More on κ-Ohio completeness

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Abstract. We study closed subspaces of κ-Ohio complete spaces and, for κ uncountable cardinal, we prove a characterization for them. We then investigate the behaviour of products of κ-Ohio complete spaces. We prove that, if the cardinal κ^+ is endowed with either the order or the discrete topology, the space $(\kappa^+)^{\kappa^+}$ is not κ-Ohio complete. As a consequence, we show that, if κ is less than the first weakly inaccessible cardinal, then neither the space ω^{κ^+}, nor the space $\mathbb{R}^{\kappa^+}$ is κ-Ohio complete.

Keywords: κ-Ohio complete, compactification, subspace, product

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1. Introduction

All spaces under discussion are Tychonoff. For all undefined notions we refer to [6].

The property of κ-Ohio completeness was introduced in [5] and it is a natural generalization of Ohio completeness, which was introduced by Arhangel'skii in [1] to study remainders in compactifications of topological spaces.

Recall that a topological space X is *κ-Ohio complete* if for every compactification γX of X there exists a G_κ-subset S of γX such that $X \subseteq S$ and for every $y \in S \setminus X$, there is a G_κ-subset of γX that contains y and misses X.

In [5] particular attention was given to sum theorems for κ-Ohio complete spaces. The aim of this paper is focusing on the behaviour that closed subspaces of κ-Ohio complete spaces and products of κ-Ohio complete spaces have. Indeed it is still an open question whether the κ-Ohio completeness property is closed-hereditary or finitely multiplicative.

The paper is divided in two parts. In the first we investigate the behaviour of closed subspaces. Our main result is a characterization of closed subspaces of κ-Ohio complete spaces, for κ uncountable cardinal. In the second part we study products of κ-Ohio complete spaces. We prove that, if the cardinal κ^+ is endowed with either the order or the discrete topology, the space $(\kappa^+)^{\kappa^+}$ is not κ-Ohio complete. From this results it follows that, for a large class of cardinals κ, neither the space ω^{κ^+} nor the space $\mathbb{R}^{\kappa^+}$ is κ-Ohio complete. For more information see [2].

2. Preliminaries

Following the notation of [4] and [5] we say that a compactification γX of a space X is *κ-good* for X if there exists a G_κ-subset S of γX such that $X \subseteq S$ and for every $y \in S \setminus X$, there is a G_κ-subset of γX that contains y and misses X. We denote with the symbol $\kappa\mathcal{O}(X)$ the collection of all κ-good compactifications of X. Similarly, we say that a G_κ-subset of a compactification γX of X is a *G_κ-good subset* for X if it contains X, and if every point of $S \setminus X$ can be separated from X by a G_κ-subset of γX. If $\kappa = \omega$ we omit the symbol ω.

Observe that any space is κ-Ohio complete, for some large enough κ. Recall that the Čech-number of a space X, denoted by $\check{\mathrm{C}}(X)$, is the smallest cardinality of a collection $\mathcal{U}$ of open subsets of γX such that $X = \bigcap \mathcal{U}$, where γX is any compactification of X. Therefore, if X is any space, it follows that X is $\check{\mathrm{C}}(X)$-Ohio complete. On the other hand, for every infinite cardinal κ, there exist spaces which are not κ-Ohio complete, as it is shown in the next example (see also [4, Example 5.2]).

Example 2.1. Consider the cardinal κ^+ endowed with the discrete topology and its one point-compactification $\kappa^+ \cup \{\infty\}$. The example is the subspace X of the product $Z = (\kappa^+ \cup \{\infty\}) \times (\kappa^+ \cup \{\infty\})$ where $X = (\kappa^+ \times \kappa^+) \cup \{(\infty, \infty)\}$.

If G is a G_κ-subset of Z that contains the point (∞, ∞), then $G \cap (Z \setminus X)$ is non-empty, so X is not a G_κ-subset of Z. Similarly, $Z \setminus X$ contains no non-empty G_κ-subset of Z; this clearly implies that X is not κ-Ohio complete.

It is worth noting that, for a large class of cardinals κ, the space X we have just constructed has a good compactification, even if it is not κ-Ohio complete. Indeed, assume that κ is a non-measurable cardinal number. Then, the cardinal κ^+ is non-measurable as well, and in this case it is well-known that the discrete space of cardinality κ^+ is realcompact (see [6, Exercise 3.11.D(a)]). It follows that, under this hypothesis, the space X is realcompact (see [6, Exercise 3.11.A]), therefore its Čech-Stone compactification βX is good by [6, Theorem 3.11.10].

This means that, for a fixed cardinal κ, if the Čech-Stone compactification of a space X is κ-good for X, the space X need not be κ-Ohio complete. On the other hand, if a space X has a κ-good compactification γX, then the Čech-Stone compactification βX of X is always κ-good for X, as it is shown in the next proposition.

Proposition 2.2. *Let X be a space and let $\gamma X \in \kappa\mathcal{O}(X)$. Then $\{\delta X : \delta X \in \mathcal{C}(X)$ and $\delta X \geq \gamma X\} \subseteq \kappa\mathcal{O}(X)$.*

For the simple proof see [3, Proposition 4.3].

3. A characterization of closed subspaces of κ-Ohio complete spaces

In [3] we asked whether closed subspaces of Ohio complete spaces are again Ohio complete. Unfortunately we do not know the answer, as we do not know whether closed subspaces of κ-Ohio complete spaces are again κ-Ohio complete. However, there are some positive results; we will prove them in this section.

Proposition 2.2 asserts that if a space X has a κ-good compactification γX, then every compactification greater than or equal to γX (with respect to the order $\leq$) is κ-good for X. However, if a space is a closed subspace of a κ-Ohio complete space, then a sort of complementary property holds, as we are going to show. The formulation of the result is new, but it has actually been proved in [3]. We include the proof for completeness sake.

Lemma 3.1. *Let Y be a closed subspace of X. Fix a compactification αX of X and let $\gamma Y = \overline{Y}^{\alpha X}$. Then, for every compactification δY of Y such that $\delta Y \leq \gamma Y$, there exists some compactification ϱX of X such that $\delta Y = \overline{Y}^{\varrho X}$ and $\varrho X \leq \alpha X$.*

Proof: Fix a compactification δY of Y such that $\delta Y \leq \gamma Y$. Hence, there exists a continuous map $f : \gamma Y \to \delta Y$ such that $f(y) = y$, for every $y \in Y$. Consider the adjunction space $Z = \alpha X \cup_f \delta Y$. Clearly Z is a compact Hausdorff space, since it is the image of the compact space $\alpha X \oplus \delta Y$ under a closed continuous function, that is, the natural quotient mapping π. Observe that π is closed since f is closed (see for instance [6, p. 94]).

First we shall prove that X, considered as a subspace of Z, has the original topology, by showing that $\pi\restriction X : X \to \pi(X)$ is a homeomorphism. To verify that $\pi\restriction X$ is one-to-one, pick two different points $x, y \in X$. Observe that, since Y is closed in X, we have $(\gamma Y \setminus Y) \cap X = \emptyset$. There are three different cases to consider. If $x, y \in X \setminus Y$ we have $x, y \in \alpha X \setminus \gamma Y$ and then, by construction, the equivalence classes of x and y are $\{x\}$ and $\{y\}$ respectively. If $x \in X \setminus Y$ and $y \in Y$, the equivalence classes of x and y are $\{x\}$ and $\{y\} \cup f^{-1}(y)$, respectively. Finally, if $x, y \in Y$, the equivalence classes of x and y are $\{x\} \cup f^{-1}(x)$ and $\{y\} \cup f^{-1}(y)$, respectively. In all cases we have $\pi(x) \neq \pi(y)$. This proves that $\pi\restriction X$ is one-to-one.

We will now prove that $\pi\restriction X$ is closed. As we observed before π is closed. Let D be a closed subspace of X, then we may find a closed subset C of $\alpha X \oplus \delta Y$, such that $D = C \cap X$. It follows that $\pi(D) = \pi(C \cap X) = \pi(C) \cap X$ is a closed subset of X. This shows that $\pi\restriction X$ is a homeomorphism.

In a similar way we can prove that δY as a subspace of Z has the original topology. It follows that $\overline{Y}^{Z} = \delta Y$.

Since the space Z is clearly a compactification of X such that $Z \leq \alpha X$, we are done. □

Given a space X we say that a compactification γX of X is *very κ-good* if $\{\delta X : \delta X \in \mathcal{C}(X) \text{ and } \delta X \leq \gamma X\} \subseteq \kappa\mathcal{O}(X)$. In particular, if γX is a very κ-good compactification for X, then every compactification δX of X such that $\delta X \leq \gamma X$, is very κ-good for X.

Theorem 3.2. *Let Y be a closed subspace of a space X. Assume that X has a very κ-good compactification αX. Then $\gamma Y = \overline{Y}$ (closure in αX) is a very κ-good compactification for Y.*

Proof: Fix a compactification δY of Y such that $\delta Y \leq \gamma Y$. By Lemma 3.1, there exists a compactification ϱX of X such that $\delta Y = \overline{Y}^{\varrho X}$ and $\varrho X \leq \alpha X$.

Since αX is a very κ-good compactification for X, the compactification ϱX is κ-good for X. Let S be a G_κ-subset of ϱX that is κ-good for X. Then the set $S \cap \delta Y$ is G_κ-good for Y. This completes the proof. □

An application of Theorem 3.2 is the following result, which shows that κ-Ohio completeness is hereditary with respect to closed and C^*-embedded subspaces (see also [3]).

Corollary 3.3. *Let Y be a closed C^*-embedded subspace of a κ-Ohio complete space X. Then Y is κ-Ohio complete.*

Proof: Closures are taken in βX. It follows from Theorem 3.2 that $\overline{Y}$ is a very κ-good compactification for Y. But $\overline{Y} = \beta Y$, by [6, Corollary 3.6.7]. This proves that Y is κ-Ohio complete. □

Corollary 3.4. *Let Y be a closed subspace of a κ-Ohio complete normal space X. Then Y is κ-Ohio complete.*

If $A \subseteq X$, a continuous function $f : X \to A$ is called a *retraction* of X onto A, if $f(x) = x$ for all $x \in A$. In this case A is called a *retract* of X.

Corollary 3.5.
(1) *Every clopen subspace of a κ-Ohio complete space is κ-Ohio complete.*
(2) *Every retract of a κ-Ohio complete space is κ-Ohio complete.*

Proof: This follows from the fact that clopen subspaces and retracts are closed and C^*-embedded subspaces. □

Unfortunately this does not answer to the following:

Question 3.6. *Is κ-Ohio completeness a closed-hereditary property?*

Theorem 3.2 implies in particular that a closed subspace of a κ-Ohio complete space has some very κ-good compactification. It is pretty natural to ask whether the converse is true, that is, whether, given a space having a very κ-good compactification, it can be embedded as a closed subspace in some κ-Ohio complete space.

The following theorem shows that, if κ is an uncountable cardinal number, the answer is yes.

Theorem 3.7. *Let κ be an uncountable cardinal number. The following statements are equivalent.*
(1) *Y is a closed subspace of a κ-Ohio complete space X.*
(2) *There exists a very κ-good compactification γY of Y.*

Proof: (1) $\Rightarrow$ (2) follows from Theorem 3.2.

(2) $\Rightarrow$ (1). Fix a very κ-good compactification γY of Y. Consider the ordinal space ω_1+1 and let Z be the space $(\omega_1+1) \times \gamma Y$, and let X be the subspace of Z given by

$$\big(\omega_1 \times \gamma Y\big) \cup \{\omega_1\} \times Y.$$

Then Y is clearly a closed subspace of X, so to prove the theorem it suffices to show that X is κ-Ohio complete.

First observe that $\beta X = Z$. Indeed, note that $\beta(\omega_1 \times \gamma Y) = (\omega_1+1) \times \gamma Y = Z$. This can be found in [6, Problem 3.12.20(c)]. Since $\omega_1 \times \gamma Y \subseteq X \subseteq Z$, it follows that $\beta X = Z$ by [6, Corollary 3.6.9].

To show that X is κ-Ohio complete, fix a compactification αX of X. Then $\alpha X \leq \beta X = Z$. So we may fix a continuous function $f : Z \to \alpha X$ such that $f \restriction X$ is the identity on X. We let g be the restriction of f to the set $\{\omega_1\} \times \gamma Y$. Note that since the remainder $\beta X \setminus X$ is contained in the domain of g, it follows that the remainder $\alpha X \setminus X$ is contained in the range of g. So the range of g is given by

$$W = (\omega_1 \times Y) \cup (\alpha X \setminus X).$$

Clearly, the function g witnesses the fact that $W \leq \gamma Y$. By assumption it follows that W is a κ-good compactification for $\{\omega_1\} \times Y$, so we may fix a G_κ-subset S of αX such that every point in $(W \cap S) \setminus (\{\omega_1\} \times Y)$ can be separated from $\{\omega_1\} \times Y$ by a G_κ-subset of W.

Now let $S' = (\omega_1 \times \gamma Y) \cup S$. Since $\omega_1 \times \gamma Y$ is locally compact, it is an open subset of αX and therefore S' is a G_κ-subset of αX. We claim that S' is a G_κ-good subset for X.

So pick an arbitrary point $p \in S' \setminus X$. Then $p \in S \setminus (\{\omega_1\} \times Y)$. By the choice of S, there is a G_κ-subset T of W such that $p \in T$ and $T \cap (\{\omega_1\} \times Y) = \emptyset$. Now note that $\omega_1 \times \gamma Y$ is the union of ω_1-many compact subspaces and therefore $\alpha X \setminus (\omega_1 \times \gamma Y) = W$ is a G_{ω_1}-subset and hence a G_κ-subset of αX. But then T is also a G_κ-subset of αX. Since T is disjoint from X, this set separates the point p from X. This completes the proof. $\square$

Question 3.8. *Does the equivalence of Theorem* 3.7 *also hold for* $\kappa = \omega$*?*

4. Products of κ-Ohio complete spaces

As we said in the introduction we do not know whether κ-Ohio completeness is finitely multiplicative. Actually, we do not know if even the product of a κ-Ohio complete space with a compact space is again κ-Ohio complete. However, there is some relation between these questions and Question 3.6, as the next theorem shows (see also [3, Theorem 3.4]):

Theorem 4.1. *Let κ be an infinite cardinal number. Consider the following statements.*

(1) *Preimages of κ-Ohio complete spaces under perfect mappings are κ-Ohio complete.*
(2) *The product of a κ-Ohio complete space and a compact space is always κ-Ohio complete.*
(3) *Every closed subspace of a κ-Ohio complete space is κ-Ohio complete.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3).

Proof: To prove that (1) $\Rightarrow$ (2), let X be a κ-Ohio complete space and K be a compact space. Then $\pi_X : X \times K \to X$ is a perfect mapping, so the hypothesis implies that $X \times K$ is κ-Ohio complete.

For (2) $\Rightarrow$ (3), let Y be a closed subspace of a κ-Ohio complete space X. Consider the product $Z = X \times \beta Y$ and its subspace $\Delta(Y)$. By [6, Theorem 3.6.1], Y is C^*-embedded in βY. From this fact it easily follows that $\Delta(Y)$ is a C^*-embedded copy of Y in Z. Since $\Delta(Y)$ is also closed in Z, by Corollary 3.3 it follows that if Z is κ-Ohio complete then so is Y.

We finally prove that (2) $\Rightarrow$ (1). Since (2) $\Rightarrow$ (3), it follows from [6, Theorem 3.7.26] that (1) holds. $\square$

Therefore, if Question 4.2 below has a positive answer, then Question 3.6 has a positive answer as well.

Question 4.2. *Is the product of a κ-Ohio complete space with a compact space again κ-Ohio complete?*

On the other hand, it is straightforward to see that if a product space is κ-Ohio complete, then each of its factors is κ-Ohio complete as well.

Proposition 4.3. *Let $X = \prod_{\alpha<\tau} X_\alpha$ be a κ-Ohio complete space. Then, for every $\alpha < \tau$, the space X_α is κ-Ohio complete.*

Proof: Note that every X_α is a retract of X. Now it suffices to apply Corollary 3.5(2). $\square$

The following results show that the product of κ-many κ-Ohio complete spaces has many κ-good compactifications.

Lemma 4.4. *Let $\{X_\alpha : \alpha < \kappa\}$ be a family of spaces. For every $\alpha < \kappa$, let S_α be a G_κ-subset of X_α. Then $\prod_{\alpha<\kappa} S_\alpha$ is a G_κ-subset of $X = \prod_{\alpha<\kappa} X_\alpha$.*

Proposition 4.5. *Let $\{X_\alpha : \alpha < \kappa\}$ be a family of spaces. For every $\alpha < \kappa$, let $\gamma_\alpha X_\alpha \in \kappa\mathcal{O}(X_\alpha)$. Then $\prod_{\alpha<\kappa} \gamma_\alpha X_\alpha \in \kappa\mathcal{O}(\prod_{\alpha<\kappa} X_\alpha)$.*

Proof: Since $\gamma_\alpha X_\alpha \in \kappa\mathcal{O}(X_\alpha)$, for every $\alpha < \kappa$ there exists a G_κ-subset S_α of $\gamma_\alpha X_\alpha$ which is κ-good with respect to X_α. By Lemma 4.4, the set $\prod_{\alpha<\kappa} S_\alpha$ is a G_κ-subset of $\prod_{\alpha<\kappa} \gamma_\alpha X_\alpha$ that clearly contains $\prod_{\alpha<\kappa} X_\alpha$. We will show that $\prod_{\alpha<\kappa} S_\alpha$ is κ-good with respect to $\prod_{\alpha<\kappa} X_\alpha$.

So, pick a point $p = (p_\alpha)_{\alpha<\kappa} \in \prod_{\alpha<\kappa} S_\alpha \setminus \prod_{\alpha<\kappa} X_\alpha$. Then, for some $\beta < \kappa$, we have $p_\beta \in S_\beta \setminus X_\beta$. Therefore, there exists a G_κ-subset T_β of $\gamma_\beta X_\beta$ containing p_β and missing X_β. The set $Z = \pi_\beta^{-1}(T_\beta)$ is a G_κ-subset of $\prod_{\alpha<\kappa} \gamma_\alpha X_\alpha$ that contains p and misses $\prod_{\alpha<\kappa} X_\alpha$. This proves the proposition. $\square$

The proof of the preceding proposition is based on the fact that the intersection of κ-many G_κ-subsets is again a G_κ-subset. Since this property may fail for larger intersections, we might expect that Proposition 4.5 does not generalize to products with κ^+-many factors. The next proposition shows that in fact this is the case.

Proposition 4.6. *Let Y be the cardinal κ^+ endowed with either the discrete or the order topology, and consider its one-point compactification $\omega Y = Y \cup \{\infty\}$. Then $(\omega Y)^{\kappa^+}$ is not a κ-good compactification for Y^{κ^+}.*

PROOF: Observe that the point ∞ is not a G_κ-subset of ωY. Hence, Y^{κ^+} is G_κ-dense in $(\omega Y)^{\kappa^+}$. But its remainder $(\omega Y)^{\kappa^+} \setminus Y^{\kappa^+}$ is G_κ-dense in $(\omega Y)^{\kappa^+}$ as well. So $(\omega Y)^{\kappa^+}$ cannot be a κ-good compactification for Y^{κ^+}. $\square$

Corollary 4.7. *If the cardinal κ^+ is endowed with either the order or the discrete topology, the space $(\kappa^+)^{\kappa^+}$ is not κ-Ohio complete.*

An application of this result is that the limit of an inverse system of κ-Ohio complete spaces need not be κ-Ohio complete.

Proposition 4.8. *The limit of an inverse system of κ-Ohio complete spaces need not be κ-Ohio complete.*

PROOF: If $\alpha < \kappa^+$, then it follows from [9, Proposition 1.10] that $\check{\mathrm{C}}((\kappa^+)^\alpha) \leq |\alpha| \leq \kappa$. So it is clear that $(\kappa^+)^\alpha$ is κ-Ohio complete. Now observe that $(\kappa^+)^{\kappa^+}$ can be seen as the inverse limit of the system $\{(\kappa^+)^\alpha, \pi^\alpha_\beta, \kappa^+\}$, where $\pi^\alpha_\beta : (\kappa^+)^\alpha \to (\kappa^+)^\beta$ is the usual projection. $\square$

Remark 4.9. Let us remark that the behaviour of the space $(\kappa^+)^{\kappa^+}$ can be different if we consider κ^+ endowed with the discrete or with the order topology. Indeed, if κ^+ has the discrete topology, then, for a large class of cardinals (namely all non-measurable cardinals κ, see [6, Exercise 3.11.D(a)]), the space $(\kappa^+)^{\kappa^+}$ is realcompact and then it has a κ-good compactification.

If we now consider κ^+ with the order topology and we assume that $\kappa = \omega$, then the space $\omega_1^{\omega_1}$ is pseudocompact (see, for example [6, Exercise 3.12.21.(e)]). By a well-known result of Glicksberg ([7]), we have $\beta(\omega_1^{\omega_1}) = (\beta\omega_1)^{\omega_1}$. Since $\beta\omega_1 = \omega_1 + 1$, Proposition 4.6 implies that $\beta(\omega_1^{\omega_1})$ is not a good compactification for $\omega_1^{\omega_1}$. Therefore, it follows by Proposition 2.2, that $\omega_1^{\omega_1}$ does not have any good compactification.

A natural question is then whether the space ω^{κ^+} is or is not κ-Ohio complete. Observe that the argument used in Proposition 4.6 cannot be applied to ω^{κ^+}: every product compactification of ω^{κ^+} is indeed even good. This is a consequence of the next proposition. We will however answer our question in Corollary 4.16 below.

Recall that the compact covering number of a space X, denoted by $\mathrm{kcov}(X)$, is the smallest cardinality of a collection $\mathcal{K}$ of compact subsets of X such that $X = \bigcup \mathcal{K}$. It is well-known and easy to show that for a space X and for any compactification γX of X, we have $\mathrm{kcov}(\gamma X \setminus X) = \check{\mathrm{C}}(X)$.

Proposition 4.10. *Let $X = \prod_{\alpha<\kappa} X_\alpha$, where $\mathrm{kcov}(X_\alpha) \leq \lambda$ for every $\alpha < \kappa$, and let $\gamma_\alpha X_\alpha \in \mathcal{C}(X_\alpha)$, for every $\alpha < \kappa$. Then $\prod_{\alpha<\kappa} \gamma_\alpha X_\alpha \in \lambda\mathcal{O}(X)$.*

Proof: Let $Z = \prod_{\alpha<\kappa} \gamma_\alpha X_\alpha$. We will show that Z itself is the good G_λ-subset we are looking for. Note that, since $\mathrm{kcov}(X_\alpha) \leq \lambda$, the remainder $\gamma_\alpha X_\alpha \setminus X_\alpha$ is a G_λ-subset of $\gamma_\alpha X_\alpha$, for every $\alpha < \kappa$.

Now, fix a point $x = (x_\alpha)_{\alpha<\kappa} \in Z \setminus X$. So, there exists some $\alpha < \kappa$ such that $x_\alpha \in \gamma_\alpha X_\alpha \setminus X_\alpha$. The set $W = \pi_\alpha^{-1}(\gamma_\alpha X_\alpha \setminus X_\alpha)$, is a G_λ-subset of Z that misses X. This completes the proof. □

Corollary 4.11. *If a product space has σ-compact factors, then any compactification of its product is good.*

This raises the question whether spaces like ω^{κ^+} or $\mathbb{R}^{\kappa^+}$ are κ-Ohio complete or not. Furthermore it turns out that finding a non κ-good compactification for such spaces is not trivial.

Nevertheless, using Proposition 4.6, which is a very simple but very useful result, we will be able to prove that, if κ is less than the first weakly inaccessible cardinal, neither ω^{κ^+} nor $\mathbb{R}^{\kappa^+}$ is κ-Ohio complete.

Theorem 4.12. *If X contains a closed copy of the space κ^+, endowed either with the discrete or the order topology, then X^{κ^+} is not κ-Ohio complete.*

Proof: Let us prove the theorem assuming that X contains a closed copy of the discrete space of cardinality κ^+. The other case is analogous. Since X contains a closed copy of κ^+, the space X^{κ^+} contains a closed copy of $(\kappa^+)^{\kappa^+}$. Assume, striving for a contradiction, that X^{κ^+} is κ-Ohio complete and let $Z = (\gamma X)^{\kappa^+}$, where γX is any compactification of X. Closures are taken in Z.

Our hypothesis, combined with Theorem 3.2, imply that $\overline{(\kappa^+)^{\kappa^+}}$ is a very κ-good compactification for $(\kappa^+)^{\kappa^+}$. Since $\overline{(\kappa^+)^{\kappa^+}} \geq (\omega\kappa^+)^{\kappa^+}$, the latter product is a κ-good compactification for $(\kappa^+)^{\kappa^+}$, which is a contradiction with Proposition 4.6. □

From the proof of Theorem 4.12 we get the following:

Corollary 4.13. *If X contains a closed copy of the space κ^+, endowed either with the discrete or the order topology, then no compactification of X^{κ^+} of the form $(\gamma X)^{\kappa^+}$ can be very κ-good for X^{κ^+}.*

Recall that an uncountable cardinal is called weakly inaccessible if it is a regular limit cardinal. We denote by θ the first weakly inaccessible cardinal.

Corollary 4.14. *Assume that $\kappa < \theta$. If X^{κ^+} is κ-Ohio complete, then X is countably compact.*

Proof: Observe at first that if $\kappa < \theta$, then $\kappa^+ < \theta$. If X were not countably compact, then X^{κ^+} would contain a closed copy of ω^{κ^+}. Since $\kappa^+ < \theta$, the power ω^{κ^+} contains a closed copy of the discrete space κ^+, by [8]. Then X^{κ^+} would contain a closed copy of κ^+, which is a contradiction with Theorem 4.12. □

Question 4.15. *Can we improve Corollary* 4.14 *substituting 'countably compact' by 'compact'?*

In [3] we showed that the answer is yes for $\kappa = \omega$.

Corollary 4.16. *If $\kappa < \theta$, then neither ω^{κ^+} nor $\mathbb{R}^{\kappa^+}$ is κ-Ohio complete.*

Corollary 4.17. *If $\kappa < \theta$, then no compactification of ω^{κ^+} (resp. $\mathbb{R}^{\kappa^+}$) of the form Z^{κ^+} is very κ-good for ω^{κ^+} (resp. $\mathbb{R}^{\kappa^+}$).*

Question 4.18. *Let $\kappa < \theta$. Does exist some very κ-good compactification for ω^{κ^+} (resp. $\mathbb{R}^{\kappa^+}$)?*

By Theorem 3.7 this question is equivalent to the question whether, if κ^+ is strictly less than the first weakly inaccessible cardinal, the space ω^{κ^+} (resp. $\mathbb{R}^{\kappa^+}$) can be embedded as a closed subspace in some κ-Ohio complete space. Moreover, let us point out that if Question 4.15 has a positive answer, then Question 4.18 has a negative answer.

Actually, to answer in the negative to Question 4.18 it would be enough to show that the space $(\kappa^+)^{\kappa^+}$, where κ^+ is endowed with the discrete topology does not have any very κ-good compactification. Unfortunately we do not know the answer to this.

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