, completely continuous mapping, condensing mapping, uncountably many bounded positive solutions

Classification: 34K15, 34C10

1. Introduction

Recently, it is well known that the theory of neutral delay differential equations and systems undergoes a rapid development, especially for the existence of nonoscillatory solutions of second-order and higher order neutral delay differential equations and systems, refer to [1], [3]–[5], [7]–[9], [11]–[14] and the references therein.

In 2007, Zhou [12] used the Krasnoselskii fixed point theorem to study the existence of nonoscillatory solutions of the following second-order nonlinear neutral differential equation

$$\frac{d}{dt}\Big[r(t)\frac{d}{dt}(x(t)+p(t)x(t-\tau))\Big]+\sum_{i=1}^{m}Q_i(t)f_i(x(t-\sigma_i))=0, \quad t\ge t_0, \tag{1.1}$$

where $m\ge 1$ is an integer, $\tau>0$, $\sigma_i\ge 0$, $r,p,Q_i\in C([t_0,+\infty),\mathbb{R})$ and $f_i\in C(\mathbb{R},\mathbb{R})$ for $i\in\{1,2,\dots,m\}$.

In 2002, Zhou and Zhang [14] applied the Banach contraction principle to study the following higher order neutral functional differential equation with positive and negative coefficients

$$\frac{d^n}{dt^n}\Big[x(t)+cx(t-\tau)\Big]+(-1)^{n+1}\big[P(t)x(t-\sigma)-Q(t)x(t-\delta)\big]=0, \quad t\ge t_0, \tag{1.2}$$

where $n\ge 1$ is a integer, $c\in\mathbb{R}$, $\tau,\sigma,\delta\in\mathbb{R}^+$ and $P,Q\in C([t_0,+\infty),\mathbb{R}^+)$.

In 2005, Lin [8] got some sufficient conditions for oscillation and nonoscillation for the second-order nonlinear neutral differential equation

$$\frac{d^2}{dt^2}\Big[x(t)-p(t)x(t-\tau)\Big]+q(t)f(x(t-\sigma))=0, \quad t\ge 0, \tag{1.3}$$

where $\tau,\sigma>0$, $p,q\in C([0,+\infty),\mathbb{R})$, $f\in C(\mathbb{R},\mathbb{R})$ with $q(t)\ge 0$ and $xf(x)>0$ for $t\in\mathbb{R}$, $x\in\mathbb{R}/\{0\}$.

In 2008, a system of higher order nonlinear neutral differential equations

$$\begin{aligned}\frac{d^n}{dt^n}\Big[y_i(t)-a_i(t)y_i(t-\tau_i)\Big]&=p_i(t)g_i(y_{3-i}(t-\sigma_{3-i}))+f_i(t),\\ &\qquad t\ge t_0,\ i\in\{1,2\}\end{aligned} \tag{1.4}$$

was investigated by Hanuštiaková and Olach [4], where $n\ge 1$ is an integer, $\tau_i,\sigma_i>0$, $a_i,p_i,f_i\in C([t_0,+\infty),\mathbb{R})$ and $g_i\in C(\mathbb{R},\mathbb{R})$ for $i\in\{1,2\}$. Some sufficient conditions for the existence of nonoscillatory bounded solutions of equations (1.4) were obtained by using the Krasnoselskii fixed point theorem and the Schauder fixed point theorem.

In this paper, we are concerned with the following system of third order nonlinear neutral delay differential equations:

$$\begin{aligned}&\frac{d}{dt}\Big\{r_i(t)\frac{d}{dt}\Big[\lambda_i(t)\frac{d}{dt}\Big(x_i(t)-f_i(t,x_1(t-\sigma_{i1}),x_2(t-\sigma_{i2}),x_3(t-\sigma_{i3}))\Big)\Big]\Big\}\\ &\quad+\frac{d}{dt}\Big[r_i(t)\frac{d}{dt}g_i(t,x_1(p_{i1}(t)),x_2(p_{i2}(t)),x_3(p_{i3}(t)))\Big]\\ &\quad+\frac{d}{dt}h_i(t,x_1(q_{i1}(t)),x_2(q_{i2}(t)),x_3(q_{i3}(t)))\\ &=l_i(t,x_1(\eta_{i1}(t)),x_2(\eta_{i2}(t)),x_3(\eta_{i3}(t))), \quad t\ge t_0,\ i\in\{1,2,3\},\end{aligned} \tag{1.5}$$

where $\sigma_{ij} > 0$, $r_i, \lambda_i \in C([t_0,+\infty),\mathbb{R}^+)$, $f_i, g_i, h_i, l_i \in C([t_0,+\infty) \times \mathbb{R}^3, \mathbb{R})$, $p_{ij}, q_{ij}, \eta_{ij} \in C([t_0,+\infty),\mathbb{R})$ with

$$\lim_{t\to+\infty} p_{ij}(t) = \lim_{t\to+\infty} q_{ij}(t) = \lim_{t\to+\infty} \eta_{ij}(t) = +\infty$$

for $i, j \in \{1, 2, 3\}$.

By using the Krasnoselskii fixed point theorem, the Schauder fixed point theorem, the Sadovskii fixed point theorem and the Banach contraction principle respectively, we demonstrate four existence theorems of uncountably many bounded positive solutions of equations (1.5).

2. Preliminaries

Throughout this paper, put $I = [t_0, +\infty)$ and let $C(I, \mathbb{R}^3)$ denote the Banach space of all continuous and bounded vector functions $x(t) = (x_1(t), x_2(t), x_3(t))$ on I with norm $\|x\| = \max_{1\le i\le 3} \sup_{t\in I} |x_i(t)|$. For any $a, b \in C(I, \mathbb{R}^+)$, set $\overline{a} = \sup_{t\in I} a(t)$, $\underline{a} = \inf_{t\in I} a(t)$, $\overline{b} = \sup_{t\in I} b(t)$, $\underline{b} = \inf_{t\in I} b(t)$ and

$$\Omega(a,b) = \Big\{x(t) = (x_1(t), x_2(t), x_3(t)) \in C(I,\mathbb{R}^3) : a(t) \le x_i(t) \le b(t),\\ \forall\, t \in I, i \in \{1,2,3\}\Big\}.$$

Obviously, $\Omega(a,b)$ is a bounded closed and convex subset of $C(I, \mathbb{R}^3)$. For any $D \subseteq \Omega(a,b)$ and $t \in I$, let

$$D(t) = \sup\big\{ \max_{1\le i\le 3} |x_i(t) - y_i(t)| : x(t) = (x_1(t), x_2(t), x_3(t)),\\ y(t) = (y_1(t), y_2(t), y_3(t)) \in D\big\};$$
$$\operatorname{diam} D = \sup\{\|x-y\| : x, y \in D\}.$$

It is assumed in the sequel that there exist functions $a, b, c_i, d_i, \alpha_i, \beta_i, \gamma_i, \mu_i, \tau_i, \zeta_i \in C(I, \mathbb{R}^+)$ for $i \in \{1,2,3\}$ with $a(t) < b(t)$ for $t \in I$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

(i) $\int_{t_0}^{+\infty} \max\left\{\frac{\alpha_i(s)}{\lambda_i(s)}, \frac{\beta_i(s)}{r_i(s)}, \gamma_i(s), \frac{1}{r_i(s)}, \frac{1}{\lambda_i(s)}\right\} ds < +\infty$, $i \in \{1,2,3\}$;
(ii) $|f_i(t, u_1, u_2, u_3)| \le c_i(t)$, $\forall\, t \in I$, $u_i \in [\underline{a}, \overline{b}]$, $i \in \{1,2,3\}$;
(iii) $|f_i(t, u_1, u_2, u_3) - f_i(t, v_1, v_2, v_3)| \le d_i(t) \max_{1\le j\le 3} |u_j - v_j|$, $\forall\, t \in I$, $u_j, v_j \in [\underline{a}, \overline{b}]$, $i, j \in \{1,2,3\}$;
(iv) $|g_i(t, u_1, u_2, u_3)| \le \alpha_i(t)$, $|h_i(t, u_1, u_2, u_3)| \le \beta_i(t)$, $|l_i(t, u_1, u_2, u_3)| \le \gamma_i(t)$, $\forall\, t \in I$, $u_i \in [\underline{a}, \overline{b}]$, $i \in \{1,2,3\}$;
(v) $\int_{t_0}^{+\infty} \max\left\{\frac{s\alpha_i(s)}{\lambda_i(s)}, \frac{\beta_i(s)}{r_i(s)}, \gamma_i(s), \frac{1}{r_i(s)}, \frac{s}{\lambda_i(s)}\right\} ds < +\infty$, $i \in \{1,2,3\}$;

(vi)

$$
\begin{aligned}
&\big|f_i(t,x_1(t-\sigma_{i1}),x_2(t-\sigma_{i2}),x_3(t-\sigma_{i3}))\\
&-f_i(t,y_1(t-\sigma_{i1}),y_2(t-\sigma_{i2}),y_3(t-\sigma_{i3}))\big|\\
&+\int_t^{+\infty}\frac{1}{\lambda_i(s)}|g_i(s,x_1(p_{i1}(s)),x_2(p_{i2}(s)),x_3(p_{i3}(s)))\\
&\qquad\qquad -g_i(s,y_1(p_{i1}(s)),y_2(p_{i2}(s)),y_3(p_{i3}(s)))|\,ds\\
&+\int_t^{+\infty}\int_s^{+\infty}\frac{1}{\lambda_i(s)r_i(u)}|h_i(u,x_1(q_{i1}(u)),x_2(q_{i2}(u)),x_3(q_{i3}(u)))\\
&\qquad\qquad -h_i(u,y_1(q_{i1}(u)),y_2(q_{i2}(u)),y_3(q_{i3}(u)))|\,du\,ds\\
&+\int_t^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{1}{\lambda_i(s)r_i(u)}|l_i(v,x_1(\eta_{i1}(v)),x_2(\eta_{i2}(v)),x_3(\eta_{i3}(v)))\\
&\qquad\qquad -l_i(v,y_1(\eta_{i1}(v)),y_2(\eta_{i2}(v)),y_3(\eta_{i3}(v)))|\,dv\,du\,ds\\
&\le\varphi(D(t)),\quad \forall\, D\subseteq\Omega(a,b),\ x,y\in D,\ t\in I,\ i\in\{1,2,3\};
\end{aligned}
$$

(vii) $|g_i(t,u_1,u_2,u_3)-g_i(t,v_1,v_2,v_3)|\le\mu_i(t)\max_{1\le j\le 3}|u_j-v_j|$,
$|h_i(t,u_1,u_2,u_3)-h_i(t,v_1,v_2,v_3)|\le\tau_i(t)\max_{1\le j\le 3}|u_j-v_j|$,
$|l_i(t,u_1,u_2,u_3)-l_i(t,v_1,v_2,v_3)|\le\zeta_i(t)\max_{1\le j\le 3}|u_j-v_j|$,
$\forall\, t\in I$, $u_j,v_j\in[\underline{a},\overline{b}]$, $i,j\in\{1,2,3\}$;

(viii) $\int_{t_0}^{+\infty}\max\left\{\frac{\mu_i(s)}{\lambda_i(s)},\frac{\tau_i(s)}{r_i(s)},\zeta_i(s),\frac{1}{r_i(s)},\frac{1}{\lambda_i(s)}\right\}ds<+\infty,\ \ i\in\{1,2,3\}$.

Let $\sigma=\max\{\sigma_{ij}: i,j\in\{1,2,3\}\}$. By a solution of equations (1.5), we mean a vector function $x=(x_1,x_2,x_3)$ such that for some $t_1\ge t_0$ and $i\in\{1,2,3\}$, $x_i\in C([t_1-\sigma,+\infty),\mathbb{R})$, $x_i(t)-f_i(t,x_1(t-\sigma_{i1}),x_2(t-\sigma_{i2}),x_3(t-\sigma_{i3}))$ is 3 times continuously differentiable on $[t_1,+\infty)$, $g_i(t,x_1(p_{i1}(t)),x_2(p_{i2}(t)),x_3(p_{i3}(t)))$ is 2 times continuously differentiable on $[t_1,+\infty)$, $h_i(t,x_1(q_{i1}(t)),x_2(q_{i2}(t)),x_3(q_{i3}(t)))$ is continuously differentiable on $[t_1,+\infty)$ and equations (1.5) hold for $t\ge t_1$.

The following four lemmas play significant roles in this paper.

Lemma 2.1 (Krasnoselskii Fixed Point Theorem [2]). *Let D be a nonempty bounded closed convex subset of a Banach space X and $S,Q: D\to X$ satisfy $Sx+Qy\in D$ for each $x,y\in D$. If Q is a contraction mapping and S is a completely continuous mapping, then the equation $Sx+Qx=x$ has at least one solution in D.*

Lemma 2.2 (Schauder Fixed Point Theorem [2]). *Let D be a nonempty closed convex subset of a Banach space X. Let $S: D\to D$ be a continuous mapping such that SD is a relatively compact subset of X. Then S has at least one fixed point in D.*

Lemma 2.3 (Sadovskii Fixed Point Theorem [10]). *Let D be a nonempty bounded closed convex subset of a Banach space X and $S : D \to D$ be a continuous condensing mapping. Then S has at least one fixed point in D.*

Lemma 2.4 (Banach contraction principle). *Let D be a closed subset of a completely metric space X and $S : D \to D$ be a contraction on D. Then S has at least one fixed point in D.*

3. Existence of uncountably many bounded positive solutions

In this section, we demonstrate the existence of uncountably many bounded positive solutions of equations (1.5). Let

$$c = \max_{1\le i\le 3} \sup_{t\in I} c_i(t) \quad \text{and} \quad d = \max_{1\le i\le 3} \sup_{t\in I} d_i(t).$$

Theorem 3.1. *Let $a, b \in C(I, \mathbb{R}^+)$ with $\overline{a} < \underline{b}$ and let* (i)-(iv) *hold. If $d \in (0, 1)$ and $c < \frac{\underline{b}-\overline{a}}{2}$, then equations* (1.5) *possess uncountably many bounded positive solutions in $\Omega(a, b)$.*

Proof: Set $L \in (\overline{a} + c, \underline{b} - c)$. According to (i), we deduce that there exists $T \ge t_0 + \sigma$ large enough satisfying

$$\begin{aligned}\sum_{i=1}^{3}\Bigg[\int_T^{+\infty} \frac{\alpha_i(s)}{\lambda_i(s)}\,ds + \int_T^{+\infty}\int_s^{+\infty} \frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds \\ + \int_T^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\Bigg] \\ < \min\left\{\underline{b} - c - L, L - c - \overline{a}\right\}.\end{aligned} \tag{3.1}$$

Define two mappings $Q_L, S_L : \Omega(a, b) \to C(I, \mathbb{R}^3)$ by

$$\begin{aligned}(Q_L x)(t) &= ((Q_{L1}x)(t), (Q_{L2}x)(t), (Q_{L3}x)(t)),\\ (S_L x)(t) &= ((S_{L1}x)(t), (S_{L2}x)(t), (S_{L3}x)(t))\end{aligned}$$

for $x = (x_1, x_2, x_3) \in \Omega(a, b)$ and $t \in I$, where

(3.2)

$$(Q_{Li}x)(t) = \begin{cases} L + f_i(t, x_1(t-\sigma_{i1}), x_2(t-\sigma_{i2}), x_3(t-\sigma_{i3})), & t \ge T,\\ (Q_{Li}x)(T), & t_0 \le t < T,\end{cases}$$

$$(S_{Li}x)(t) = \begin{cases} \int_t^{+\infty} \frac{g_i(s,x_1(p_{i1}(s)),x_2(p_{i2}(s)),x_3(p_{i3}(s)))}{\lambda_i(s)}\,ds \\ \quad - \int_t^{+\infty}\int_s^{+\infty} \frac{h_i(u,x_1(q_{i1}(u)),x_2(q_{i2}(u)),x_3(q_{i3}(u)))}{\lambda_i(s)r_i(u)}\,du\,ds \\ \quad - \int_t^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{l_i(v,x_1(\eta_{i1}(v)),x_2(\eta_{i2}(v)),x_3(\eta_{i3}(v)))}{\lambda_i(s)r_i(u)}\,dv\,du\,ds, \\ \qquad\qquad t \ge T,\\ (S_{Li}x)(T), \qquad t_0 \le t < T\end{cases}$$

for $i \in \{1,2,3\}$.

Firstly, we prove $Q_L x + S_L y \in \Omega(a,b)$ for all $x, y \in \Omega(a,b)$. Due to (ii), (iv), (3.1) and (3.2), we get that for each $x, y \in \Omega(a,b)$, $t \geq T$, $i \in \{1,2,3\}$,

$$
\begin{aligned}
&(Q_{Li}x + S_{Li}y)(t) \\
&\leq L + c_i(t) + \int_T^{+\infty} \frac{\alpha_i(s)}{\lambda_i(s)}\,ds + \int_T^{+\infty}\int_s^{+\infty} \frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds \\
&\quad + \int_T^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds \\
&\leq L + c + (\underline{b} - c - L) \\
&\leq b(t)
\end{aligned} \tag{3.3}
$$

and

$$
\begin{aligned}
&(Q_{Li}x + S_{Li}y)(t) \\
&\geq L - c_i(t) - \Bigg[\int_T^{+\infty} \frac{\alpha_i(s)}{\lambda_i(s)}\,ds + \int_T^{+\infty}\int_s^{+\infty} \frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds \\
&\quad + \int_T^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\Bigg] \\
&\geq L - c - (L - c - \overline{a}) \\
&\geq a(t).
\end{aligned} \tag{3.4}
$$

It follows from (3.3) and (3.4) that $Q_L\Omega(a,b) + S_L\Omega(a,b) \subseteq \Omega(a,b)$.

Secondly, we demonstrate that Q_L is a contraction mapping. According to (3.2) and (iii), we derive that

$$
\begin{aligned}
&|(Q_{Li}x)(t) - (Q_{Li}y)(t)| \\
&= |f_i(t, x_1(t-\sigma_{i1}), x_2(t-\sigma_{i2}), x_3(t-\sigma_{i3})) \\
&\quad - f_i(t, y_1(t-\sigma_{i1}), y_2(t-\sigma_{i2}), y_3(t-\sigma_{i3}))| \\
&\leq d_i(t) \max_{1\leq j\leq 3} |x_j(t-\sigma_{ij}) - y_j(t-\sigma_{ij})| \\
&\leq d\|x - y\|, \quad \forall\, x, y \in \Omega(a,b),\ t \geq T,\ i \in \{1,2,3\},
\end{aligned}
$$

which implies that

$$
\|Q_L x - Q_L y\| \leq d\|x - y\|, \quad \forall\, x, y \in \Omega(a,b).
$$

That is, Q_L is a contraction mapping by $d \in (0,1)$.

Thirdly, we show that S_L is completely continuous. Now we demonstrate S_L is continuous in $\Omega(a,b)$. Let $x_0 = (x_{01}, x_{02}, x_{03}) \in \Omega(a,b)$ and $\{x_k\}_{k\geq 0} =$

$(\{x_{k1}\}_{k\geq 0}, \{x_{k2}\}_{k\geq 0}, \{x_{k3}\}_{k\geq 0}) \subset \Omega(a,b)$ with $x_k \to x_0$ as $k \to +\infty$. (3.2) yields that

$$
\begin{aligned}
&\|S_L x_k - S_L x_0\| = \max_{1\leq i\leq 3} \sup_{t\in I} |(S_{Li}x_k)(t) - (S_{Li}x_0)(t)| \\
&\leq \max_{1\leq i\leq 3} \sup_{t\geq T} \Bigg\{ \int_t^{+\infty} \frac{1}{\lambda_i(s)} |g_i(s, x_{k1}(p_{i1}(s)), x_{k2}(p_{i2}(s)), x_{k3}(p_{i3}(s))) \\
&\qquad - g_i(s, x_{01}(p_{i1}(s)), x_{02}(p_{i2}(s)), x_{03}(p_{i3}(s)))|\, ds \\
&\quad + \int_t^{+\infty} \int_s^{+\infty} \frac{1}{\lambda_i(s) r_i(u)} |h_i(u, x_{k1}(q_{i1}(u)), x_{k2}(q_{i2}(u)), x_{k3}(q_{i3}(u))) \\
&\qquad - h_i(u, x_{01}(q_{i1}(u)), x_{02}(q_{i2}(u)), x_{03}(q_{i3}(u)))|\, du\, ds \\
&\quad + \int_t^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{1}{\lambda_i(s) r_i(u)} |l_i(v, x_{k1}(\eta_{i1}(v)), x_{k2}(\eta_{i2}(v)), x_{k3}(\eta_{i3}(v))) \\
&\qquad - l_i(v, x_{01}(\eta_{i1}(v)), x_{02}(\eta_{i2}(v)), x_{03}(\eta_{i3}(v)))|\, dv\, du\, ds \Bigg\} \\
&\leq \max_{1\leq i\leq 3} \Bigg[\int_T^{+\infty} \frac{1}{\lambda_i(s)} |g_i(s, x_{k1}(p_{i1}(s)), x_{k2}(p_{i2}(s)), x_{k3}(p_{i3}(s))) \\
&\qquad - g_i(s, x_{01}(p_{i1}(s)), x_{02}(p_{i2}(s)), x_{03}(p_{i3}(s)))|\, ds \\
&\quad + \int_T^{+\infty} \int_s^{+\infty} \frac{1}{\lambda_i(s) r_i(u)} |h_i(u, x_{k1}(q_{i1}(u)), x_{k2}(q_{i2}(u)), x_{k3}(q_{i3}(u))) \\
&\qquad - h_i(u, x_{01}(q_{i1}(u)), x_{02}(q_{i2}(u)), x_{03}(q_{i3}(u)))|\, du\, ds \\
&\quad + \int_T^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{1}{\lambda_i(s) r_i(u)} |l_i(v, x_{k1}(\eta_{i1}(v)), x_{k2}(\eta_{i2}(v)), x_{k3}(\eta_{i3}(v))) \\
&\qquad - l_i(v, x_{01}(\eta_{i1}(v)), x_{02}(\eta_{i2}(v)), x_{03}(\eta_{i3}(v)))|\, dv\, du\, ds \Bigg].
\end{aligned} \tag{3.5}
$$

Note that

$$
\begin{aligned}
&|g_i(s, x_{k1}(p_{i1}(s)), x_{k2}(p_{i2}(s)), x_{k3}(p_{i3}(s))) \\
&\qquad - g_i(s, x_{01}(p_{i1}(s)), x_{02}(p_{i2}(s)), x_{03}(p_{i3}(s)))| \leq 2\alpha_i(s), \\
&|h_i(u, x_{k1}(q_{i1}(u)), x_{k2}(q_{i2}(u)), x_{k3}(q_{i3}(u))) \\
&\qquad - h_i(u, x_{01}(q_{i1}(u)), x_{02}(q_{i2}(u)), x_{03}(q_{i3}(u)))| \leq 2\beta_i(u), \\
&|l_i(v, x_{k1}(\eta_{i1}(v)), x_{k2}(\eta_{i2}(v)), x_{k3}(\eta_{i3}(v))) \\
&\qquad - l_i(v, x_{01}(\eta_{i1}(v)), x_{02}(\eta_{i2}(v)), x_{03}(\eta_{i3}(v)))| \leq 2\gamma_i(v),
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
&|g_i(s, x_{k1}(p_{i1}(s)), x_{k2}(p_{i2}(s)), x_{k3}(p_{i3}(s))) \\
&\quad - g_i(s, x_{01}(p_{i1}(s)), x_{02}(p_{i2}(s)), x_{03}(p_{i3}(s)))| \to 0, \\
&|h_i(u, x_{k1}(q_{i1}(u)), x_{k2}(q_{i2}(u)), x_{k3}(q_{i3}(u))) \\
&\quad - h_i(u, x_{01}(q_{i1}(u)), x_{02}(q_{i2}(u)), x_{03}(q_{i3}(u)))| \to 0, \\
&|l_i(v, x_{k1}(\eta_{i1}(v)), x_{k2}(\eta_{i2}(v)), x_{k3}(\eta_{i3}(v))) \\
&\quad - l_i(v, x_{01}(\eta_{i1}(v)), x_{02}(\eta_{i2}(v)), x_{03}(\eta_{i3}(v)))| \to 0
\end{aligned}
\tag{3.7}
$$

as $k \to +\infty$ for $s, u, v \in [T, +\infty)$ and $i \in \{1, 2, 3\}$. It follows from (3.5), (3.6), (3.7) and Lebesgue dominated convergence theorem that $\|S_L x_k - S_L x_0\| \to 0$ as $k \to +\infty$. Hence S_L is continuous in $\Omega(a, b)$. Now we prove that $S_L \Omega(a, b)$ is relatively compact. In view of (i), (iv) and (3.2), we deduce that

$$
\begin{aligned}
\|S_L x\| &= \max_{1 \le i \le 3} \sup_{t \in I} |(S_{Li} x)(t)| \\
&\le \sum_{i=1}^{3} \Bigg[\int_T^{+\infty} \frac{\alpha_i(s)}{\lambda_i(s)}\, ds + \int_T^{+\infty} \int_s^{+\infty} \frac{\beta_i(u)}{\lambda_i(s) r_i(u)}\, du\, ds \\
&\quad + \int_T^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{\gamma_i(v)}{\lambda_i(s) r_i(u)}\, dv\, du\, ds \Bigg], \quad \forall\, x \in \Omega(a, b).
\end{aligned}
$$

That is, $S_L \Omega(a, b)$ is uniformly bounded. For the equicontinuity of $S_L \Omega(a, b)$ on I, according to Levitans result [6], it suffices to prove that for any given $\epsilon > 0$, I can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have change of amplitude less than ϵ. Let $\epsilon > 0$. By (i), there exists $T_* > T$ such that

$$
\begin{aligned}
&\sum_{i=1}^{3} \Bigg[\int_{T_*}^{+\infty} \frac{\alpha_i(s)}{\lambda_i(s)}\, ds + \int_{T_*}^{+\infty} \int_s^{+\infty} \frac{\beta_i(u)}{\lambda_i(s) r_i(u)}\, du\, ds \\
&\quad + \int_{T_*}^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{\gamma_i(v)}{\lambda_i(s) r_i(u)}\, dv\, du\, ds \Bigg] < \frac{\epsilon}{2}\,.
\end{aligned}
\tag{3.8}
$$

It follows from (iv), (3.2) and (3.8) that for all $x \in \Omega(a, b)$, $t_2 \ge t_1 \ge T_*$ and $i \in \{1, 2, 3\}$,

$$
\begin{aligned}
&|(S_{Li} x)(t_1) - (S_{Li} x)(t_2)| \\
&\le \int_{t_1}^{+\infty} \frac{\alpha_i(s)}{\lambda_i(s)}\, ds + \int_{t_1}^{+\infty} \int_s^{+\infty} \frac{\beta_i(u)}{\lambda_i(s) r_i(u)}\, du\, ds \\
&\quad + \int_{t_1}^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{\gamma_i(v)}{\lambda_i(s) r_i(u)}\, dv\, du\, ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{t_2}^{+\infty}\frac{\alpha_i(s)}{\lambda_i(s)}\,ds+\int_{t_2}^{+\infty}\int_s^{+\infty}\frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds\\
&+\int_{t_2}^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\\
\le{}& 2\left[\int_{T_*}^{+\infty}\frac{\alpha_i(s)}{\lambda_i(s)}\,ds+\int_{T_*}^{+\infty}\int_s^{+\infty}\frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds\right.\\
&\left.+\int_{T_*}^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\right]\\
<{}&\epsilon.
\end{aligned}
$$

For each $x\in\Omega(a,b), T\le t_1\le t_2\le T_*$ and $i\in\{1,2,3\}$, by (iv) and (3.2), we infer that

$$
\begin{aligned}
&|(S_{Li}x)(t_1)-(S_{Li}x)(t_2)|\\
&\le\int_{t_1}^{t_2}\frac{\alpha_i(s)}{\lambda_i(s)}\,ds+\int_{t_1}^{t_2}\int_s^{+\infty}\frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds\\
&\quad+\int_{t_1}^{t_2}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\\
&\le M_i|t_1-t_2|,
\end{aligned}
\tag{3.9}
$$

where

$$
M_i=\max_{T\le s\le T_*}\left\{\frac{\alpha_i(s)}{\lambda_i(s)}+\int_s^{+\infty}\frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du+\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\right\}.
$$

(3.9) implies that there exists $\delta=\frac{\epsilon}{1+M_i}>0$ such that $|(S_{Li}x)(t_1)-(S_{Li}x)(t_2)|<\epsilon$ for any $t_1,t_2\in[T,T_*]$ with $|t_1-t_2|<\delta$ and $x\in\Omega(a,b)$.

For $x\in\Omega(a,b), t_0\le t_1\le t_2\le T$ and $i\in\{1,2,3\}$, due to (3.2), we infer that

$$|(S_{Li}x)(t_1)-(S_{Li}x)(t_2)|=0.$$

Hence Lemma 2.1 ensures that there exists $x\in\Omega(a,b)$ with $Q_Lx+S_Lx=x$. It is easy to verify that x is a bounded positive solution of equations (1.5).

Finally, we investigate that equations (1.5) possess uncountably many bounded positive solutions. Let $L_1,L_2\in(\overline{a}+c,\underline{b}-c)$ with $L_1\neq L_2$. For each $j\in\{1,2\}$, we choose a constant $T_j>t_0+\sigma$ and two mappings Q_{L_j} and S_{L_j} satisfying (3.1) and (3.2), where L and T are replaced by L_j and T_j, respectively, and

$$
\begin{aligned}
&\sum_{i=1}^{3}\left[\int_{T_3}^{+\infty}\frac{\alpha_i(s)}{\lambda_i(s)}\,ds+\int_{T_3}^{+\infty}\int_s^{+\infty}\frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds\right.\\
&\quad\left.+\int_{T_3}^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\right]<\frac{|L_1-L_2|}{2}
\end{aligned}
\tag{3.10}
$$

for some $T_3 > \max\{T_1, T_2\}$. Obviously, the mappings $Q_{L_1} + S_{L_1}$ and $Q_{L_2} + S_{L_2}$ have the fixed points $x, y \in \Omega(a, b)$, respectively. That is, x and y are bounded positive solutions of equations (1.5) in $\Omega(a, b)$. In order to show that equations (1.5) possess uncountably many bounded positive solutions in $\Omega(a, b)$, we need only to prove that $x \neq y$. Indeed, by (3.2) we gain that for $t \geq T_3$, $i \in \{1, 2, 3\}$,

$$
\begin{aligned}
x_i(t) = L_1 &+ f_i(t, x_1(t-\sigma_{i1}), x_2(t-\sigma_{i2}), x_3(t-\sigma_{i3})) \\
&+ \int_t^{+\infty} \frac{g_i(s, x_1(p_{i1}(s)), x_2(p_{i2}(s)), x_3(p_{i3}(s)))}{\lambda_i(s)}\, ds \\
&- \int_t^{+\infty}\int_s^{+\infty} \frac{h_i(u, x_1(q_{i1}(u)), x_2(q_{i2}(u)), x_3(q_{i3}(u)))}{\lambda_i(s) r_i(u)}\, du\, ds \\
&- \int_t^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{l_i(v, x_1(\eta_{i1}(v)), x_2(\eta_{i2}(v)), x_3(\eta_{i3}(v)))}{\lambda_i(s) r_i(u)}\, dv\, du\, ds
\end{aligned}
$$

and

$$
\begin{aligned}
y_i(t) = L_2 &+ f_i(t, y_1(t-\sigma_{i1}), y_2(t-\sigma_{i2}), y_3(t-\sigma_{i3})) \\
&+ \int_t^{+\infty} \frac{g_i(s, y_1(p_{i1}(s)), y_2(p_{i2}(s)), y_3(p_{i3}(s)))}{\lambda_i(s)}\, ds \\
&- \int_t^{+\infty}\int_s^{+\infty} \frac{h_i(u, y_1(q_{i1}(u)), y_2(q_{i2}(u)), y_3(q_{i3}(u)))}{\lambda_i(s) r_i(u)}\, du\, ds \\
&- \int_t^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{l_i(v, y_1(\eta_{i1}(v)), y_2(\eta_{i2}(v)), y_3(\eta_{i3}(v)))}{\lambda_i(s) r_i(u)}\, dv\, du\, ds,
\end{aligned}
$$

which together with (iv) and (3.10) yield that

$$
\begin{aligned}
&\big| x_i(t) - y_i(t) - (f_i(t, x_1(t-\sigma_{i1}), x_2(t-\sigma_{i2}), x_3(t-\sigma_{i3})) \\
&\quad - f_i(t, y_1(t-\sigma_{i1}), y_2(t-\sigma_{i2}), y_3(t-\sigma_{i3})))\big| \\
&\geq |L_1 - L_2| - 2\Big[\int_{T_3}^{+\infty} \frac{\alpha_i(s)}{\lambda_i(s)}\, ds + \int_{T_3}^{+\infty}\int_s^{+\infty} \frac{\beta_i(u)}{\lambda_i(s) r_i(u)}\, du\, ds \\
&\quad + \int_{T_3}^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{\gamma_i(v)}{\lambda_i(s) r_i(u)}\, dv\, du\, ds\Big] \\
&> 0, \quad \forall\, t \geq T_3, i \in \{1, 2, 3\},
\end{aligned}
$$

that is, $x \neq y$. This completes the proof. □

Theorem 3.2. *Let $a, b \in C(I, \mathbb{R}^+)$ with $\overline{a} < \underline{b}$ and let* (iv) *and* (v) *hold. Then equations* (1.5) *with $f_i(t, u_1, u_2, u_3) = u_i$ for $i \in \{1, 2, 3\}$ possess uncountably many bounded positive solutions in $\Omega(a, b)$.*

PROOF: Let $L \in (\overline{a}, \underline{b})$. According to (v), we deduce that there exists sufficiently large $T \geq t_0 + \sigma$ satisfying

$$\begin{aligned}(3.11)\quad &\sum_{i=1}^{3}\sum_{j=1}^{+\infty}\Big[\int_{T+j\sigma}^{+\infty}\frac{\alpha_i(s)}{\lambda_i(s)}\,ds + \int_{T+j\sigma}^{+\infty}\int_{s}^{+\infty}\frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds\\ &\quad + \int_{T+j\sigma}^{+\infty}\int_{s}^{+\infty}\int_{u}^{+\infty}\frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\Big]\\ &< \min\{\underline{b} - L, L - \overline{a}\}.\end{aligned}$$

Define a mapping $Q_L : \Omega(a,b) \to C(I, \mathbb{R}^3)$ by

$$(3.12)\qquad (Q_L x)(t) = ((Q_{L1}x)(t), (Q_{L2}x)(t), (Q_{L3}x)(t)),$$

where
(3.13)

$$(Q_{Li}x)(t) = \begin{cases} L - \sum_{j=1}^{+\infty}\Big[\int_{t+j\sigma}^{+\infty}\frac{g_i(s,x_1(p_{i1}(s)),x_2(p_{i2}(s)),x_3(p_{i3}(s)))}{\lambda_i(s)}\,ds \\ \quad - \int_{t+j\sigma}^{+\infty}\int_{s}^{+\infty}\frac{h_i(u,x_1(q_{i1}(u)),x_2(q_{i2}(u)),x_3(q_{i3}(u)))}{\lambda_i(s)r_i(u)}\,du\,ds \\ \quad - \int_{t+j\sigma}^{+\infty}\int_{s}^{+\infty}\int_{u}^{+\infty}\frac{l_i(v,x_1(\eta_{i1}(v)),x_2(\eta_{i2}(v)),x_3(\eta_{i3}(v)))}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\Big], \\ \hfill t \geq T, \\ (Q_{Li}x)(T), \hfill t_0 \leq t < T \end{cases}$$

for $i \in \{1,2,3\}$.

First of all, we prove $Q_L x \in \Omega(a,b)$ for all $x \in \Omega(a,b)$. Due to (iv) and (3.13), we derive that for each $x \in \Omega(a,b)$ and $i \in \{1,2,3\}$,

$$\begin{aligned}&(Q_{Li}x)(t)\\ &\leq L + \sum_{j=1}^{+\infty}\Big[\int_{T+j\sigma}^{+\infty}\frac{\alpha_i(s)}{\lambda_i(s)}\,ds + \int_{T+j\sigma}^{+\infty}\int_{s}^{+\infty}\frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds\\ &\quad + \int_{T+j\sigma}^{+\infty}\int_{s}^{+\infty}\int_{u}^{+\infty}\frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\Big]\\ &\leq L + (\underline{b} - L)\\ &\leq b(t), \qquad t \geq T,\\ &(Q_{Li}x)(t)\\ &\geq L - \sum_{j=1}^{+\infty}\Big[\int_{T+j\sigma}^{+\infty}\frac{\alpha_i(s)}{\lambda_i(s)}\,ds + \int_{T+j\sigma}^{+\infty}\int_{s}^{+\infty}\frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds\end{aligned}$$

$$+\int_{T+j\sigma}^{+\infty}\int_{s}^{+\infty}\int_{u}^{+\infty}\frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\Bigg]$$
$$\geq L-(L-\overline{a})$$
$$\geq a(t), \qquad t\geq T.$$

Therefore, $Q_L\Omega(a,b)\subseteq\Omega(a,b)$.

Next, we demonstrate that Q_L is completely continuous. It is claimed that Q_L is continuous. Indeed, let $x_0=(x_{01},x_{02},x_{03})\in\Omega(a,b)$ and $\{x_k\}_{k\geq 0}=(\{x_{k1}\}_{k\geq 0},\{x_{k2}\}_{k\geq 0},\{x_{k3}\}_{k\geq 0})\subset\Omega(a,b)$ with $x_k\to x_0$ as $k\to+\infty$. (3.13) yields that

(3.14)
$$\begin{aligned}
&\|Q_Lx_k-Q_Lx_0\|\\
&=\max_{1\leq i\leq 3}\sup_{t\in I}|(Q_{Li}x_k)(t)-(Q_{Li}x_0)(t)|\\
&\leq\max_{1\leq i\leq 3}\sup_{t\in I}\Bigg\{\sum_{j=1}^{+\infty}\Bigg[\int_{t+j\sigma}^{+\infty}\frac{1}{\lambda_i(s)}|g_i(s,x_{k1}(p_{i1}(s)),x_{k2}(p_{i2}(s)),x_{k3}(p_{i3}(s)))\\
&\qquad-g_i(s,x_{01}(p_{i1}(s)),x_{02}(p_{i2}(s)),x_{03}(p_{i3}(s)))|\,ds\\
&\quad+\int_{t+j\sigma}^{+\infty}\int_{s}^{+\infty}\frac{1}{\lambda_i(s)r_i(u)}|h_i(u,x_{k1}(q_{i1}(u)),x_{k2}(q_{i2}(u)),x_{k3}(q_{i3}(u)))\\
&\qquad-h_i(u,x_{01}(q_{i1}(u)),x_{02}(q_{i2}(u)),x_{03}(q_{i3}(u)))|\,du\,ds\\
&\quad+\int_{t+j\sigma}^{+\infty}\int_{s}^{+\infty}\int_{u}^{+\infty}\frac{1}{\lambda_i(s)r_i(u)}|l_i(v,x_{k1}(\eta_{i1}(v)),x_{k2}(\eta_{i2}(v)),x_{k3}(\eta_{i3}(v)))\\
&\qquad-l_i(v,x_{01}(\eta_{i1}(v)),x_{02}(\eta_{i2}(v)),x_{03}(\eta_{i3}(v)))|\,dv\,du\,ds\Bigg]\Bigg\}\\
&\leq\max_{1\leq i\leq 3}\sum_{j=1}^{+\infty}\Bigg[\int_{T+j\sigma}^{+\infty}\frac{1}{\lambda_i(s)}|g_i(s,x_{k1}(p_{i1}(s)),x_{k2}(p_{i2}(s)),x_{k3}(p_{i3}(s)))\\
&\qquad-g_i(s,x_{01}(p_{i1}(s)),x_{02}(p_{i2}(s)),x_{03}(p_{i3}(s)))|\,ds\\
&\quad+\int_{T+j\sigma}^{+\infty}\int_{s}^{+\infty}\frac{1}{\lambda_i(s)r_i(u)}|h_i(u,x_{k1}(q_{i1}(u)),x_{k2}(q_{i2}(u)),x_{k3}(q_{i3}(u)))\\
&\qquad-h_i(u,x_{01}(q_{i1}(u)),x_{02}(q_{i2}(u)),x_{03}(q_{i3}(u)))|\,du\,ds\\
&\quad+\int_{T+j\sigma}^{+\infty}\int_{s}^{+\infty}\int_{u}^{+\infty}\frac{1}{\lambda_i(s)r_i(u)}|l_i(v,x_{k1}(\eta_{i1}(v)),x_{k2}(\eta_{i2}(v)),x_{k3}(\eta_{i3}(v)))\\
&\qquad-l_i(v,x_{01}(\eta_{i1}(v)),x_{02}(\eta_{i2}(v)),x_{03}(\eta_{i3}(v)))|\,dv\,du\,ds\Bigg].
\end{aligned}$$

In light of (3.6), (3.7), (3.14) and Lebesgue dominated convergence theorem, we infer that $\|Q_Lx_k-Q_Lx_0\|\to 0$ as $k\to+\infty$, which means that Q_L is continuous.

Now we show that $Q_L\Omega(a,b)$ is relatively compact. On account of $Q_L\Omega(a,b) \subseteq \Omega(a,b)$, Q_L is uniformly bounded. Because of (v), for any $\epsilon > 0$, choose $T_* > T$ large enough such that

$$\sum_{i=1}^{3}\sum_{j=1}^{+\infty}\Big[\int_{T_*+j\sigma}^{+\infty}\frac{\alpha_i(s)}{\lambda_i(s)}\,ds+\int_{T_*+j\sigma}^{+\infty}\int_s^{+\infty}\frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds \tag{3.15}$$
$$+\int_{T_*+j\sigma}^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\Big]<\frac{\epsilon}{2}\,.$$

By (iv), (3.13) and (3.15), for $x \in \Omega(a,b), t_2 \geq t_1 \geq T_*$ and $i \in \{1,2,3\}$, we have

$$\begin{aligned}
&|(Q_{Li}x)(t_1)-(Q_{Li}x)(t_2)|\\
&\leq \sum_{j=1}^{+\infty}\Big[\int_{t_1+j\sigma}^{+\infty}\frac{\alpha_i(s)}{\lambda_i(s)}\,ds+\int_{t_1+j\sigma}^{+\infty}\int_s^{+\infty}\frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds\\
&\quad+\int_{t_1+j\sigma}^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\Big]\\
&\quad+\sum_{j=1}^{+\infty}\Big[\int_{t_2+j\sigma}^{+\infty}\frac{\alpha_i(s)}{\lambda_i(s)}\,ds+\int_{t_2+j\sigma}^{+\infty}\int_s^{+\infty}\frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds\\
&\quad+\int_{t_2+j\sigma}^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\Big]\\
&<\epsilon.
\end{aligned}$$

For $T \leq t_1 \leq t_2 \leq T_*$, choose a sufficiently large integer $w \geq 1$ satisfying $T + j\sigma \geq T_*$ with $j \geq w$. For $x \in \Omega(a,b)$ and $i \in \{1,2,3\}$, we get that

$$\begin{aligned}
&|(Q_{Li}x)(t_1)-(Q_{Li}x)(t_2)|\\
&\leq \sum_{j=1}^{+\infty}\Big[\int_{t_1+j\sigma}^{t_2+j\sigma}\frac{\alpha_i(s)}{\lambda_i(s)}\,ds+\int_{t_1+j\sigma}^{t_2+j\sigma}\int_s^{+\infty}\frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds\\
&\quad+\int_{t_1+j\sigma}^{t_2+j\sigma}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\Big]\\
&=\sum_{j=1}^{w}\Big[\int_{t_1+j\sigma}^{t_2+j\sigma}\frac{\alpha_i(s)}{\lambda_i(s)}\,ds+\int_{t_1+j\sigma}^{t_2+j\sigma}\int_s^{+\infty}\frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds\\
&\quad+\int_{t_1+j\sigma}^{t_2+j\sigma}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\Big]\\
&\quad+\sum_{j=w+1}^{+\infty}\Big[\int_{t_1+j\sigma}^{t_2+j\sigma}\frac{\alpha_i(s)}{\lambda_i(s)}\,ds+\int_{t_1+j\sigma}^{t_2+j\sigma}\int_s^{+\infty}\frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_1+j\sigma}^{t_2+j\sigma}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\Bigg] \\
&\le \sum_{j=1}^{w}\Bigg[\int_{t_1+j\sigma}^{t_2+j\sigma}\frac{\alpha_i(s)}{\lambda_i(s)}\,ds+\int_{t_1+j\sigma}^{t_2+j\sigma}\int_s^{+\infty}\frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds \\
&\quad+\int_{t_1+j\sigma}^{t_2+j\sigma}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\Bigg] \\
&\quad+\sum_{j=1}^{+\infty}\Bigg[\int_{T_*+j\sigma}^{+\infty}\frac{\alpha_i(s)}{\lambda_i(s)}\,ds+\int_{T_*+j\sigma}^{+\infty}\int_s^{+\infty}\frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds \\
&\quad+\int_{T_*+j\sigma}^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\Bigg] \\
&< W_i|t_1-t_2|+\frac{\epsilon}{2}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
W_i=\max_{T+\sigma\le s\le T_*+w\sigma}\Bigg\{&\sum_{j=1}^{w}\Bigg[\frac{\alpha_i(s)}{\lambda_i(s)}+\int_s^{+\infty}\frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du \\
&+\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\Bigg]\Bigg\},
\end{aligned}
$$

which implies that there exists $\delta=\frac{\epsilon}{2(1+W_i)}>0$ such that $|(Q_{Li}x)(t_1)-(Q_{Li}x)(t_2)|<\epsilon$ for any $t_1,t_2\in[T,T_*]$ with $|t_1-t_2|<\delta$ and $x\in\Omega(a,b)$.

For $x\in\Omega(a,b)$, $t_0\le t_1\le t_2\le T$ and $i\in\{1,2,3\}$, it follows from (3.13) that

$$|(Q_{Li}x)(t_1)-(Q_{Li}x)(t_2)|=0.$$

Thus Lemma 2.2 ensures that there exists $x\in\Omega(a,b)$ with $Q_Lx=x$. That is, for $i\in\{1,2,3\}$,

$$
x_i(t)=\begin{cases}
L-\sum_{j=1}^{+\infty}\Big[\int_{t+j\sigma}^{+\infty}\frac{g_i(s,x_1(p_{i1}(s)),x_2(p_{i2}(s)),x_3(p_{i3}(s)))}{\lambda_i(s)}\,ds \\
\quad-\int_{t+j\sigma}^{+\infty}\int_s^{+\infty}\frac{h_i(u,x_1(q_{i1}(u)),x_2(q_{i2}(u)),x_3(q_{i3}(u)))}{\lambda_i(s)r_i(u)}\,du\,ds \\
\quad-\int_{t+j\sigma}^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{l_i(v,x_1(\eta_{i1}(v)),x_2(\eta_{i2}(v)),x_3(\eta_{i3}(v)))}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\Big], \\
\hfill t\ge T, \\
x_i(T), \hfill t_0\le t<T.
\end{cases}
$$

It follows that for $t \geq T$ and $i \in \{1,2,3\}$,

$$\begin{aligned} x_i(t) - x_i(t-\sigma) &= \int_t^{+\infty} \frac{g_i(s, x_1(p_{i1}(s)), x_2(p_{i2}(s)), x_3(p_{i3}(s)))}{\lambda_i(s)}\,ds \\ &\quad - \int_t^{+\infty}\int_s^{+\infty} \frac{h_i(u, x_1(q_{i1}(u)), x_2(q_{i2}(u)), x_3(q_{i3}(u)))}{\lambda_i(s) r_i(u)}\,du\,ds \\ &\quad - \int_t^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{l_i(v, x_1(\eta_{i1}(v)), x_2(\eta_{i2}(v)), x_3(\eta_{i3}(v)))}{\lambda_i(s) r_i(u)}\,dv\,du\,ds. \end{aligned}$$

It is easy to verify that x is a bounded positive solution of equations (1.5).

Finally, we investigate that equations (1.5) possess uncountably many bounded positive solutions. Let $L_1, L_2 \in (\overline{a}+c, \underline{b}-c)$ with $L_1 \neq L_2$. For each $j \in \{1,2\}$, choose a constant $T_j > t_0 + \sigma$ and a mapping Q_{L_j} to satisfy (3.11), (3.12) and (3.13), where L and T are replaced by L_j and T_j, respectively, and

$$\begin{aligned} \sum_{i=1}^{3}\sum_{j=1}^{+\infty} \Bigg[\int_{T_3+j\sigma}^{+\infty} \frac{\alpha_i(s)}{\lambda_i(s)}\,ds + \int_{T_3+j\sigma}^{+\infty}\int_s^{+\infty} \frac{\beta_i(u)}{\lambda_i(s) r_i(u)}\,du\,ds \\ + \int_{T_3+j\sigma}^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{\gamma_i(v)}{\lambda_i(s) r_i(u)}\,dv\,du\,ds \Bigg] < \frac{|L_1 - L_2|}{2} \end{aligned} \tag{3.16}$$

for some $T_3 > \max\{T_1, T_2\}$. Obviously, the mappings Q_{L_1} and Q_{L_2} have the fixed points $x, y \in \Omega(a,b)$, respectively. That is, x and y are bounded positive solutions of equations (1.5). Next we need only to prove that $x \neq y$. As a matter of fact, by (3.13) we get that for $t \geq T_3$ and $i \in \{1,2,3\}$,

$$\begin{aligned} x_i(t) = L_1 - \sum_{j=1}^{+\infty} \Bigg[&\int_{t+j\sigma}^{+\infty} \frac{g_i(s, x_1(p_{i1}(s)), x_2(p_{i2}(s)), x_3(p_{i3}(s)))}{\lambda_i(s)}\,ds \\ &- \int_{t+j\sigma}^{+\infty}\int_s^{+\infty} \frac{h_i(u, x_1(q_{i1}(u)), x_2(q_{i2}(u)), x_3(q_{i3}(u)))}{\lambda_i(s) r_i(u)}\,du\,ds \\ &- \int_{t+j\sigma}^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{l_i(v, x_1(\eta_{i1}(v)), x_2(\eta_{i2}(v)), x_3(\eta_{i3}(v)))}{\lambda_i(s) r_i(u)}\,dv\,du\,ds \Bigg], \\ y_i(t) = L_2 - \sum_{j=1}^{+\infty} \Bigg[&\int_{t+j\sigma}^{+\infty} \frac{g_i(s, y_1(p_{i1}(s)), y_2(p_{i2}(s)), y_3(p_{i3}(s)))}{\lambda_i(s)}\,ds \\ &- \int_{t+j\sigma}^{+\infty}\int_s^{+\infty} \frac{h_i(u, y_1(q_{i1}(u)), y_2(q_{i2}(u)), y_3(q_{i3}(u)))}{\lambda_i(s) r_i(u)}\,du\,ds \\ &- \int_{t+j\sigma}^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{l_i(v, y_1(\eta_{i1}(v)), y_2(\eta_{i2}(v)), y_3(\eta_{i3}(v)))}{\lambda_i(s) r_i(u)}\,dv\,du\,ds \Bigg], \end{aligned}$$

which together with (iv) and (3.16) yield that

$$\begin{aligned}
\big|x_i(t)-y_i(t)\big| \geq |L_1-L_2| - 2\sum_{j=1}^{+\infty}\Big[&\int_{T_3+j\sigma}^{+\infty}\frac{\alpha_i(s)}{\lambda_i(s)}\,ds \\
&+\int_{T_3+j\sigma}^{+\infty}\int_s^{+\infty}\frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds \\
&+\int_{T_3+j\sigma}^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\Big] \\
&> 0, \quad \forall\, t\geq T_3, i\in\{1,2,3\},
\end{aligned}$$

that is, $x\neq y$. This completes the proof. □

Theorem 3.3. *Let $a,b\in C(I,\mathbb{R}^+)$ with $\overline{a}<\underline{b}$ and let* (i), (ii), (iv) *and* (vi) *hold. If $c<\frac{\underline{b}-\overline{a}}{2}$ and φ is nondecreasing with $\varphi(t+)<t$ for each $t>0$, then equations* (1.5) *possess uncountably many bounded positive solutions in $\Omega(a,b)$.*

Proof: Put $L\in(\overline{a}+c,\underline{b}-c)$. In view of (i), there exists sufficiently large $T\geq t_0+\sigma$ satisfying (3.1). Define a mapping $Q_L:\Omega(a,b)\to C(I,\mathbb{R}^3)$ by (3.12), where

(3.17)
$$(Q_{Li}x)(t)=\begin{cases} L+f_i(t,x_1(t-\sigma_{i1}),x_2(t-\sigma_{i2}),x_3(t-\sigma_{i3})) \\ \quad +\int_t^{+\infty}\frac{g_i(s,x_1(p_{i1}(s)),x_2(p_{i2}(s)),x_3(p_{i3}(s)))}{\lambda_i(s)}\,ds \\ \quad -\int_t^{+\infty}\int_s^{+\infty}\frac{h_i(u,x_1(q_{i1}(u)),x_2(q_{i2}(u)),x_3(q_{i3}(u)))}{\lambda_i(s)r_i(u)}\,du\,ds \\ \quad -\int_t^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{l_i(v,x_1(\eta_{i1}(v)),x_2(\eta_{i2}(v)),x_3(\eta_{i3}(v)))}{\lambda_i(s)r_i(u)}\,dv\,du\,ds, \\ \qquad\qquad t\geq T, \\ (Q_{Li}x)(T), \qquad t_0\leq t<T \end{cases}$$

for $i\in\{1,2,3\}$.

Firstly, we assure that $Q_Lx\in\Omega(a,b)$ for all $x\in\Omega(a,b)$. In terms of (ii), (iv), (3.1) and (3.17), we infer that for each $x\in\Omega(a,b)$ and $i\in\{1,2,3\}$,

$$\begin{aligned}
&(Q_{Li}x)(t) \\
&\leq L+c_i(t)+\bigg(\int_T^{+\infty}\frac{\alpha_i(s)}{\lambda_i(s)}\,ds+\int_T^{+\infty}\int_s^{+\infty}\frac{\beta_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds \\
&\quad+\int_T^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{\gamma_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\bigg) \qquad (3.18)\\
&\leq L+c+(\underline{b}-c-L) \\
&\leq b(t), \quad t\geq T,
\end{aligned}$$

$$
\begin{aligned}
&(Q_{Li}x)(t)\\
&\ge L - c_i(t) - \left(\int_T^{+\infty} \frac{\alpha_i(s)}{\lambda_i(s)}\, ds + \int_T^{+\infty} \int_s^{+\infty} \frac{\beta_i(u)}{\lambda_i(s) r_i(u)}\, du\, ds \right.\\
&\qquad \left. + \int_T^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{\gamma_i(v)}{\lambda_i(s) r_i(u)}\, dv\, du\, ds \right)\\
&\ge L - c - (L - c - \overline{a})\\
&\ge a(t), \quad t \ge T.
\end{aligned}
\tag{3.19}
$$

Thus $Q_L \Omega(a,b) \subseteq \Omega(a,b)$.

Secondly, we claim that

$$
\lim_{t \to 0^+} \varphi(t) = 0 = \varphi(0).
\tag{3.20}
$$

Because $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing and nonnegative, we deduce that

$$
0 \le \varphi(0) \le \varphi(t) \le \varphi(s), \quad \forall\, s > t > 0,
$$

which together with $\varphi(t+) < t$ for each $t > 0$ ensures that

$$
0 \le \varphi(0) \le \varphi(t) \le \lim_{s \to t^+} \varphi(s) = \varphi(t+) < t, \quad \forall\, t > 0.
$$

Letting $t \to 0^+$ in the above inequalities, we get that (3.20) holds.

Thirdly, we prove that Q_L is continuous. Let $x_0 = (x_{01}, x_{02}, x_{03}) \in \Omega(a,b)$ and $\{x_k\}_{k \ge 0} = (\{x_{k1}\}_{k \ge 0}, \{x_{k2}\}_{k \ge 0}, \{x_{k3}\}_{k \ge 0}) \subset \Omega(a,b)$ with $x_k \to x_0$ as $k \to +\infty$. Let $D_k = \{x_k, x_0\}$ for $k \ge 1$. It follows from (vi), (3.17) and (3.20) that

$$
\begin{aligned}
&\|Q_L x_k - Q_L x_0\| = \max_{1 \le i \le 3} \sup_{t \in I} |(Q_{Li} x_k)(t) - (Q_{Li} x_0)(t)|\\
&\le \max_{1 \le i \le 3} \sup_{t \ge T} \Big[|f_i(t, x_{k1}(t - \sigma_{i1}), x_{k2}(t - \sigma_{i2}), x_{k3}(t - \sigma_{i3}))\\
&\qquad - f_i(t, x_{01}(t - \sigma_{i1}), x_{02}(t - \sigma_{i2}), x_{03}(t - \sigma_{i3}))|\\
&\quad + \int_t^{+\infty} \frac{1}{\lambda_i(s)} |g_i(s, x_{k1}(p_{i1}(s)), x_{k2}(p_{i2}(s)), x_{k3}(p_{i3}(s)))\\
&\qquad - g_i(s, x_{01}(p_{i1}(s)), x_{02}(p_{i2}(s)), x_{03}(p_{i3}(s)))|\, ds\\
&\quad + \int_t^{+\infty} \int_s^{+\infty} \frac{1}{\lambda_i(s) r_i(u)} |h_i(u, x_{k1}(q_{i1}(u)), x_{k2}(q_{i2}(u)), x_{k3}(q_{i3}(u)))\\
&\qquad - h_i(u, x_{01}(q_{i1}(u)), x_{02}(q_{i2}(u)), x_{03}(q_{i3}(u)))|\, du\, ds\\
&\quad + \int_t^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{1}{\lambda_i(s) r_i(u)} |l_i(v, x_{k1}(\eta_{i1}(v)), x_{k2}(\eta_{i2}(v)), x_{k3}(\eta_{i3}(v)))
\end{aligned}
$$

$$
\begin{aligned}
&\quad - l_i(v, x_{01}(\eta_{i1}(v)), x_{02}(\eta_{i2}(v)), x_{03}(\eta_{i3}(v)))|\, dv\, du\, ds\Big] \\
&\le \sup_{t\ge T} \varphi(D_k(t)) = \sup_{t\ge T} \varphi\big(\max_{1\le j\le 3} |x_{kj}(t) - x_{0j}(t)|\big) \\
&\le \varphi(\|x_k - x_0\|) \to 0 \ \text{ as } \ k \to +\infty.
\end{aligned}
$$

Thereupon, Q_L is continuous in $\Omega(a,b)$.

Lastly, we demonstrate that Q_L is a condensing mapping. Let $\epsilon > 0$. For any nonempty subset D of $\Omega(a,b)$ with $\alpha(D) > 0$, where α denotes the Kuratowski measure of noncompactness, there exist finitely many subsets $D_1, D_2, \dots, D_n$ of $\Omega(a,b)$ such that

$$
D \subseteq \bigcup_{m=1}^{n} D_m, \ \operatorname{diam} D_m \le \alpha(D) + \epsilon, \quad \forall\, m \in \{1,2,\dots,n\}. \tag{3.21}
$$

It follows from (vi) and (3.17) that for any $x, y \in D_m$, $m \in \{1,2,\dots,n\}$,

$$
\begin{aligned}
&\|Q_L x - Q_L y\| = \max_{1\le i\le 3} \sup_{t\in I} |(Q_{Li}x)(t) - (Q_{Li}y)(t)| \\
&\le \max_{1\le i\le 3} \sup_{t\ge T} \Big[|f_i(t, x_1(t-\sigma_{i1}), x_2(t-\sigma_{i2}), x_3(t-\sigma_{i3})) \\
&\qquad - f_i(t, y_1(t-\sigma_{i1}), y_2(t-\sigma_{i2}), y_3(t-\sigma_{i3}))| \\
&\quad + \int_t^{+\infty} \frac{1}{\lambda_i(s)} |g_i(s, x_1(p_{i1}(s)), x_2(p_{i2}(s)), x_3(p_{i3}(s))) \\
&\qquad - g_i(s, y_1(p_{i1}(s)), y_2(p_{i2}(s)), y_3(p_{i3}(s)))|\, ds \\
&\quad + \int_t^{+\infty} \int_s^{+\infty} \frac{1}{\lambda_i(s) r_i(u)} |h_i(u, x_1(q_{i1}(u)), x_2(q_{i2}(u)), x_3(q_{i3}(u))) \\
&\qquad - h_i(u, y_1(q_{i1}(u)), y_2(q_{i2}(u)), y_3(q_{i3}(u)))|\, du\, ds \\
&\quad + \int_t^{+\infty} \int_s^{+\infty} \int_u^{+\infty} \frac{1}{\lambda_i(s) r_i(u)} |l_i(v, x_1(\eta_{i1}(v)), x_2(\eta_{i2}(v)), x_3(\eta_{i3}(v))) \\
&\qquad - l_i(v, y_1(\eta_{i1}(v)), y_2(\eta_{i2}(v)), y_3(\eta_{i3}(v)))|\, dv\, du\, ds \Big] \\
&\le \sup_{t\ge T} \varphi(D_m(t)) \\
&\le \varphi(\operatorname{diam} D_m),
\end{aligned}
$$

which means that

$$
\operatorname{diam}(Q_L D_m) \le \varphi(\operatorname{diam} D_m), \quad \forall\, m \in \{1,2,\dots,n\}. \tag{3.22}
$$

According to (3.21) and (3.22), we derive that

$$\begin{aligned}\alpha(Q_L D) &\le \alpha\Big(\bigcup_{m=1}^{n} Q_L D_m\Big) = \max_{1\le m\le n}\{\alpha(Q_L D_m)\} \le \max_{1\le m\le n} \operatorname{diam}(Q_L D_m)\\ &\le \max_{1\le m\le n} \varphi(\operatorname{diam} D_m) \le \varphi(\alpha(D)+\epsilon).\end{aligned}$$

Setting $\epsilon \to 0$ in the above inequality, we gain that

$$\alpha(Q_L D) \le \varphi(\alpha(D)+0) < \alpha(D),$$

which implies that Q_L is condensing. Lemma 2.3 ensures that there exists $x \in \Omega(a,b)$ with $Q_L x = x$, which is a solution of equations (1.5). The rest of the proof is similar to that of Theorem 3.1. This completes the proof. $\square$

Theorem 3.4. *Let $a, b \in C(I, \mathbb{R}^+)$ with $\overline{a} < \underline{b}$ and let* (i)–(iv), (vii) *and* (viii) *hold. If $c < \frac{\underline{b}-\overline{a}}{2}$ and $d \in (0,1)$, then equations* (1.5) *possess uncountably many bounded positive solutions in $\Omega(a,b)$.*

Proof: Put $L \in (\overline{a}+c, \underline{b}-c)$. Due to (i) and (viii), we derive that there exists $T \ge t_0 + \sigma$ large enough satisfying (3.1) and

$$\begin{aligned}\sum_{i=1}^{3}\Bigg[\int_T^{+\infty}\frac{\mu_i(s)}{\lambda_i(s)}\,ds + \int_T^{+\infty}\int_s^{+\infty}\frac{\tau_i(u)}{\lambda_i(s)r_i(u)}\,du\,ds\\ + \int_T^{+\infty}\int_s^{+\infty}\int_u^{+\infty}\frac{\zeta_i(v)}{\lambda_i(s)r_i(u)}\,dv\,du\,ds\Bigg] < \frac{1-d}{2}.\end{aligned}\tag{3.23}$$

Define a mapping $Q_L : \Omega(a,b) \to C(I, \mathbb{R}^3)$ by (3.12) and (3.17). Just as (3.18) and (3.19), we can demonstrate that Q_L is a self-mapping on $\Omega(a,b)$ by (ii), (iv) and (3.1).

We now investigate that Q_L is a contraction mapping. According to (iii), (vii) and (3.23), we get that

$$\begin{aligned}&|(Q_{Li}x)(t) - (Q_{Li}y)(t)|\\ &\le |f_i(t, x_1(t-\sigma_{i1}), x_2(t-\sigma_{i2}), x_3(t-\sigma_{i3}))\\ &\qquad - f_i(t, y_1(t-\sigma_{i1}), y_2(t-\sigma_{i2}), y_3(t-\sigma_{i3}))|\\ &\quad + \int_t^{+\infty}\frac{1}{\lambda_i(s)}|g_i(s, x_1(p_{i1}(s)), x_2(p_{i2}(s)), x_3(p_{i3}(s)))\\ &\qquad - g_i(s, y_1(p_{i1}(s)), y_2(p_{i2}(s)), y_3(p_{i3}(s)))|\,ds\\ &\quad + \int_t^{+\infty}\int_s^{+\infty}\frac{1}{\lambda_i(s)r_i(u)}|h_i(u, x_1(q_{i1}(u)), x_2(q_{i2}(u)), x_3(q_{i3}(u)))\end{aligned}$$

$$
\begin{aligned}
&\qquad - h_i(u, y_1(q_{i1}(u)), y_2(q_{i2}(u)), y_3(q_{i3}(u)))|\, du\, ds \\
&\quad + \int_t^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{1}{\lambda_i(s) r_i(u)} |l_i(v, x_1(\eta_{i1}(v)), x_2(\eta_{i2}(v)), x_3(\eta_{i3}(v))) \\
&\qquad - l_i(v, y_1(\eta_{i1}(v)), y_2(\eta_{i2}(v)), y_3(\eta_{i3}(v)))|\, dv\, du\, ds\Big] \\
&\le d_i(t) \max_{1\le j\le 3} |x_j(t-\sigma_{ij}) - y_j(t-\sigma_{ij})| \\
&\quad + \int_t^{+\infty} \frac{\mu_i(s)\max_{1\le j\le 3}|x_j(p_{ij}(s)) - y_j(p_{ij}(s))|}{\lambda_i(s)}\, ds \\
&\quad + \int_t^{+\infty}\int_s^{+\infty} \frac{\tau_i(u)\max_{1\le j\le 3}|x_j(q_{ij}(u)) - y_j(q_{ij}(u))|}{\lambda_i(s) r_i(u)}\, du\, ds \\
&\quad + \int_t^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{\zeta_i(v)\max_{1\le j\le 3}|x_j(\eta_{ij}(v)) - y_j(\eta_{ij}(v))|}{\lambda_i(s) r_i(u)}\, dv\, du\, ds \\
&\le \Bigg(d + \int_T^{+\infty} \frac{\mu_i(s)}{\lambda_i(s)}\, ds + \int_T^{+\infty}\int_s^{+\infty} \frac{\tau_i(u)}{\lambda_i(s) r_i(u)}\, du\, ds \\
&\quad + \int_T^{+\infty}\int_s^{+\infty}\int_u^{+\infty} \frac{\zeta_i(v)}{\lambda_i(s) r_i(u)}\, dv\, du\, ds\Bigg) \|x - y\| \\
&< \frac{1+d}{2}\|x - y\|, \qquad t \ge T, i \in \{1, 2, 3\},
\end{aligned}
$$

which implies that $\|Q_L x - Q_L y\| < \frac{1+d}{2}\|x - y\|$ for any $x, y \in \Omega(a, b)$. Clearly, Q_L is a contraction mapping by $d \in (0, 1)$. Consequently, Q_L has a unique fixed point $x \in \Omega(a, b)$, which is a bounded positive solution of equations (1.5). The rest of the proof is similar to that of Theorem 3.1 and is omitted. This completes the proof. $\square$

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