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On a parabolic integrodifferential equation of Barbashin type

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Abstract. In the present paper we study some basic qualitative properties of solutions of a nonlinear parabolic integrodifferential equation of Barbashin type which occurs frequently in applications. The fundamental integral inequality with explicit estimate is used to establish the results.

Keywords: parabolic integrodifferential equation, Barbashin type, integral inequality, explicit estimate, approximate solutions, continuous dependence on parameters

Classification: 34K10, 35R10

1. Introduction

In [2], E.A. Barbashin first initiated the study of the integrodifferential equation of the form

$$\frac{\partial}{\partial t}u(t,x) = c(t,x)u(t,x) + \int_a^b k(t,x,y)u(t,y)\,dy + f(t,x), \tag{1.0}$$

which arises in mathematical modelling of many applied problems (see [1, Section 19]). The equation (1.0) has been studied by many authors and is now known in the literature as integrodifferential equation of Barbashin type or simply Barbashin equation (see [1, p. 1]). In this paper we consider the nonlinear integrodifferential equation of Barbashin type

$$\frac{\partial}{\partial t}u(t,x) = f(t,x,u(t,x)) + \int_a^b g(t,x,y,u(t,y))\,dy + h(t,x), \tag{1.1}$$

which satisfies the initial condition

$$u\,(0,x) = u_0\,(x)\,, \tag{1.2}$$

for $(t,x) \in \Delta$, where $f \in C(\Delta \times \mathbb{R}, \mathbb{R})$, $g \in C(\Delta \times I \times \mathbb{R}, \mathbb{R})$, $h \in C(\Delta, \mathbb{R})$, $u_0 \in C(I, \mathbb{R})$ are given functions and u is the unknown function to be found. Here $I = [a,b]$ $(a < b)$, $\mathbb{R}_+ = [0,\infty)$ are the given subsets of $\mathbb{R}$, the set of real numbers, $\Delta = \mathbb{R}_+ \times I$ and $C(D_1, D_2)$ denotes the class of continuous functions from the set D_1 to the set D_2. The problem of existence of solutions of (1.1)–(1.2) can be dealt with the method employed in [4]–[6] (see also [3, p. 273]). For detailed account on the study of such equations, see [1] and the references cited therein.

In the general case, solving (1.1)–(1.2) is a highly nontrivial problem. In dealing with the equations like (1.1)–(1.2), the basic questions to be answered are:

(i) if solutions do exist, what are their nature ?
(ii) how can we find them or closely approximate them ?

The study of such questions is interesting and needs a fresh outlook for handling the qualitative properties of solutions of (1.1)–(1.2). In the present work, we study some fundamental qualitative properties of solutions of (1.1)–(1.2) by using a certain variant of the integral inequality with explicit estimate established by the present author in [8]. A particular feature of our approach here is that it is elementary and provide some basic results for future advanced studies in the field.

2. A basic integral inequality

We require the following variant of the integral inequality given in [8].

Lemma. *Let $w, p \in C(\Delta, \mathbb{R}_+)$, $q \in C(\Delta \times I, \mathbb{R}_+)$ and let $c \geq 0$ be a constant. If*

$$w(t,x) \leq c + \int_0^t \left[p(s,x)w(s,x) + \int_a^b q(s,x,y)w(s,y)\,dy \right] ds \tag{2.1}$$

for $(t,x) \in \Delta$, then

$$w(t,x) \leq cP(t,x) \exp\left(\int_0^t \int_a^b q(s,x,y)\,P(s,y)\,dy\,ds \right) \tag{2.2}$$

for $(t,x) \in \Delta$, where

$$P(t,x) = \exp\left(\int_0^t p(s,x)\,ds \right). \tag{2.3}$$

Proof: Define a function $m(t,x)$ by

$$m(t,x) = c + \int_0^t \int_a^b q(s,x,y)w(s,y)\,dy\,ds. \tag{2.4}$$

Then (2.1) can be restated as

$$w(t,x) \leq m(t,x) + \int_0^t p(s,x)\,w(s,x)\,ds. \tag{2.5}$$

It is easy to observe that $m(t,x)$ is nonnegative for $(t,x) \in \Delta$ and nondecreasing in $t \in \mathbb{R}_+$ for every $x \in I$. Treating (2.5) as one-dimensional integral inequality in $t \in \mathbb{R}_+$ for every $x \in I$ and a suitable application of the inequality given in [4, p. 12 and also pp. 325–326] yields

$$w(t,x) \le m(t,x)\,P(t,x)\,. \tag{2.6}$$

From (2.4) and (2.6), we observe that

$$m(t,x) \le c + \int_0^t \int_a^b q(s,x,y)P(s,y)m(s,y)\,dy\,ds. \tag{2.7}$$

Setting

$$e(s) = \int_a^b q(s,x,y)P(s,y)m(s,y)\,dy, \tag{2.8}$$

for every $x \in I$, the inequality (2.7) can be restated as

$$m(t,x) \le c + \int_0^t e(s)\;ds. \tag{2.9}$$

Let

$$z(t) = c + \int_0^t e(s)\,ds. \tag{2.10}$$

Then $z(0) = c$ and

$$m(t,x) \le z(t) \tag{2.11}$$

for $t \in \mathbb{R}_+$ and for every $x \in I$. From (2.10), (2.8), (2.11), we observe that

$$\begin{aligned} z'(t) = e(t) &= \int_a^b q(t,x,y)P(t,y)m(t,y)\,dy \\ &\le z(t)\int_a^b q(t,x,y)P(t,y)\,dy. \end{aligned} \tag{2.12}$$

The inequality (2.12) implies

$$z(t) \le c\exp\left(\int_0^t \int_a^b q(s,x,y)P(s,y)\,dy\,ds\right). \tag{2.13}$$

The required inequality in (2.2) follows from (2.13), (2.10) and (2.6). □

3. Estimates of the solutions

First we shall give the following theorem concerning the estimate of the solution of (1.1)–(1.2).

Theorem 1. *Suppose that the functions f, g, h, u_0 in (1.1)–(1.2) satisfy the conditions*

$$|f(t,x,u) - f(t,x,\bar{u})| \le c(t,x)\,|u-\bar{u}|\,, \tag{3.1}$$

$$|g(t,x,y,u) - g(t,x,y,\bar{u})| \le k(t,x,y)\,|u-\bar{u}|\,, \tag{3.2}$$

and

$$d = \sup_{(t,x)\in\Delta} \left| \phi(t,x) + \int_0^t \left[f(s,x,0) + \int_a^b g(s,x,y,0)\,dy \right] ds \right| < \infty, \tag{3.3}$$

where $c \in C(\Delta, \mathbb{R}_+)$, $k \in C(\Delta \times I, \mathbb{R}_+)$ and

$$\phi(t,x) = u_0(x) + \int_0^t h(s,x)\,ds. \tag{3.4}$$

If $u(t,x)$ is any solution of (1.1)–(1.2) on Δ, then

$$|u(t,x)| \le dC(t,x) \exp\left(\int_0^t \int_a^b k(s,x,y) C(s,y)\,dy\,ds \right) \tag{3.5}$$

for $(t,x) \in \Delta$, where

$$C(t,x) = \exp\left(\int_0^t c(s,x)\,ds \right). \tag{3.6}$$

Proof: Using the fact that $u(t,x)$ is a solution of (1.1)–(1.2) on Δ and the hypotheses, we observe that

$$\begin{aligned}
|u(t,x)| &= \Bigg| \phi(t,x) + \int_0^t \Bigg[\{f(s,x,u(s,x)) - f(s,x,0) + f(s,x,0)\} \\
&\qquad + \int_a^b \{g(s,x,y,u(s,y)) - g(s,x,y,0) + g(s,x,y,0)\}\,dy \Bigg] ds \Bigg| \\
&\le \left| \phi(t,x) + \int_0^t \left[f(s,x,0) + \int_a^b g(s,x,y,0)\,dy \right] ds \right| \\
&\qquad + \int_0^t \Bigg[|f(s,x,u(s,x)) - f(s,x,0)| \\
&\qquad + \int_a^b |g(s,x,y,u(s,y)) - g(s,x,y,0)|\,dy \Bigg] ds \\
&\le d + \int_0^t \left[c(s,x)\,|u(s,x)| + \int_a^b k(s,x,y)\,|u(s,y)|\,dy \right] ds.
\end{aligned} \tag{3.7}$$

Now an application of Lemma to (3.7) yields (3.5). □

Remark 1. We note that the estimate obtained in (3.5) provides a bound on the solution $u(t,x)$ of (1.1)–(1.2) on Δ. In Theorem 1, if we assume that

$$\int_0^\infty c(s,x)\,ds < \infty, \int_0^\infty \int_a^b k(s,x,y)C(s,y)\,dy\,ds < \infty,$$

then the solution $u(t,x)$ of (1.1)–(1.2) is bounded on Δ.

A slight variant of Theorem 1 is embodied in the following theorem.

Theorem 2. *Suppose that the functions f, g, h, u_0 in* (1.1)–(1.2) *satisfy the conditions* (3.1), (3.2) *and*

$$\bar{d} = \int_0^\infty \left[|f(s,x,\phi(s,x))| + \int_a^b |g(s,x,y,\phi(s,y))|\,dy \right] ds < \infty, \tag{3.8}$$

where $\phi(t,x)$ is defined by (3.4). *If $u(t,x)$ is any solution of* (1.1)–(1.2) *on Δ, then*

$$|u(t,x) - \phi(t,x)| \le \bar{d}C(t,x)\exp\left(\int_0^t \int_a^b k(s,x,y)C(s,y)\,dy\,ds \right), \tag{3.9}$$

for $(t,x) \in \Delta$, where $C(t,x)$ is given by (3.6).

The proof follows by the similar arguments as in the proof of Theorem 1 with suitable modifications. We omit the details.

4. Approximate solutions

In this section we focus on the approximation of solutions to (1.1)–(1.2). We obtain conditions under which we can estimate errors between true solutions and approximate solutions.

Let $u(t,x) \in C(\Delta, \mathbb{R})$, let $\frac{\partial}{\partial t}u(t,x)$ exist on Δ and satisfy the inequality

$$\left| \frac{\partial}{\partial t}u(t,x) - f(t,x,u(t,x)) - \int_a^b g(t,x,y,u(t,y))\,dy - h(t,x) \right| \le \varepsilon,$$

for a given constant $\varepsilon \ge 0$, where it is supposed that (1.2) holds. Then we call $u(t,x)$ the ε-approximate solution with respect to the equation (1.1).

Our main result in this section concerning the estimate on the difference between the two approximate solutions of (1.1)–(1.2) is given in the following theorem.

Theorem 3. *Suppose that the functions f, g in (1.1) satisfy the conditions (3.1), (3.2). Let $u_i(t,x)$ $(i=1,2)$, $(t,x) \in \Delta$ be respectively ε_i-approximate solutions of (1.1) with*

$$u_i(0,x) = \bar{u}_i(x), \tag{4.1}$$

where $\bar{u}_i \in C(I,\mathbb{R})$, and let

$$\phi_i(t,x) = \bar{u}_i(x) + \int_0^t h(s,x)\,ds. \tag{4.2}$$

Suppose that

$$|\phi_1(t,x) - \phi_2(t,x)| \leq \delta, \tag{4.3}$$

where $\delta \geq 0$ is a constant and

$$M = \sup_{t\in\mathbb{R}_+} [(\varepsilon_1+\varepsilon_2)t+\delta] < \infty. \tag{4.4}$$

Then

$$|u_1(t,x) - u_2(t,x)| \leq MC(t,x)\exp\left(\int_0^t\int_a^b k(s,x,y)C(s,y)\,dy\,ds\right) \tag{4.5}$$

for $(t,x) \in \Delta$, where $C(t,x)$ is given by (3.6).

Proof: Since $u_i(t,x)$ $(i=1,2)$, $(t,x)\in\Delta$ are respectively ε_i-approximate solutions of (1.1) with (4.1) on Δ, we have

$$\left|\frac{\partial}{\partial t}u_i(t,x) - f(t,x,u_i(t,x)) - \int_a^b g(t,x,y,u_i(t,y))\,dy - h(t,x)\right| \leq \varepsilon_i. \tag{4.6}$$

By taking $t=s$ in (4.6) and integrating both sides with respect to s from 0 to t for $t\in\mathbb{R}_+$, we get

(4.7)

$$\begin{aligned}
\varepsilon_i t &\geq \int_0^t \left|\frac{\partial}{\partial s}u_i(s,x) - f(s,x,u_i(s,x)) - \int_a^b g(s,x,y,u_i(s,y))\,dy - h(s,x)\right| ds \\
&\geq \left|\int_0^t \left\{\frac{\partial}{\partial s}u_i(s,x) - f(s,x,u_i(s,x)) - \int_a^b g(s,x,y,u_i(s,y))\,dy - h(s,x)\right\} ds\right| \\
&= \left|u_i(t,x) - \phi_i(t,x) - \int_0^t \left[f(s,x,u_i(s,x)) + \int_a^b g(s,x,y,u_i(s,y))\,dy\right] ds\right|.
\end{aligned}$$

From (4.7) and using the elementary inequalities

$$|v - z| \le |v| + |z|, \quad |v| - |z| \le |v - z|, \tag{4.8}$$

for $v, z \in \mathbb{R}_+$, we observe that

(4.9)

$$\begin{aligned}
(\varepsilon_1 + \varepsilon_2)\, t \ge & \Bigg| u_1(t,x) - \phi_1(t,x) \\
& - \int_0^t \left[f(s,x,u_1(s,x)) + \int_a^b g(s,x,y,u_1(s,y))\, dy \right] ds \Bigg| \\
& + \left| u_2(t,x) - \phi_2(t,x) - \int_0^t \left[f(s,x,u_2(s,x)) + \int_a^b g(s,x,y,u_2(s,y))\, dy \right] ds \right| \\
\ge & \Bigg| \left\{ u_1(t,x) - \phi_1(t,x) - \int_0^t \left[f(s,x,u_1(s,x)) + \int_a^b g(s,x,y,u_1(s,y))\, dy \right] ds \right\} \\
& - \left\{ u_2(t,x) - \phi_2(t,x) - \int_0^t \left[f(s,x,u_2(s,x)) + \int_a^b g(s,x,y,u_2(s,y))\, dy \right] ds \right\} \Bigg| \\
\ge & |u_1(t,x) - u_2(t,x)| - |\phi_1(t,x) - \phi_2(t,x)| \\
& - \Bigg| \int_0^t \left[f(s,x,u_1(s,x)) + \int_a^b g(s,x,y,u_1(s,y))\, dy \right] ds \\
& - \int_0^t \left[f(s,x,u_2(s,x)) + \int_a^b g(s,x,y,u_2(s,y))\, dy \right] ds \Bigg| .
\end{aligned}$$

Let $u(t,x) = |u_1(t,x) - u_2(t,x)|$, $(t,x) \in \Delta$. From (4.9) and using the hypotheses, we observe that

$$\begin{aligned}
u(t,x) &\le (\varepsilon_1 + \varepsilon_2)\, t + \delta + \int_0^t \left[c(s,x)u(s,x) + \int_a^b k(s,x,y)u(s,y)\, dy \right] ds \\
&\le M + \int_0^t \left[c(s,x)u(s,x) + \int_a^b k(s,x,y)u(s,y)\, dy \right] ds.
\end{aligned} \tag{4.10}$$

Now an application of Lemma to (4.10) yields (4.5). □

Remark 2. In case $u_1(t,x)$ is a solution of (1.1) with $u_1(0,x) = \bar{u}_1(x)$, we have $\varepsilon_1 = 0$ and from (4.5), we see that $u_2(t,x) \to u_1(t,x)$ as $\varepsilon_2 \to \varepsilon_1$ and $\delta \to 0$. Furthermore, if we put (i) $\varepsilon_1 = \varepsilon_2 = 0$, $\bar{u}_1(x) = \bar{u}_2(x)$ in (4.5), then the uniqueness of solutions of (1.1) is established and (ii) $\varepsilon_1 = \varepsilon_2 = 0$ in (4.5), then we get the bound which shows the dependency of solutions of (1.1) on given initial values.

Consider (1.1)–(1.2) together with the following integrodifferential equation

$$\frac{\partial}{\partial t} v(t,x) = \bar{f}(t,x,v(t,x)) + \int_a^b \bar{g}(t,x,y,v(t,y))\,dy + \bar{h}(t,x), \tag{4.11}$$

with the given initial condition

$$v(0,x) = v_0(x) \tag{4.12}$$

for $(t,x) \in \Delta$, where $\bar{f} \in C(\Delta \times \mathbb{R}, \mathbb{R})$, $\bar{g} \in C(\Delta \times I \times \mathbb{R}, \mathbb{R})$, $\bar{h} \in C(\Delta, \mathbb{R})$, $v_0 \in C(I, \mathbb{R})$.

In the following theorem we provide conditions concerning the closedness of solutions of (1.1)–(1.2) and (4.11)–(4.12).

Theorem 4. *Suppose that the functions f, g in* (1.1) *satisfy the conditions* (3.1), (3.2), *and that there exist constants $\bar{\varepsilon}_i \geq 0$, $\bar{\delta}_i \geq 0$ $(i = 1, 2)$ such that*

$$\left|f(t,x,u) - \bar{f}(t,x,u)\right| \leq \bar{\varepsilon}_1, \tag{4.13}$$

$$|g(t,x,y,u) - \bar{g}(t,x,y,u)| \leq \bar{\varepsilon}_2, \tag{4.14}$$

$$\left|h(t,x) - \bar{h}(t,x)\right| \leq \bar{\delta}_1, \tag{4.15}$$

$$|u_0(x) - v_0(x)| \leq \bar{\delta}_2, \tag{4.16}$$

where f, g, h, u_0 and $\bar{f}, \bar{g}, \bar{h}, v_0$ are the functions in (1.1)–(1.2) *and* (4.11)–(4.12) *and*

$$\bar{M} = \sup_{t \in \mathbb{R}_+} \left[\bar{\delta}_2 + \left[\bar{\delta}_1 + \bar{\varepsilon}_1 + \bar{\varepsilon}_2(b-a)\right] t\right] < \infty. \tag{4.17}$$

Let $u(t,x)$ and $v(t,x)$ be respectively the solutions of (1.1)–(1.2) *and* (4.11)–(4.12) *for $(t,x) \in \Delta$. Then*

$$|u(t,x) - v(t,x)| \leq \bar{M} C(t,x) \exp\left(\int_0^t \int_a^b k(s,x,y) C(s,y)\,dy\,ds\right), \tag{4.18}$$

for $(t,x) \in \Delta$, where $C(t,x)$ is given by (3.6).

Proof: Let $z(t,x) = |u(t,x) - v(t,x)|$, $(t,x) \in \Delta$. Using the facts that $u(t,x)$, $v(t,x)$ are respectively the solutions of (1.1)–(1.2), (4.11)–(4.12) on Δ and the hypotheses, we observe that

(4.19)

$$z(t,x) \leq \left|u_0(x) + \int_0^t h(s,x)\,ds - v_0(x) - \int_0^t \bar{h}(s,x)\,ds\right|$$
$$+ \int_0^t \left[|f(s,x,u(s,x)) - f(s,x,v(s,x))| + \left|f(s,x,v(s,x)) - \bar{f}(s,x,v(s,x))\right|\right.$$

$$
\begin{aligned}
&+\int_a^b \{|g(s,x,y,u(s,y))-g(s,x,y,v(s,y))| \\
&+\ |g(s,x,y,v(s,y))-\bar{g}(s,x,y,v(s,y))|\}\ dy]\, ds \\
\le\ & |u_0(x)-v_0(x)|+\int_0^t \left|h(s,x)-\bar{h}\,(s,x)\right|\, ds \\
&+\int_0^t \left[c(s,x)z(s,x)+\bar{\varepsilon}_1+\int_a^b \{k(s,x,y)z(s,y)+\bar{\varepsilon}_2\}\ dy\right] ds \\
\le\ & \left[\bar{\delta}_2+\bar{\delta}_1 t+\bar{\varepsilon}_1 t+\bar{\varepsilon}_2(b-a)t\right]+\int_0^t \left[c(s,x)z(s,x)+\int_a^b k(s,x,y)z(s,y)\, dy\right] ds \\
\le\ & \bar{M}+\int_0^t \left[c(s,x)z(s,x)+\int_a^b k(s,x,y)z(s,y)\, dy\right] ds.
\end{aligned}
$$

Now an application of Lemma to (4.19) yields (4.18). □

Remark 3. We note that the result given in Theorem 4 relates the solutions of (1.1)–(1.2) and (4.11)–(4.12) in the sense that if f, g, h, u_0 are respectively close to $\bar{f}, \bar{g}, \bar{h}, v_0$; then the solutions of (1.1)–(1.2) and (4.11)–(4.12) are also close together.

5. Continuous dependence on parameters

In this section we shall deal with the continuous dependence of solutions of equations of the form (1.1) on parameters.

We consider the following integrodifferential equations of Barbashin type

$$
\frac{\partial}{\partial t}u(t,x)=f(t,x,u(t,x),\mu)+\int_a^b g(t,x,y,u(t,y),\mu)\, dy+h(t,x), \tag{5.1}
$$

$$
\frac{\partial}{\partial t}u(t,x)=f(t,x,u(t,x),\mu_0)+\int_a^b g(t,x,y,u(t,y),\mu_0)\, dy+h(t,x), \tag{5.2}
$$

with the given initial condition (1.2), where $f \in C(\Delta\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $g \in C(\Delta\times I\times \mathbb{R}\times\mathbb{R},\mathbb{R})$, $h\in C(\Delta,\mathbb{R})$ and μ, μ_0 are parameters.

The following theorem deals with the dependency of solutions of (5.1)–(1.2) and (5.2)–(1.2) on parameters.

Theorem 5. *Suppose that the functions f, g in (5.1), (5.2) satisfy the conditions*

$$
|f(t,x,u,\mu)-f(t,x,\bar{u},\mu)| \le \bar{c}(t,x)\,|u-\bar{u}|\,, \tag{5.3}
$$

$$
|f(t,x,u,\mu)-f\,(t,x,u,\mu_0)| \le n_1(t,x)\,|\mu-\mu_0|\,, \tag{5.4}
$$

$$
|g(t,x,y,u,\mu)-g(t,x,y,\bar{u},\mu)| \le \bar{k}(t,x,y)\,|u-\bar{u}|\,, \tag{5.5}
$$

$$
|g(t,x,y,u,\mu)-g\,(t,x,y,u,\mu_0)| \le n_2(t,x,y)\,|\mu-\mu_0|\,, \tag{5.6}
$$

where $\bar{c}, n_1 \in C(\Delta, \mathbb{R}_+)$, $\bar{k}, n_2 \in C(\Delta \times I, \mathbb{R}_+)$ *and*

$$(5.7) \qquad N = \sup_{(t,x)\in\Delta} \int_0^t \left[n_1(s,x) + \int_a^b n_2(s,x,y)\,dy \right] ds < \infty.$$

Let $u_1(t,x)$ *and* $u_2(t,x)$ *be respectively, the solutions of* (5.1)–(1.2) *and* (5.2)–(1.2) *on* Δ. *Then*

$$(5.8) \quad |u_1(t,x) - u_2(t,x)| \le |\mu - \mu_0|\,\bar{C}(t,x) \exp\left(\int_0^t \int_a^b \bar{k}(s,x,y)\bar{C}(s,y)\,dy\,ds \right)$$

for $(t,x) \in \Delta$, *where* $\bar{C}(t,x)$ *is given by* (3.6) *with* $c(t,x)$ *replaced by* $\bar{c}(t,x)$.

The proof can be completed by following the proofs of the above given results and closely looking at the proof of Theorem 3.3 given in [7]. We leave the details to the reader.

Remark 4. We note that, by obtaining a suitable variant of Lemma one can obtain results similar to those given above to the following Barbashin type integrodifferential equation

$$(5.9) \qquad \frac{\partial}{\partial t} u(t,x) = f(t,x,u(t,x)) + \int_G k(t,x,y,u(t,y))\,dy + h(t,x),$$

with the given initial condition, under some suitable conditions on the functions involved in (5.10), where G is a compact subset of $\mathbb{R}^n$. For more results on the equations of the type (5.10), we refer the interested readers to [1], [4]–[6] and the references cited therein.

6. Remarks and comments

In [1], the topological, monotonicity and variational methods are used to study the various Barbashin type integrodifferential equations and also partial integral equations of Barbashin type. The results obtained in this paper point out the well known truth that the technique of inequalities with explicit estimates is really a powerful tool for the study of important qualitative properties of solutions of (1.1)–(1.2). Suppose we are interested in finding a solution u of the integrodifferential equation of Barbashin type (1.0) satisfying the initial condition (1.2). Putting $\frac{\partial}{\partial t} u(t,x) := w(t,x)$ in (1.0)–(1.2), we arrive at the equation (see [1, p. 5])

$$(6.1) \quad w(t,x) = h(t,x) + \int_0^t c(t,x) w(\tau,x)\,d\tau + \int_0^t \int_a^b k(t,x,\sigma) w(\tau,\sigma)\,d\sigma\,d\tau,$$

where

$$(6.2) \qquad h(t,x) = f(t,x) + c(t,x)u_0(x) + \int_a^b k(t,x,\sigma)u_0(\sigma)\,d\sigma.$$

Problems of this type lead to partial integral equations of the form (see [1, p. 476])

$$u(t,x) = f(t,x) + Ku(t,x), \tag{6.3}$$

where

$$\begin{aligned} Ku(t,x) &= e(x)\int_0^t l(t,\tau)u(\tau,x)\,d\tau + c(t)\int_a^b m(x,\sigma)u(t,\sigma)\,d\sigma \\ &\quad + d(t,x)\int_0^t\int_a^b n(t,\tau)m(x,\sigma)u(\tau,\sigma)\,d\sigma\,d\tau, \end{aligned} \tag{6.4}$$

or Ku may be written in the more general forms as noted in the introduction of [1]. As far as equations of the forms (1.0)–(1.2) and (6.3) are concerned, numerous models arise whose detailed treatment is desired. Usually, it is hoped and expected that the analysis presented in Theorems 1–5 will also be equally useful in the study of equations like (6.3). In fact, it involves the task of designing a new inequality similar to the one given in Lemma, which will allow applications in the discussion of equations like (6.3). Indeed, it is not an easy matter but a challenging problem for future investigations.

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