On π–caliber and an application of Prikry's partial order

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Abstract. We study the concept of π-caliber as an alternative to the well known concept of caliber. π-caliber and caliber values coincide for regular cardinals greater than or equal to the Souslin number of a space. Unlike caliber, π-caliber may take on values below the Souslin number of a space. Under Martin's axiom, 2^ω is a π-caliber of $\mathbb{N}^*$. Prikry's poset is used to settle a problem by Fedeli regarding possible values of very weak caliber.

Keywords: nowhere dense, point-κ family, π-caliber

Classification: Primary 54A38, 54A15; Secondary 03E35

Let κ be a cardinal number. A family $\mathcal{P}$ of non-empty subsets of a space X is a *point-κ family* if for every $x \in X$, $|\{U \in \mathcal{P} : x \in U\}| < \kappa$.

A cardinal number κ is a *caliber* of a space X if every *point-κ* family of non-empty open subsets of X has cardinality less than κ. Since its inception (N. Šanin, [12], [13]), caliber (and its variations) has been the object of intense study in general topology, set theory, and combinatorics (cf. [2], [3]).

A cardinal number κ is a *π-caliber* of a space X if for every point-κ family $\mathcal{P}$ of non-empty open subsets of X and for every non-empty open set $G \subseteq X$ there exists a non-empty open set $V \subseteq G$ such that $|\{U \in \mathcal{P} : V \cap U \neq \emptyset\}| < \kappa$.

It is obvious that if κ is a caliber of a space X, then κ is a π-caliber of X. The converse implication does not hold: suffice to notice that if κ is a π-caliber of X_α for each α, then κ is going to be a π-caliber of the disjoint union of all the spaces X_α. Thus π-caliber constitutes a proper generalization of caliber because the values for caliber are bounded from below by the Souslin number of a space whereas values for π-caliber are not. The distinction between π-caliber and caliber can only occur for spaces with large (relative to π-caliber) Souslin number. For we show that if κ is a regular uncountable cardinal and the Souslin number of a space X is less than or equal to κ, then κ is a caliber of X if and only if κ is a π-caliber of X.

Let κ be an infinite cardinal. A space X is called *κ-Baire* if for each family $\{E_\alpha : \alpha < \kappa\}$ of nowhere dense subsets of X and for each non-empty open subset U of X, $U - \bigcup\{E_\alpha : \alpha < \kappa\} \neq \emptyset$. ω-Baire spaces are known as *Baire spaces*.

The cardinal ω cannot be the value of caliber of any infinite Hausdorff space. The Fletcher-Lindgren theorem ([6]; see also [10]) asserts that ω is a π-caliber of X if and only if X is a Baire space. However the existence of a normal ultrafilter on an uncountable cardinal κ implies the existence of a compact Hausdorff extremally disconnected space X that is λ-Baire for every $\lambda < \kappa$ but ω_1 is not a π-caliber of X. It follows that any infinite regular cardinal $\lambda < \kappa$, in particular, ω_1, is a very weak caliber of the space X but ω_1 is not a π-caliber of X. This settles, modulo measurable cardinals, a conjecture by Fedeli [5].

1. π-caliber and κ-Baire spaces

Characterizing κ-Baire spaces in terms of κ being a possible π-caliber, as indicated in the Fletcher-Lindgren theorem, ends there. At first, let us notice that, other than ω, a value for the π-caliber has no bearing on the Baire type of the space. Take, e.g., X to be a T_1 countable space without isolated points. Then X is not a Baire space and yet any cardinal of uncountable cofinality is going to be a (π-)caliber of X. Now, we are going to construct an example demonstrating impossibility of the converse.

Let us begin by recalling several pertinent notions and facts. All other undefined terms can be found in [7].

We say that a family $\mathfrak{F}$ is a *λ-complete filter over a set X* if $\mathfrak{F}$ is a family of infinite subsets of X such that:

(j) $\bigcap \mathfrak{F} = \emptyset$ and for each $A \subseteq \mathfrak{F}$, if $|A| < \lambda$, then $\bigcap A \in \mathfrak{F}$;

(jj) if $a \in \mathfrak{F}$ and $a \subseteq b \subseteq X$, then $b \in \mathfrak{F}$.

A filter $\mathfrak{F}$ over a cardinal κ is *normal* if $\mathfrak{F}$ is closed under diagonal intersections[1]. A cardinal κ is *measurable* if it is uncountable and there exists an ultrafilter over κ which is also κ-complete. We will need the following two known facts (cf. [7]).

Theorem 1. (1) *If κ is a measurable cardinal, then there exists an ultrafilter over κ that is κ-complete and normal.*

(2) *Let $\mathfrak{F}$ be an ultrafilter over κ that is κ-complete and normal. If $\mathcal{P}$ is a partition of $[\kappa]^{<\omega}$ into less than κ pieces, then there exists $A \in \mathfrak{F}$ such that for each natural number n there is $B \in \mathcal{P}$ such that $[A]^n \subseteq B$.*

Let $\mathfrak{F}$ be a filter over a cardinal κ. The following definition of a partially ordered set $P(\mathfrak{F}, \kappa)$ is due to K. Prikry [11]. The underlying set of $P(\mathfrak{F}, \kappa)$ is the collection of all pairs (s, F) such that $s \in [\kappa]^{<\omega}$, $F \in \mathfrak{F}$, and $\alpha < \beta$ whenever $\alpha \in s$ and $\beta \in F$; $(t, E) \leq (s, F)$ if s is an initial segment of t, i.e., $s = t \cap \gamma$ for some $\gamma < \kappa$, $E \subseteq F$, and $t - s \subseteq F$.

Prikry's poset plays a very important role in forcing considerations involving measurable cardinals (cf. [7], [9]). In our discussion that follows we are going to refrain from making any forcing references and present our arguments in purely topological fashion.

A partially ordered set $(P, <)$ is separative if for all $p, q \in P$, if $p \nleq q$ then there exists an $c \leq p$ that is incompatible with q. The following lemma is pretty straightforward but for the sake of completeness we prove it here.

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[1] The *diagonal intersection* of a family $\{A_\alpha : \alpha < \kappa\}$ of subsets of the cardinal κ is the set $\Delta\{A_\alpha : \alpha < \kappa\} = \{\beta < \kappa : \beta \in \bigcap\{A_\alpha : \alpha < \beta\}\}$.

Lemma 1. *Let $\mathfrak{F}$ be a κ-complete filter over κ. Then the partially ordered set $P(\mathfrak{F},\kappa)$ is separative.*

Proof: Suppose that $(t,E) \not\leq (s,F)$.

Case 1. s is not an initial segment of t.

Subcase 1.1. $\max s \leq \max t$. We set $c=(t,E)$. Trivially c is incompatible with (s,F).

Subcase 1.2. $\max s > \max t$. Pick α,β such that $\max s < \alpha < \beta < \kappa$ and $\alpha \in E$. We set $c=(t\cup\{\alpha\}, E-\beta)$. Then $c<(t,E)$. Trivially, $\max s < \max(t \cup \{\alpha\})$ and since s is not an initial segment of t, s is not an initial segment of $t\cup\{\alpha\}$ either. Thus c is incompatible with (s,F).

Case 2. s is an initial segment of t but $t-s \not\subseteq F$. We set $c=(t,E)$. Trivially c is incompatible with (s,F).

Case 3. s is an initial segment of t and $t-s\subseteq F$ but $E \not\subseteq F$. Pick α,β such that $\max s<\alpha<\beta<\kappa$ and $\alpha \in E-F$. We set $c=(t\cup\{\alpha\},E-\beta)$. Since $\max t < \min E$, $c<(t,E)$. To show that c is incompatible with (s,F), take any $(r,G)\in P(\mathfrak{F},\kappa)$ such that $(r,G)\leq(t\cup\{\alpha\},E-\beta)$. Then, in particular, $t\cup\{\alpha\}$ is an initial segment of r. Hence $\alpha\in r-s$ and so $r-s\not\subseteq F$. $\square$

For a non-empty subset D of $P(\mathfrak{F},\kappa)$, let $\mathrm{pr}(D)=\{s : \exists_F\ (s,F)\in D\}$.

Lemma 2. *If D is dense in $(s,F)\in P(\mathfrak{F},\kappa)$, then there exists $E\in\mathfrak{F}$ such that $\{0<n<\omega : s\cup[E]^n \subseteq \mathrm{pr}(D)\}$ is infinite. Here, $s\cup[E]^n$ stands for the set $\{s\cup t : t\in[E]^n\}$.*

Proof: For the two-element partition

$$\{\{t-s : t\in \mathrm{pr}(D)\}, [\kappa]^{<\omega}-\{t-s : t\in\mathrm{pr}(D)\}\}$$

of $[\kappa]^{<\omega}$ take $A\in\mathfrak{F}$ that satisfies (2) of Theorem 1 and set $E=F\cap A$. Fix a natural number m, pick an arbitrary subset t of E of size $m+1$, and take $(s\cup t, E-\max t)$. By density of D, there exists $(r,H)\in D$ such that $(r,H)\leq(s\cup t, E-\max t)$. Thus $r-s\subset E$ and $n=|r-s|>m$. Hence $s\cup[E]^n\subseteq\mathrm{pr}(D)$. $\square$

Theorem 2. *If κ is a measurable cardinal, then there exists a compact Hausdorff extremally disconnected space X such that X is λ-Baire for each $\lambda<\kappa$ and ω_1 is not a π-caliber of X.*

Proof: Let $P(\mathfrak{F},\kappa)$ be the Prikry partially ordered set, where $\mathfrak{F}$ is an ultrafilter over κ that is κ-complete and normal. Since $P(\mathfrak{F},\kappa)$ is separative, it is a dense subset of a complete Boolean algebra $\mathbb{B}$ (see [7]). We take X to be the Stone space of $\mathbb{B}$. Thus X is a compact Hausdorff extremally disconnected space. For $a\in\mathbb{B}$ let $[a]=\{x\in X : a\in x\}$. The sets $[a]$ are closed and open subsets of X. Moreover, for each dense subset D of $\mathbb{B}$, $[D]=\{[a] : a\in D\}$ is a π-base for X.

Let us show that ω_1 is not a π-caliber of X. Towards this goal, for each $n < \omega$, one can construct $\mathcal{R}_n$ so that:

(1) $\mathcal{R}_n \subseteq [P(\mathfrak{F},\kappa)]$, $\mathcal{R}_n$ is a pairwise disjoint family, and $\bigcup \mathcal{R}_n$ is dense in X;
(2) if $m < n$, then $\mathcal{R}_n$ is a refinement of $\mathcal{R}_m$; moreover, if $[(s,F)] \in \mathcal{R}_m$, $[(t,E)] \in \mathcal{R}_n$, and $[(t,E)] \subseteq [(s,F)]$, then $|t| > |s|$.

Let $\mathcal{P} = \bigcup\{\mathcal{R}_n : n < \omega\}$. Clearly, $\mathcal{P}$ is a point-ω_1 open family in X. We shall show that each non-empty open subset of X is intersected by exactly κ elements of $\mathcal{P}$.

Assume not. There exists $(s,F) \in P(\mathfrak{F},\kappa)$ such that $[(s,F)]$ intersects less than κ elements of $\mathcal{P}$. Hence

$$A = F \cap \bigcap \{E : [(t,E)] \in \mathcal{P} \text{ and } [(s,F)] \cap [(t,E)] \neq \emptyset\} \in \mathfrak{F}.$$

Pick arbitrary $\alpha \in A$. Then $(s \cup \{\alpha\}, A - (\alpha+1)) \in P(\mathfrak{F},\kappa)$ and $(s \cup \{\alpha\}, A - (\alpha+1)) < (s,F)$. Notice that if $[(t,E)] \in \mathcal{P}$, $[(s,F)] \cap [(t,E)] \neq \emptyset$, and $s \neq t$, then $(s \cup \{\alpha\}, A - (\alpha+1))$ is incompatible with (t,E). Consequently, by (2), $[(s \cup \{\alpha\}, A - (\alpha+1))]$ is disjoint with each element of $\mathcal{R}_n$, whenever $n > |s|$. This contradicts (1).

Let us show that X is λ-Baire for each $\lambda < \kappa$. Let $\{N_\alpha : \alpha < \lambda\}$ be a family of nowhere dense subsets of X, where $\lambda < \kappa$. Fix $(s,F) \in P(\mathfrak{F},\kappa)$ and set

$$D_\alpha = \{(t,E) \in P(\mathfrak{F},\kappa) : (t,E) \leq (s,F) \text{ and } [(s,F)] \cap N_\alpha = \emptyset\}.$$

Each of the sets D_α, $\alpha < \lambda$, is dense in (s,F). By Lemma 2, for each $\alpha < \lambda$ there exists $E_\alpha \in \mathfrak{F}$ such that $\{0 < n < \omega : s \cup [E_\alpha]^n \subseteq \mathrm{pr}(D_\alpha)\}$ is infinite. Set $E = \bigcap\{E_\alpha : \alpha < \lambda\}$ and $L_n = \{\alpha < \lambda : s \cup [E]^n \subseteq \mathrm{pr}(D_\alpha)\}$ for each $0 < n < \omega$. Then $E \in \mathfrak{F}$ and $\lambda = \bigcup\{L_n : 0 < n < \omega\}$. Let $n(0) < n(1) < \ldots n(i) < \ldots$ be such that $\lambda = \bigcup\{L_{n(i)} : i < \omega\}$ and $L_{n(i)} \neq \emptyset$ for each $i < \omega$.

Pick a subset s_0 of E of size $n(0)$. For each $\xi \in L_{n(0)}$ select $F_\xi \in \mathfrak{F}$ so that $(s \cup s_0, F_\xi) \in D_\xi$ and set $\Upsilon_0 = E \cap \bigcap\{F_\xi : \xi \in L_{n(0)}\}$. Thus $(s \cup s_0, \Upsilon_0) \leq (s,F)$ and $[(s \cup s_0, \Upsilon_0)] \cap N_\xi = \emptyset$ for each $\xi \in L_{n(0)}$.

Pick a subset s_1 of Υ_0 of size $n(1) - n(0)$. For each $\xi \in L_{n(1)}$ select $F_\xi \in \mathfrak{F}$ so that $(s \cup s_0 \cup s_1, F_\xi) \in D_\xi$ and set $\Upsilon_1 = \Upsilon_0 \cap \bigcap\{F_\xi : \xi \in L_{n(1)}\}$. Thus $(s \cup s_0 \cup s_1, \Upsilon_1) \leq (s \cup s_0, \Upsilon_0)$ and $[(s \cup s_0 \cup s_1, \Upsilon_1)] \cap N_\xi = \emptyset$ for each $\xi \in L_{n(1)}$.

Pick a subset s_2 of Υ_1 of size $n(2) - n(1)$. For each $\xi \in L_{n(2)}$ select $F_\xi \in \mathfrak{F}$ so that $(s \cup s_0 \cup s_1 \cup s_2, F_\xi) \in D_\xi$ and set $\Upsilon_2 = \Upsilon_1 \cap \bigcap\{F_\xi : \xi \in L_{n(2)}\}$. Thus $(s \cup s_0 \cup s_1 \cup s_2, \Upsilon_2) \leq (s \cup s_0 \cup s_1, \Upsilon_1)$ and $[(s \cup s_0 \cup s_1 \cup s_2, \Upsilon_2)] \cap N_\xi = \emptyset$ for each $\xi \in L_{n(2)}$.

The construction goes on. Consequently, we get a nested downward sequence $\{(s \cup s_0 \cup s_1 \cup \cdots \cup s_k, \Upsilon_k)\}_{k=0}^{\infty}$ of elements of $P(\mathfrak{F},\kappa)$ such that

$$[(s \cup s_0 \cup s_1 \cdots \cup s_k, \Upsilon_k)] \cap \bigcup\{N_\xi : \xi \in L_{n(k)}\} = \emptyset$$

for each $k \in \omega$. Since $(s \cup s_0, \Upsilon_0) \leq (s, F)$,

$$\emptyset \neq \bigcap\{[(s \cup s_0 \cup s_1 \cdots \cup s_k, \Upsilon_k)] : k \in \omega\} \subseteq [(s, F)] - \bigcup\{N_\alpha : \alpha < \lambda\}.$$

□

Following A. Fedeli [5], a cardinal number κ is a *very weak caliber* of a space X if for every open point-κ family $\mathcal{P}$ of cardinality at most κ and for every non-empty open set $G \subseteq X$ there exists a non-empty open set $V \subseteq G$ such that $|\{U \in \mathcal{P} : V \cap U \neq \emptyset\}| < \kappa$. It is easy to see (cf. [4] or Lemma 4) that if X is a κ-(semi)Baire space, then κ is a very weak caliber of X. In [5], 421_8, Fedeli writes: "*It would be interesting, for a regular cardinal κ, to know whether there exists a space which has very weak caliber κ but has not π-caliber*[2] *κ.*" The space X constructed in Theorem 2 is such a space for the measurable cardinal κ.

Problem 1. Construct a consistent example of small cardinality, e.g., by using a precipitous ideal on ω_2, or even an example in ZFC, of a space X that is λ-Baire and ω_1 is not a π-caliber of X, for some regular cardinal $\lambda > \omega_1$.

2. π-calibers of some spaces

A space X is called *κ-semibaire* if for each family $\{E_\alpha : \alpha < \kappa\}$ of nowhere dense subsets of X and for each non-empty open subset U of X there exists $A \subseteq \kappa$, $|A| = \kappa$, such that $U - \bigcup\{E_\alpha : \alpha \in A\} \neq \emptyset$.

Observe that any κ-Baire space is a κ-semibaire space and that any ω-semibaire space is also a Baire space. Thus ω-semibaire = Baire. Let us notice also the following lemmas.

Lemma 3. *Let $\{E_\alpha : \alpha < \lambda\}$ be a family of nowhere dense subsets of X such that $E_\alpha \subseteq E_\beta$ whenever $\alpha \leq \beta < \lambda$. If the set $\bigcup\{E_\alpha : \alpha < \lambda\}$ contains a non-empty open subset of X, then X is not a κ-semibaire space for any cardinal κ of cofinality λ.*

Lemma 4. *Let X be a κ-semibaire space and let $\mathcal{P}$ be a point-κ open family in X such that $\mathcal{P} \leq \kappa$. If $G \subseteq X$ non-empty open set, then there exists a non-empty open set $V \subseteq G$ such that $|\{U \in \mathcal{P} : V \cap U \neq \emptyset\}| < \kappa$. Thus κ is a very weak caliber of X.*

Proof: If $|\mathcal{P}| < \kappa$, then there is nothing to prove. If $|\mathcal{P}| = \kappa$, faithfully index $\mathcal{P}$, say $\mathcal{P} = \{U_\alpha : \alpha < \kappa\}$, and set $E_\alpha = G - \bigcup\{U_\xi : \alpha \leq \xi < \kappa\}$. Thus $\{E_\alpha : \alpha < \kappa\}$ is a nested upward family of closed subset of G such that $\bigcup\{E_\alpha : \alpha < \kappa\} = G$. Since X is κ-semibaire, there must exist an $\alpha < \kappa$ and a non-empty open set V such that $V \subseteq E_\alpha$. Since E_α intersects at most $|\alpha| < \kappa$ elements of the family $\mathcal{P}$, V does too and we are done. □

Proposition 1. *If κ is a regular cardinal and κ is a π-caliber of X, then X is a κ-semibaire space.*

[2] The original has *weak caliber*.

Proof: Suppose to the contrary that there exist nowhere dense sets E_α, $\alpha < \kappa$, and a non-empty open set G such that $G \subseteq \bigcup\{E_\alpha : \alpha \in A\}$ for each $A \subseteq \kappa$ such that $|A| = \kappa$. Then $\mathcal{P} = \{G - \operatorname{cl} E_\alpha : \alpha < \kappa\}$ is a point-κ family of dense open subsets of G, and thus each non-empty open subset of G intersects every element of $\mathcal{P}$. Since κ is a π-caliber of X, $\mathcal{P}$ has to be of cardinality less than κ. Since κ is a regular cardinal, there exists $A \subseteq \kappa$, $|A| = \kappa$, such that $\operatorname{cl} E_\alpha \cap G = \operatorname{cl} E_\beta \cap G$ for each $\alpha, \beta \in A$. Let $\gamma \in A$. Then $G \cap \bigcup\{E_\alpha : \alpha \in A\} \subseteq \operatorname{cl} E_\gamma \neq G$; a contradiction. $\square$

In light of Proposition 1, while trying to establish that a regular cardinal κ is a π-caliber of X, it is necessary to assume that the space X is a κ-semibaire space.

Following Comfort and Negrepontis [1], the *Souslin number* of X, $S(X)$, is the smallest cardinal κ such that no family of pairwise disjoint non-empty open subsets of X has κ elements. Spaces with the Souslin number ω_1 are usually called *ccc spaces*. By the theorem of Erdös-Tarski theorem [1], if X is an infinite space, then $S(X)$ is an uncountable regular cardinal.

Theorem 3. *Let κ be a regular infinite cardinal and let X be a κ-semibaire space such that $S(X) \leq \kappa^+$. Then κ is a π-caliber of X.*

Proof: Let $\mathcal{P}$ be a point-κ open family in X and let G be a non-empty open subset of X. Assume to the contrary that for each non-empty open subset V of G, $|\{U \in \mathcal{P} : U \cap V \neq \emptyset\}| \geq \kappa$. By Lemma 4,

(+) If V is a non-empty open subset of G, then $|\{U \in \mathcal{P} : U \cap V \neq \emptyset\}| > \kappa$.

For each $\alpha < \kappa$ we are going to define $\mathcal{R}_\alpha$ and f_α so that:

1. $\mathcal{R}_\alpha$ is a family of pairwise disjoint non-empty opens subsets of X and $\bigcup \mathcal{R}_\alpha$ is dense in G;
2. f_α is a one-to-one function and $\operatorname{Dom}(f_\alpha) = \bigcup\{\mathcal{R}_\xi : \xi \leq \alpha\}$ and $\operatorname{Range}(f_\alpha) \subseteq \mathcal{P}$;
3. $f_\alpha(W) \supset W$ for each $W \in \operatorname{Dom}(f_\alpha)$;
4. $f_\alpha \subseteq f_\beta$ if $\alpha < \beta < \kappa$.

Suppose that $\mathcal{R}_\alpha$ and f_α have already been defined for each $\alpha < \beta$, where $\beta < \kappa$. Set $\mathcal{Q} = \mathcal{P} - \bigcup\{f_\alpha(\mathcal{R}_\alpha) : \alpha < \beta\}$. Notice that since $|f_\alpha(\mathcal{R}_\alpha)| \leq \kappa$ for each $\alpha < \beta$, the family $\mathcal{Q}$ also satisfies condition (+). We proceed to constructing $\mathcal{R}_\beta$ and f_β.

Let $\{U_\xi : \xi < \lambda\}$ be an enumeration of $\mathcal{Q}$. We set $W_0 = U_0 \cap G$ and $W_\alpha = [G - \operatorname{cl}(\bigcup\{U_\xi : \xi < \alpha\})] \cap U_\alpha$ for each α, $0 < \alpha < \lambda$. Clearly, the open sets W_α, $\alpha < \lambda$, are pairwise disjoint and, because of property (+), $\bigcup\{W_\alpha : \alpha < \lambda\}$ is a dense subset of G. Finally, we set $\mathcal{R}_\beta = \{W_\alpha : \alpha < \lambda \text{ and } W_\alpha \neq \emptyset\}$ and $f_\beta = \bigcup\{f_\alpha : \alpha < \beta\} \cup \{(W_\alpha, U_\alpha) : \alpha < \lambda \text{ and } W_\alpha \neq \emptyset\}$. One can easily see that $\mathcal{R}_\alpha$ and f_α satisfy the conditions (1)–(4) for every $\alpha \leq \beta$; the construction is finished.

Since X is κ-semibaire, there exists $A \subseteq \kappa$, $|A| = \kappa$, such that $G - \bigcup\{E_\alpha : \alpha \in A\} \neq \emptyset$, where E_α denotes the nowhere dense set $G - \bigcup \mathcal{R}_\alpha$. Pick a point p from the set $G \cap \bigcap\{\bigcup \mathcal{R}_\alpha : \alpha \in A\}$. For every $\alpha \in A$ let W_α be the unique element of $\mathcal{R}_\alpha$ that contains p. Thus $p \in \bigcap\{f_\alpha(W_\alpha) : \alpha \in A\}$. It would follow from condition (2) that $|U \in \mathcal{P} : p \in U| \geq \kappa$; a contradiction. □

Theorem 4. *If X is a κ-semibaire space and $S(X) \leq \kappa$, then κ is a π-caliber of X.*

Proof: Suppose to the contrary that there exist a point-κ open family $\mathcal{P}$ in X and a non-empty open set $G \subseteq X$ such that if V is a non-empty open subset of G, then

$$(\blacklozenge) \qquad |\{U \in \mathcal{P} : U \cap V \neq \emptyset\}| \geq \kappa.$$

By Theorem 3, κ is a singular cardinal. Since $S(X)$ is a regular cardinal, $S(X) < \kappa$. Virtually the same way as in the proof of Theorem 3, one can construct $\mathcal{R}_\alpha$ and f_α for every $\alpha < \kappa$ (the construction goes through since the cardinality of every $\mathcal{R}_\alpha$ is $< S(X) < \kappa$ and because of condition ($\blacklozenge$)). This leads to a contradiction with $\mathcal{P}$ being point-κ. □

From Proposition 1 and Theorem 3 we get the following

Corollary 1. *Let κ be a regular infinite cardinal and let X be a space such that $S(X) \leq \kappa^+$. X is a κ-semibaire space if and only if κ is a π-caliber of X.*

Let $\mathbb{N}^*$ denote the remainder of the Čech-Stone compactification of a countable discrete space. If $\mathbf{p} = 2^\omega$ (e.g., assuming Martin's axiom), then $\mathbb{N}^*$ is a 2^ω-Baire space (cf. [8]). Hence By Theorem 3,

Corollary 2. *If $\mathbf{p} = 2^\omega$, then 2^ω is a π-caliber of $\mathbb{N}^*$.*

Corollary 3. *For a regular infinite cardinal κ and for arbitrary space X the following conditions are equivalent:*

(a) *κ is a caliber of X;*
(b) *$S(X) \leq \kappa$ and for each increasing sequence $\{E_\alpha : \alpha < \kappa\}$ of nowhere dense subsets of X, $\bigcup\{E_\alpha : \alpha < \kappa\}$ is a boundary subset of X;*
(c) *$S(X) \leq \kappa$ and κ is a π-caliber of X;*
(d) *$S(X) \leq \kappa$ and X is κ-semibaire.*

Proof: The equivalence (a)$\longleftrightarrow$(b) is known (cf. [8]).

The equivalence (c)$\longleftrightarrow$(d) has been established in Theorem 3.

The implication (a)$\longrightarrow$(d) is proved in Proposition 1. We shall prove the implication (c)$\longrightarrow$(a).

Assume that κ is a π-caliber of X and that the cardinality of any cellular family in X is $< \kappa$. Let $\mathcal{P} = \{U_\alpha : \alpha < \kappa\}$ be a family of non-empty open subsets of X. Assume to the contrary that for each $A \subseteq \kappa$, $|A| = \kappa$, $\bigcap\{U_\alpha : \alpha \in A\} = \emptyset$. Thus $\mathcal{P}$ is a point-κ open family in X. Let $\mathcal{R}$ be a maximal cellular family in X such that each member of $\mathcal{R}$ intersects $< \kappa$ members of $\mathcal{P}$. Since κ is a π-caliber of X, $\bigcup \mathcal{R}$ is a dense subset of X. Let $\mathcal{P}_V = \{U \in \mathcal{P} : U \cap V \neq \emptyset\}$. Clearly, $\bigcup\{\mathcal{P}_V : V \in \mathcal{R}\} = \mathcal{P}$ and $|\mathcal{P}_V| < \kappa$ for each $V \in \mathcal{R}$. Since we have assumed that κ is a regular cardinal, $|\mathcal{P}| < \kappa$; a contradiction. □

Corollary 4 (F. Tall [14]). *If X is ccc ω_1-Baire space, then ω_1 is a caliber of X.*

Corollary 5. *If κ is a regular cardinal and X is a κ-semibaire space such that $S(X) \leq \kappa^+$, then each open point-κ family in X has cardinality $\leq \kappa$.*

Proof: Let $\mathcal{P}$ be an open point-κ family in X. Let $\mathcal{R}$ be a maximal cellular family in X such that each member of $\mathcal{R}$ intersects $< \kappa$ members of $\mathcal{P}$. By Theorem 3, $\bigcup \mathcal{R}$ is a dense subset of X. Let $\mathcal{P}_V = \{U \in \mathcal{P} : U \cap V \neq \emptyset\}$. Clearly, $\bigcup\{\mathcal{P}_V : V \in \mathcal{R}\} = \mathcal{P}$ and $|\mathcal{P}_V| < \kappa$ for each $V \in \mathcal{R}$. Since $|\mathcal{R}| \leq \kappa$ and κ is a regular cardinal, $|\mathcal{P}| \leq \kappa$. □

A π-*base* for X is a family $\mathcal{C}$ of non-empty open subsets of X such that each non-empty open subset of X contains a member of the family $\mathcal{C}$. The cardinal number $\pi w(X) = \inf\{|\mathcal{C}| : \mathcal{Q}$ is a π-base for $X\}$ is called the π-*weight* of X.

Theorem 5. *If κ is a regular cardinal number and X is a κ-semibaire Hausdorff space such that $\pi w(X) \leq \kappa^+$, then κ is a π-caliber of X.*

Proof: Assume otherwise. Then there exist a point-κ open family $\mathcal{P}$ in X and a non-empty open set $G \subseteq X$ such that if V is a non-empty open subset of G, then $|\{U \in \mathcal{P} : U \cap V \neq \emptyset\}| \geq \kappa$. In fact, by Lemma 4, we can assume that $|\{U \in \mathcal{P} : U \cap V \neq \emptyset\}| \geq \kappa^+$, and, by Theorem 3, that $S(V) \geq \kappa^{++}$.

Let $\mathcal{Q}$ be a π-base for X such that $|\mathcal{Q}| \leq \kappa^+$ and let $\mathcal{C} = \{U \in \mathcal{Q} : \emptyset \neq U \subseteq G\}$. Since $S(G) \geq \kappa^{++}$, $|\mathcal{C}| = \kappa^+$. Index faithfully $\mathcal{C}$, say $\mathcal{C} = \{W_\alpha : \alpha < \kappa^+\}$. Since $|\{U \in \mathcal{P} : U \cap V \neq \emptyset\}| \geq \kappa^+$ whenever V is a non-empty open subset of G, one can (by induction) easily construct a one-to-one function $f : \kappa^+ \to \mathcal{P}$ so that:

$$(*) \qquad \text{For each } \alpha < \kappa^+,\ V_\alpha = W_\alpha \cap f(\alpha) \neq \emptyset.$$

The condition $(*)$ implies that the family $\{V_\alpha : \alpha < \kappa^+\}$ is both a point-κ open family in X and a π-base for G. For each $\alpha < \kappa^+$, let $\{V_{\alpha\xi} : \xi < \kappa\}$ be a family of pairwise disjoint non-empty open subsets of X such that $V_{\alpha\xi} \subseteq V_\alpha$ for each $\xi < \kappa$. We set $F_\beta = G - \bigcup\{V_{\alpha\xi} : \alpha < \kappa^+ \text{ and } \beta \leq \xi < \kappa\}$. Then $\{F_\beta : \beta < \kappa\}$ is a nested upward sequence of closed subsets of G. To get a contradiction, we are going to show that each set F_β is nowhere dense and that $\bigcup\{F_\beta : \beta < \kappa\} = G$.

To prove that F_β is nowhere dense, take any non-empty open set $V \subseteq G$. There exists $\alpha < \kappa^+$ such that $V_\alpha \subseteq V$. So, if $\beta < \kappa$, then $\emptyset \neq V_{\alpha\beta} \subseteq V$ and $V_{\alpha\beta} \cap F_\beta = \emptyset$.

To prove that the sets F_β, $\beta < \kappa$, cover G, take any point $y \in G$. For each $\alpha < \kappa^+$, let $y(\alpha) = 0$ in case $y \notin \bigcup\{V_{\alpha\xi} : \xi < \kappa\}$, or in case $y \in \bigcup\{V_{\alpha\xi} : \xi < \kappa\}$, let $y(\alpha)$ be the unique ξ such that $y \in V_{\alpha\xi}$. Since the family $\{V_\alpha : \alpha < \kappa^+\}$ is a point-κ family, there are less than κ non-zero $y(\alpha)$'s. Since κ is a regular cardinal number, there exists $\beta < \kappa$ such that $y(\alpha) < \beta$ for each $\alpha < \kappa^+$. □

Acknowledgment. Thanks to the anonymous referee for the valuable suggestions, in particular, for suggesting a part of Problem 1, and corrections.

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(*Received* February 6, 2011, *revised* July 21, 2011)