Hom-Akivis algebras

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Abstract. Hom-Akivis algebras are introduced. The commutator-Hom-associator algebra of a non-Hom-associative algebra (i.e. a Hom-nonassociative algebra) is a Hom-Akivis algebra. It is shown that Hom-Akivis algebras can be obtained from Akivis algebras by twisting along algebra endomorphisms and that the class of Hom-Akivis algebras is closed under self-morphisms. It is pointed out that a Hom-Akivis algebra associated to a Hom-alternative algebra is a Hom-Malcev algebra.

Keywords: Akivis algebra, Hom-associative algebra, Hom-Lie algebra, Hom-Akivis algebra, Hom-Malcev algebra

Classification: 17A30, 17D10, 17D99

1. Introduction

The theory of Hom-algebras originated from Hom-Lie algebras introduced by J.T. Hartwig, D. Larsson, and S.D. Silvestrov in [9] in the study of quasi-deformations of Lie algebras of vector fields, including q-deformations of Witt algebras and Virasoro algebras. The connection between the theory of Hom-algebras and deformation theory and other trends in mathematics attracted attention of researchers (see, e.g., [5], [6], [7], [8], [12], [13], [15], [20]). Generalizing the relation between Lie algebras and associative algebras, the notion of a Hom-associative algebra is introduced by A. Makhlouf and S.D. Silvestrov in [14], where it is shown that the commutator algebra (with the twisting map) of a Hom-associative algebra is a Hom-Lie algebra. By twisting defining identities, other Hom-type algebras such as Hom-alternative algebras, Hom-Jordan algebras [12], Hom-Novikov algebras [21], or Hom-Malcev algebras [22] are introduced and discussed.

As for Hom-alternative algebras or Hom-Novikov algebras, we consider in this paper a twisted version of the Akivis identity which defines the so-called Akivis algebras. We call "Hom-Akivis algebra" this twisted Akivis algebra. It is known [3] that the commutator-associator algebra of a nonassociative algebra is an Akivis algebra. This led us to consider "non-Hom-associative algebras" i.e. Hom-nonassociative algebras or nonassociative Hom-algebras ([13], [14], [19]) and we point out that the commutator-Hom-associator algebra of a non-Hom-associative algebra has a Hom-Akivis structure. In this setting, Akivis algebras are special cases of Hom-Akivis algebras in which the twisting map is the identity map. Also the class of Hom-Akivis algebras contains the one of Hom-Lie algebras in the same way as the class of Akivis algebras contains the one of Lie algebras.

Akivis algebras were introduced by M.A. Akivis ([1], [2], [3]) as a tool in the study of some aspects of web geometry and its connection with loop theory. These algebras were originally called "W-algebras" [3]. Later, K.H. Hofmann and K. Strambach [10] introduced the term "Akivis algebras" for such algebraic objects.

The rest of the present paper is organized as follows. In Section 2 we recall basic definitions and useful results about Akivis algebras, Hom-Lie algebras and Hom-associative algebras. In Section 3 we consider non-Hom-associative algebras (one observes the counterpart of the generalization of associative algebras by nonassociative ones). We construct examples of non-Hom-associative algebras. In Section 4 Hom-Akivis algebras are considered. Two methods of producing Hom-Akivis algebras are provided, starting with either non-Hom-associative algebras (Theorem 4.2) or usual Akivis algebras along with twisting maps (Corollary 4.5). Examples illustrating Corollary 4.5 are provided. Theorem 4.8 gives a construction method of a sequence of Hom-Akivis algebras from a given Akivis algebra. Hom-Akivis algebras are shown to be closed under twisting by self-morphisms (Theorem 4.4). In Section 5, Hom-Akivis algebras associated to Hom-alternative algebras are shown to be Hom-Malcev algebras (these later algebraic objects are recently introduced by D. Yau [22]). This could be seen as a generalization of the Malcev construction of Moufang-Lie algebras (i.e. Malcev algebras) from alternative algebras [16].

Throughout this paper, all vector spaces and algebras are meant over a ground field $\mathbb{K}$ of characteristic 0.

2. Preliminaries

We recall useful definitions and results that are for further use. We begin with Akivis algebras.

An *Akivis algebra* $(\mathcal{A}, [-,-], \langle -,-,- \rangle)$ is a vector space $\mathcal{A}$ together with a bilinear skew-symmetric binary operation $(x,y) \mapsto [x,y]$ and a trilinear ternary operation $(x,y,z) \mapsto \langle x,y,z \rangle$ that are linked by the identity

$$\circlearrowleft_{(x,y,z)}[[x,y],z] = \circlearrowleft_{(x,y,z)}\langle x,y,z \rangle - \circlearrowleft_{(x,y,z)}\langle y,x,z \rangle, \tag{2.1}$$

where here, and in the sequel, $\circlearrowleft_{(x,y,z)}$ denotes the sum over cyclic permutation of x, y, z.

The relation (2.1) is called the *Akivis identity*.

In loop theory, roughly, Akivis algebras are for local smooth loops what are Lie algebras for local Lie groups. However Akivis algebras originated from web geometry ([1], [2]; see also [4] for a survey of the subject). The connection between web geometry and loop theory is briefly sketched as follows.

First we recall that a *quasigroup* is a groupoid $(Q, \cdot)$ in which the equation $x \cdot y = z$ is uniquely solvable with respect to x and y for any $x, y, z \in Q$, and that a *loop* is a quasigroup with an identity element.

Let X and Y be r-dimensional differentiable manifolds and $M = X \times Y$ the $2r$-dimensional manifold whose points are pairs (x, y) with $x \in X$ and $y \in Y$. Consider in M three families of r-dimensional surfaces given by equations $x = a$, $y = b$ and $z = q(x, y)$, where $q(x, y) = z$ defines a quasigroup. These families of surfaces constitute on M a *three-web* W if they satisfy the following properties.

(a) Two surfaces of the same family are disjoint.
(b) Two surfaces of different families are incident to one point of M.
(c) Any point $(x, y) \in M$ is incident to just one surface of each family.

The quasigroup defined by $q(x, y) = z$ is then called the *coordinate* (*local differentiable*) *quasigroup* of the web W (see, e.g., [2], [4]).

Conversely, if on a $2r$-dimensional manifold M three families of r-dimensional surfaces satisfying (a), (b) and (c) above are given, i.e. a three-web of r-dimensional surfaces is defined on M, then this web generates six local differentiable quasigroups that are parastrophic each to other ([2]). In fact, any two of the three families of surfaces can be chosen as coordinates and their equations are written as $x = a$ and $y = b$; therefore the equation of the third family is written as $q(x, y) = c$. Thus, by properties (a), (b), (c), the mapping $z = q(x, y)$ satisfies all the conditions in the definition of a quasigroup. Since there are six different ways of choosing an ordered pair from three families of surfaces of a three-web, one gets six coordinate quasigroups for a given three-web. Also recall that there are six quasigroups parastrophic to each given quasigroup.

Thus there is a one-to-one correspondence between the class of quasigroups and the class of three-webs.

In local coordinates, the quasigroup equation $z = q(x, y)$ is written as

$$z^i = q^i(x^j, y^k). \tag{2.2}$$

Using the Taylor formula, the equation (2.2) can be expanded in a neighborhood of a fixed point of the local differentiable quasigroup [2]. Then M.A. Akivis introduced the so-called *fundamental tensors* α^i_{jk}, β^i_{jkl} of the quasigroup (they are expressed through the coefficients in the Taylor expansion of (2.2)). These tensors are related by the following formula

$$\beta^i_{[jkl]} = 2\alpha^s_{[jk}\alpha^i_{|s|l]} \tag{2.3}$$

i.e.

$$\begin{aligned}&\frac{1}{2}\beta^i_{jkl} - \frac{1}{2}\beta^i_{kjl} + \frac{1}{2}\beta^i_{klj} - \frac{1}{2}\beta^i_{lkj} + \frac{1}{2}\beta^i_{ljk} - \frac{1}{2}\beta^i_{jlk}\\ &= 2(\alpha^s_{jk}\alpha^i_{sl} + \alpha^s_{kl}\alpha^i_{sj} + \alpha^s_{lj}\alpha^i_{sk}).\end{aligned}$$

We note that the relation (2.3) is first obtained in [1], where the exterior differential forms and exterior derivations are used in the study of three-webs (the three families of multidimensional surfaces of the considered three-web are given by three systems of Pfaffian forms). It is now easy to see that (2.3) is the tensor form of the Akivis identity (2.1).

It is well known that the commutator algebra of an associative algebra is a Lie algebra. M.A. Akivis [3] generalized this construction to nonassociative algebras. Recall that an algebra $(\mathcal{A}, \cdot)$ is said to be *nonassociative* if there is at least a triple $x, y, z \in \mathcal{A}$ such that $(x \cdot y) \cdot z \neq x \cdot (y \cdot z)$. The following holds.

Theorem 2.1 ([3]). *The commutator-associator algebra of a nonassociative algebra is an Akivis algebra.*

The Akivis algebra constructed by Theorem 2.1 is said to be *associated* (to a given nonassociative algebra) [10].

In Section 4 (Theorem 4.2) we give the Hom-counterpart of Theorem 2.1. Beforehand we recall some facts about Hom-algebras.

A *Hom-module* [20] is a pair $(\mathcal{A}, \alpha_{\mathcal{A}})$, where $\mathcal{A}$ is a vector space and $\alpha_{\mathcal{A}} : \mathcal{A} \mapsto \mathcal{A}$ a linear map.

A *Hom-associative algebra* [14] is a triple $(\mathcal{A}, \mu_{\mathcal{A}}, \alpha_{\mathcal{A}})$ in which $(\mathcal{A}, \alpha_{\mathcal{A}})$ is a Hom-module and $\mu_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \mapsto \mathcal{A}$ is a bilinear operation on $\mathcal{A}$ such that

$$\mu_{\mathcal{A}}(\mu_{\mathcal{A}}(x, y), \alpha_{\mathcal{A}}(z)) = \mu_{\mathcal{A}}(\alpha_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y, z)), \tag{2.4}$$

for all $x, y, z \in \mathcal{A}$.

The relation (2.4) is called the *Hom-associativity* for $(\mathcal{A}, \mu_{\mathcal{A}}, \alpha_{\mathcal{A}})$. If $\alpha_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}}$, then (2.4) is just the associativity.

Using the abbreviation xy for $\mu_{\mathcal{A}}(x, y)$, the Hom-associativity (2.4) reads

$$(xy)\alpha_{\mathcal{A}}(z) = \alpha_{\mathcal{A}}(x)(yz).$$

The Hom-associative algebra $(\mathcal{A}, \mu_{\mathcal{A}}, \alpha_{\mathcal{A}})$ is said to be *multiplicative* if $\alpha_{\mathcal{A}}$ is an endomorphism (i.e. a self-morphism) of $(\mathcal{A}, \mu_{\mathcal{A}})$. Hom-associative algebras are closely related to Hom-Lie algebras.

A *Hom-Lie algebra* is a triple $(\mathcal{A}, [-,-], \alpha_{\mathcal{A}})$ in which $(\mathcal{A}, \alpha_{\mathcal{A}})$ is a Hom-module and $[-,-] : \mathcal{A} \times \mathcal{A} \mapsto \mathcal{A}$ is a bilinear skew-symmetric operation on $\mathcal{A}$ such that

$$\circlearrowleft_{(x,y,z)} [[x, y], \alpha_{\mathcal{A}}(z)] = 0, \tag{2.5}$$

for $x, y, z \in \mathcal{A}$.

The relation (2.5) is called the *Hom-Jacobi identity*. If, moreover, $\alpha_{\mathcal{A}}$ is an endomorphism of $(\mathcal{A}, [-,-])$, then $(\mathcal{A}, [-,-], \alpha_{\mathcal{A}})$ is said to be *multiplicative*. Examples of Hom-Lie algebras could be found in [9], [14], [20].

In the Hom-Lie setting, the Hom-associative algebras play the role of associative algebras in the Lie setting in this sense that the commutator algebra of a Hom-associative algebra is a Hom-Lie algebra [14]. In [20] it is shown how arbitrary associative (resp. Lie) algebras give rise to Hom-associative (resp. Hom-Lie) algebras via endomorphisms. These constructions are considered here in the Hom-Akivis setting.

3. Non-Hom-associative algebras. Examples

In [14], [19] the notion of a Hom-associative algebra is extended to the one of a Hom-nonassociative algebra (or nonassociative Hom-algebra, or just Hom-algebra), i.e. a Hom-type algebra in which the Hom-associativity (as defined by (2.4)) does not necessarily hold. In this section and in the rest of the paper, such a Hom-type algebra will be called "non-Hom-associative" (although this terminology seems to be somewhat cumbersome) in order to stress the Hom-counterpart of the generalization of associative algebras by the nonassociative ones. We provide some examples.

Definition 3.1 ([13], [14], [19]). A *multiplicative non-Hom-associative* (i.e. not necessarily Hom-associative) *algebra* is a triple $(\mathcal{A}, \mu, \alpha)$ such that

(i) $(\mathcal{A}, \alpha)$ is a Hom-module;
(ii) $\mu : \mathcal{A} \times \mathcal{A} \mapsto \mathcal{A}$ is a bilinear operation on $\mathcal{A}$;
(iii) α is an endomorphism of $(\mathcal{A}, \mu)$ (multiplicativity).

Thus, by the non-Hom-associativity of $(\mathcal{A}, \mu, \alpha)$ with a given endomorphism α, we mean that there is at least a triple $x, y, z \in \mathcal{A}$ such that $\mu(\mu(x,y), \alpha(z)) \neq \mu(\alpha(x), \mu(y,z))$.

If α is the identity map in Definition 3.1, then $(\mathcal{A}, \mu, \alpha)$ reduces to a nonassociative algebra $(\mathcal{A}, \mu)$.

For a non-Hom-associative algebra, it makes sense to consider the so-called Hom-associators [12] (see also [14]), just as associators are considered in a nonassociative algebra.

Let $(\mathcal{A}, \mu, \alpha)$ be a non-Hom-associative algebra, where $\mathcal{A}$ is a $\mathbb{K}$-linear space, μ a bilinear operation on $\mathcal{A}$ and α a twisting map. For any $x, y, z \in \mathcal{A}$, the *Hom-associator* is defined by

$$\mathfrak{as}(x,y,z) = \mu(\mu(x,y), \alpha(z)) - \mu(\alpha(x), \mu(y,z)). \tag{3.1}$$

Then $(\mathcal{A}, \mu, \alpha)$ is said to be:

Hom-flexible, if $\mathfrak{as}(x,y,x) = 0$;
Hom-alternative, if $\mathfrak{as}(x,y,z)$ is skew-symmetric in x, y, z.

The following example of a non-Hom-associative algebra is derived from an example in [12].

Example 3.2. Let $\{u, v, w\}$ be a basis of a three-dimensional vector space $\mathcal{A}$. Define on $\mathcal{A}$ the operation μ and the linear map α as follows:

$$\begin{aligned} &\mu(u,u) = au, && \mu(v,v) = av \\ &\mu(u,v) = \mu(v,u) = av, && \mu(v,w) = bw \\ &\mu(u,w) = \mu(w,u) = bw, && \mu(w,v) = \mu(w,w) = 0, \end{aligned}$$

where $a, b \in \mathbb{K}$, $a \neq 0$, $b \neq 0$ and $a \neq b$. Then $(\mathcal{A}, \mu)$ is nonassociative since $\mu(\mu(v,v),w) \neq \mu(v, \mu(v,w))$. Let

$$\alpha(u) = av,\ \alpha(v) = aw,\ \alpha(w) = bu.$$

Now we have

$$\mu(\mu(u,v), \alpha(w)) = \mu(av, bu) = a^2bv$$

and

$$\mu(\alpha(u), \mu(v,w)) = \mu(av, bw) = ab^2w$$

so $\mu(\mu(u,v),\alpha(w)) \neq \mu(\alpha(u), \mu(v,w))$, i.e. the Hom-associativity fails for the triple $(\mathcal{A}, \mu, \alpha)$, and thus $(\mathcal{A}, \mu, \alpha)$ is non-Hom-associative.

Other examples follow.

Example 3.3. The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ has a basis $\{u, v, w\}$ with multiplication:

$$[u,v] = -2u,\ [u,w] = v,\ [v,w] = -2w.$$

Define $\alpha : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{sl}(2,\mathbb{C})$ by setting:

$$\alpha(u) = w,\ \alpha(v) = -v,\ \alpha(w) = u.$$

Then α is a self-morphism of $(\mathfrak{sl}(2,\mathbb{C}), [-,-])$. Next $[\alpha(u), [w,w]] = 0$ while $[[u,w], \alpha(w)] = 2u$ so that $(\mathfrak{sl}(2,\mathbb{C}), [-,-], \alpha)$ is non-Hom-associative.

Example 3.4. There is a five-dimensional nonassociative (flexible) algebra $(\mathcal{A}, \cdot)$ with basis $\{e_1, \dots, e_5\}$ and multiplication:

$$\begin{aligned} e_1 \cdot e_2 &= e_5 + \frac{1}{2}e_4, & e_1 \cdot e_4 &= \frac{1}{2}e_1 = -e_4 \cdot e_1, \\ e_2 \cdot e_1 &= e_5 - \frac{1}{2}e_4, & e_2 \cdot e_4 &= -\frac{1}{2}e_2 = -e_4 \cdot e_2, \\ e_3 \cdot e_4 &= \frac{1}{2}e_3 = -e_4 \cdot e_3, & e_4 \cdot e_4 &= -e_5, \end{aligned}$$

and all other products are 0 (see [17, p. 29, Example 1.5]). Define $\alpha : \mathcal{A} \to \mathcal{A}$ by:

$$\alpha(e_1) = e_2,\ \alpha(e_2) = e_1,\ \alpha(e_3) = 0,\ \alpha(e_4) = -e_4,\ \alpha(e_5) = e_5.$$

Then α is a self-morphism of $(\mathcal{A}, \cdot)$. Moreover $(\mathcal{A}, \cdot, \alpha)$ is non-Hom-associative since $\alpha(e_3) \cdot (e_4 \cdot e_4) = 0$ while $(e_3 \cdot e_4) \cdot \alpha(e_4) = -\frac{1}{4}e_3$. However, by Theorem 4.4 in [22], $(\mathcal{A}, \cdot, \alpha)$ is Hom-flexible.

Example 3.5. The commutator algebra $\mathcal{A}^-$ of the algebra $\mathcal{A}$ of Example 3.4 is defined by:

$$\begin{aligned} e_1 \star e_2 &= e_4 = -e_2 \star e_1, & e_1 \star e_4 &= e_1 = -e_4 \star e_1, \\ e_2 \star e_4 &= -e_2 = -e_4 \star e_2, & e_3 \star e_4 &= e_3 = -e_4 \star e_3, \end{aligned}$$

and all other products are 0. Then $(\mathcal{A}^-, \star)$ is a Malcev algebra ([17, p. 29, Example 1.5]) and, as observed in [17], $(\mathcal{A}^-, \star)$ is isomorphic to the unique five-dimensional non-Lie solvable Malcev algebra found by E. N. Kuzmin [11]. Next, define the map $\alpha : \mathcal{A}^- \to \mathcal{A}^-$ as in Example 3.4 above. Then α is a self-morphism of $(\mathcal{A}^-, \star)$ and $(\mathcal{A}^-, \star, \alpha)$ is non-Hom-associative since $\alpha(e_3) \star (e_4 \star e_4) = 0$ and $(e_3 \star e_4) \star \alpha(e_4) = -e_3$.

One observes that the nonassociativity of $(\mathcal{A}, \mu)$ may not be enough for the failure of Hom-associativity in $(\mathcal{A}, \mu, \alpha)$ and thus the choice of the twisting map is of prime significance in the definition of Hom-type algebras. In other words, there is a freedom on how to twist. This is observed and investigated in [6]. In the present paper, we consider only twisting maps which ensures the failure of the Hom-associativity as defined by (2.4).

A construction of Hom-associative algebras from associative algebras is given by D. Yau in [20]. This result could be reported to the case of nonassociative algebras, but here an additional condition on the twisting map is needed (in fact, the twisting map must be an automorphism).

Example 3.6. Let R be a unital nonassociative algebra over $\mathbb{K}$ and let R_n denote the algebra of $n \times n$ matrices with entries in R and matrix multiplication denoted by $\mu(x, y) = xy$. Then $\mathcal{A} := (R_n, \mu)$ is also a unital nonassociative algebra. Denote by $N(\mathcal{A})$ the nucleus of $\mathcal{A}$ and suppose that there is an invertible element $u \in N(\mathcal{A})$ and $u^{-1} \in N(\mathcal{A})$. Then the map $\alpha(u) : \mathcal{A} \to \mathcal{A}$ defined by $\alpha(u)(x) = uxu^{-1}$ for $x \in R_n$, is an automorphism of $\mathcal{A}$.

Define $\mu_{\alpha(u)}(x, y) = u(xy)u^{-1}$, for all $x, y \in R_n$. Then one checks that $\mathcal{A}_u = (R_n, \mu_{\alpha(u)}, \alpha(u))$ is a multiplicative non-Hom-associative algebra and $\alpha(u)$ is an automorphism of $(R_n, \mu_{\alpha(u)})$. In this way we get a family $\{\mathcal{A}_u : u \in R_n$ invertible and $u, u^{-1} \in N(\mathcal{A})\}$ of multiplicative non-Hom-associative algebras.

An example, similar to Example 3.7 above, is given in [20] describing Hom-associative deformations by inner automorphisms.

4. Hom-Akivis algebras. Construction

In this section we give the notion of a (multiplicative) Hom-Akivis algebra that could be seen as a generalization of an Akivis algebra and we point out that such a notion does fit with the one of a non-Hom-associative algebra given in Section 3. In fact we prove the analogue of the Akivis construction (see Theorem 2.1) that the commutator-Hom-associator algebra of a given non-Hom-associative algebra is a Hom-Akivis algebra (Theorem 4.2). Theorem 4.4 shows that the class of Hom-Akivis algebras is closed under self-morphisms of such algebras. Moreover, following [20] for Hom-associative algebras and Hom-Lie algebras, we give a procedure for the construction of Hom-Akivis algebras from Akivis algebras and their algebra endomorphisms (Corollary 4.5, with a generalization by Theorem 4.8).

Definition 4.1. A *Hom-Akivis algebra* is a quadruple $(\mathcal{A}, [-,-], [-,-,-], \alpha)$, where $\mathcal{A}$ is a vector space, $[-,-] : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ a skew-symmetric bilinear map,

$[-,-,-] : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ a trilinear map and $\alpha : \mathcal{A} \to \mathcal{A}$ a linear map such that

$$\circlearrowleft_{(x,y,z)}[[x,y],\alpha(z)] = \circlearrowleft_{(x,y,z)}[x,y,z] - \circlearrowleft_{(x,y,z)}[y,x,z], \tag{4.1}$$

for all x, y, z in $\mathcal{A}$.

A Hom-Akivis algebra $(\mathcal{A}, [-,-], [-,-,-], \alpha)$ is said to be *multiplicative* if α is an endomorphism with respect to $[-,-]$ and $[-,-,-]$.

In analogy with Lie and Akivis cases, let us call (4.1) the *Hom-Akivis identity*.

Remark. (1) If $\alpha = \mathrm{id}_{\mathcal{A}}$, the Hom-Akivis identity (4.1) is the usual Akivis identity (2.1).
(2) The Hom-Akivis identity (4.1) reduces to the Hom-Jacobi identity (2.5), when $[x,y,z] = 0$, for all x, y, z in $\mathcal{A}$.

The following result shows how one can get Hom-Akivis algebras from non-Hom-associative algebras.

Theorem 4.2. *The commutator-Hom-associator algebra of a multiplicative non-Hom-associative algebra is a multiplicative Hom-Akivis algebra.*

Proof: Let $(\mathcal{A}, \mu, \alpha)$ be a multiplicative non-Hom-associative algebra. For any x, y, z in $\mathcal{A}$, define the operations

$[x,y] := \mu(x,y) - \mu(y,x)$ (commutator)

$[x,y,z]_\alpha := \mathfrak{as}(x,y,z)$ (Hom-associator; see (3.1)).

For simplicity, set xy for $\mu(x,y)$. Then

$$[[x,y],\alpha(z)] = (xy)\alpha(z) - (yx)\alpha(z) - \alpha(z)(xy) + \alpha(z)(yx)$$

and

$$[x,y,z]_\alpha - [y,x,z]_\alpha = (xy)\alpha(z) - \alpha(x)(yz) - (yx)\alpha(z) + \alpha(y)(xz).$$

Expanding $\sigma[[x,y],\alpha(z)]$ and $\sigma([x,y,z]_\alpha - [y,x,z]_\alpha)$ respectively, one gets (4.1) and so $(\mathcal{A}, [-,-], [-,-,-]_\alpha, \alpha)$ is a Hom-Akivis algebra. The multiplicativity of $(\mathcal{A}, [-,-], [-,-,-]_\alpha, \alpha)$ follows from the one of $(\mathcal{A}, \mu, \alpha)$. □

The remarks above and Theorem 4.2 show that Definition 4.1 fits with the non-Hom-associativity. The Hom-Akivis algebra constructed by Theorem 4.2 is said to be *associated* (to a given non-Hom-associative algebra). Starting from other considerations and using other notations, D. Yau has come to Theorem 4.2 above (see [22, Lemma 3.16]).

For the next results we need the following

Definition 4.3. Let $(\mathcal{A}, [-,-], [-,-,-], \alpha)$ and $(\tilde{\mathcal{A}}, \{-,-\}, \{-,-,-\}, \tilde{\alpha})$ be Hom-Akivis algebras. A morphism $\phi : \mathcal{A} \to \tilde{\mathcal{A}}$ of Hom-Akivis algebras is a linear map of $\mathbb{K}$-vector spaces $\mathcal{A}$ and $\tilde{\mathcal{A}}$ such that

$$\phi([x,y]) = \{\phi(x), \phi(y)\},$$

$$\phi([x,y,z]) = \{\phi(x),\phi(y),\phi(z)\}.$$

For example, if we take $(\mathcal{A},[-,-],[-,-,-],\alpha)$ as a multiplicative Hom-Akivis algebra, then the twisting self-map α is itself an endomorphism of $(\mathcal{A},[-,-],[-,-,-])$.

The following result holds.

Theorem 4.4. *Let $\mathcal{A}_\alpha := (\mathcal{A},[-,-],[-,-,-],\alpha)$ be a Hom-Akivis algebra and $\beta : \mathcal{A} \to \mathcal{A}$ a self-morphism of $\mathcal{A}_\alpha$. Let $\beta^0 = \mathrm{id}_{\mathcal{A}}$, $\beta^n = \beta \circ \beta^{n-1}$ for any integer $n \geq 1$ and define on $\mathcal{A}$ a bilinear operation $[-,-]_{\beta^n}$ and a trilinear operation $[-,-,-]_{\beta^n}$ by*

$$[x,y]_{\beta^n} := \beta^n([x,y]),$$
$$[x,y,z]_{\beta^n} := \beta^{2n}([x,y,z]),$$

for all $x,y,z \in \mathcal{A}$. Then $\mathcal{A}_{\beta^n} := (\mathcal{A},[-,-]_{\beta^n},[-,-,-]_{\beta^n},\beta^n \circ \alpha)$ is a Hom-Akivis algebra.

Moreover, if $\mathcal{A}_\alpha$ is multiplicative and β commutes with α, then $\mathcal{A}_{\beta^n}$ is multiplicative.

Proof: Clearly $[-,-]_{\beta^n}$ (resp. $[-,-,-]_{\beta^n}$) is a bilinear (resp. trilinear) map and the skew-symmetry of $[-,-]$ in $\mathcal{A}_\alpha$ implies the skew-symmetry of $[-,-]_{\beta^n}$ in $\mathcal{A}_{\beta^n}$.

Next, we have (by the Hom-Akivis identity (4.1)),

$$\begin{aligned}
&\circlearrowleft_{(x,y,z)}[[x,y]_{\beta^n},(\beta^n\circ\alpha)(z)]_{\beta^n} = \beta^{2n}(\circlearrowleft_{(x,y,z)}[[x,y],\alpha(z)])\\
&= \beta^{2n}(\circlearrowleft_{(x,y,z)}[x,y,z] - \circlearrowleft_{(x,y,z)}[y,x,z])\\
&= \circlearrowleft_{(x,y,z)}(\beta^{2n}([x,y,z]) - \beta^{2n}([y,x,z]))\\
&= \circlearrowleft_{(x,y,z)}([x,y,z]_{\beta^n} - [y,x,z]_{\beta^n}),
\end{aligned}$$

which means that $\mathcal{A}_{\beta^n}$ is a Hom-Akivis algebra.

The second assertion is proved as follows:

$$\begin{aligned}
&[(\beta^n\circ\alpha)(x),(\beta^n\circ\alpha)(y)]_{\beta^n} = \beta^n([(\beta^n\circ\alpha)(x),(\beta^n\circ\alpha)(y)])\\
&= \beta^n([(\alpha\circ\beta^n)(x),(\alpha\circ\beta^n)(y)])\\
&= (\beta^n\circ\alpha)([\beta^n(x),\beta^n(y)])\\
&= (\beta^n\circ\alpha)(\beta^n([x,y])) = (\beta^n\circ\alpha)([x,y]_{\beta^n})
\end{aligned}$$

and

$$\begin{aligned}
&[(\beta^n\circ\alpha)(x),(\beta^n\circ\alpha)(y),(\beta^n\circ\alpha)(z)]_{\beta^n}\\
&= \beta^{2n}([(\beta^n\circ\alpha)(x),(\beta^n\circ\alpha)(y),(\beta^n\circ\alpha)(z)])\\
&= \beta^{2n}([(\alpha\circ\beta^n)(x),(\alpha\circ\beta^n)(y),(\alpha\circ\beta^n)(z)]) = (\beta^{2n}\circ\alpha)([\beta^n(x),\beta^n(y),\beta^n(z)])\\
&= ((\beta^{2n}\circ\alpha)\circ\beta^n)([x,y,z]) = ((\alpha\circ\beta^{2n})\circ\beta^n)([x,y,z])
\end{aligned}$$

$$= ((\alpha \circ \beta^n) \circ \beta^{2n})([x,y,z]) = (\alpha \circ \beta^n)(\beta^{2n}([x,y,z]))$$
$$= (\beta^n \circ \alpha)([x,y,z]_{\beta^n}).$$

This completes the proof. □

Corollary 4.5. *Let $(\mathcal{A}, [-,-], [-,-,-])$ be an Akivis algebra and β an endomorphism of $(\mathcal{A}, [-,-], [-,-,-])$. Define on $\mathcal{A}$ a bilinear operation $[-,-]_\beta$ and a trilinear operation $[-,-,-]_\beta$ by*

$$[x,y]_\beta := [\beta(x), \beta(y)] \ \ (= \beta([x,y])),$$
$$[x,y,z]_\beta := [\beta^2(x), \beta^2(y), \beta^2(z)] \ \ (= \beta^2([x,y,z])),$$

for all $x, y, z \in \mathcal{A}$, where $\beta^2 = \beta \circ \beta$. Then $(\mathcal{A}, [-,-]_\beta, [-,-,-]_\beta, \beta)$ is a multiplicative Hom-Akivis algebra.

Moreover, suppose that $\tilde{A}$ is another Akivis algebra and that $\tilde{\beta}$ is an endomorphism of $\tilde{A}$. If $f : \mathcal{A} \to \tilde{A}$ is an Akivis algebra morphism satisfying $f \circ \beta = \tilde{\beta} \circ f$, then $f : (\mathcal{A}, [-,-]_\beta, [-,-,-]_\beta, \beta) \to (\tilde{A}, [-,-]_{\tilde{\beta}}, [-,-,-]_{\tilde{\beta}}, \tilde{\beta})$ is a morphism of multiplicative Hom-Akivis algebras.

Proof: The first part of this theorem is a special case of Theorem 4.4 above when $\alpha = \mathrm{id}$ and $n = 1$. The second part is proved in the same way as in Theorem 4.4 when $n = 1$. For completeness, we repeat it as follows.

$$[f(x), f(y)]_{\tilde{\beta}} = \tilde{\beta}([f(x), f(y)]) = (\tilde{\beta} \circ f)([x,y])$$
$$= (f \circ \beta)([x,y]) = f([\beta(x), \beta(y)]) = f([x,y]_\beta)$$

and

$$[f(x), f(y), f(z)]_{\tilde{\beta}} = \tilde{\beta}^2([f(x), f(y), f(z)]) = (\tilde{\beta}^2 \circ f)([x,y,z])$$
$$= (\tilde{\beta} \circ (\tilde{\beta} \circ f))([x,y,z]) = (\tilde{\beta} \circ (f \circ \beta))([x,y,z]) = ((\tilde{\beta} \circ f) \circ \beta)([x,y,z])$$
$$= ((f \circ \beta) \circ \beta)([x,y,z]) = (f \circ \beta^2)([x,y,z]) = f(\beta^2([x,y,z])) = f([x,y,z]_\beta).$$

This completes the proof. □

Remark. (1) The gist of Theorem 4.4 is that the category of Hom-Akivis algebras is closed under twisting by self-morphisms.

(2) Corollary 4.5 is the Akivis algebra analogue of a result in the Hom-Lie setting [20]. It shows how Hom-Akivis algebras can be constructed from Akivis algebras. This procedure was first given by Yau [20] in the construction of Hom-associative (resp. Hom-Lie) algebras starting from associative (resp. Lie) algebras. Such a procedure has been further extended to coalgebras [15] and to other systems (see, e.g., [5], [12]).

As an illustration, we use Corollary 4.5 to construct examples of Hom-Akivis algebras starting from given Akivis algebras.

Example 4.6. Let $(\mathcal{A}, \cdot)$ be the four-dimensional anticommutative algebra with basis $\{e_1, e_2, e_3, e_4\}$ and multiplication table

$$\begin{aligned}
&e_1 \cdot e_2 = e_2 = -e_2 \cdot e_1,\ e_1 \cdot e_3 = e_3 = -e_3 \cdot e_1,\ e_1 \cdot e_4 = -e_4 = -e_4 \cdot e_1,\\
&e_2 \cdot e_3 = -2e_4 = -e_3 \cdot e_2
\end{aligned}$$

and all other products are 0. Then $(\mathcal{A}, \cdot)$ is a non-Lie Malcev algebra ([18, Example 3.1]). Then, by Theorem 2.1, the Akivis algebra $(\mathcal{A}, [-,-], [-,-,-])$ associated to $(\mathcal{A}, \cdot)$ is given by:

$$\begin{aligned}
&[e_1, e_2] = 2e_2,\ [e_1, e_3] = 2e_3,\ [e_1, e_4] = -2e_4,\\
&[e_2, e_3] = -4e_4,\\
&[e_1, e_1, e_2] = -e_2,\ [e_1, e_1, e_3] = -e_3,\ [e_1, e_1, e_2] = e_4,\\
&[e_1, e_2, e_3] = -2e_4,\\
&[e_1, e_3, e_2] = 4e_4,\\
&[e_1, e_4, e_4] = -2e_4,\\
&[e_2, e_1, e_1] = e_2,\ [e_2, e_1, e_3] = 4e_4,\\
&[e_2, e_3, e_1] = -4e_4,\\
&[e_3, e_1, e_1] = e_3,\ [e_3, e_1, e_2] = -4e_4,\\
&[e_3, e_2, e_1] = 4e_4,\\
&[e_4, e_1, e_1] = e_4.
\end{aligned}$$

Now define a linear map $\beta : \mathcal{A} \mapsto \mathcal{A}$ by

$$\beta(e_1) = e_1 + e_4,\ \beta(e_2) = e_3,\ \beta(e_3) = e_3,\ \beta(e_4) = 0.$$

Then β is an endomorphism of $(\mathcal{A}, \cdot)$ and, subsequently, by linearity, also an endomorphism of the associated Akivis algebra $(\mathcal{A}, [-,-], [-,-,-])$. Next, define on $\mathcal{A}$ the operations $[-,-]_\beta$ and $[-,-,-]_\beta$ by

$$\begin{aligned}
&[e_1, e_2]_\beta = 2e_3,\ [e_1, e_3]_\beta = 2e_3,\\
&[e_1, e_1, e_2]_\beta = -e_3,\ [e_1, e_1, e_3]_\beta = -e_3,\\
&[e_2, e_1, e_1]_\beta = e_3,\\
&[e_3, e_1, e_1]_\beta = e_3
\end{aligned}$$

and all missing products are 0. Then Corollary 4.5 implies that $(\mathcal{A}, [-,-]_\beta, [-,-,-]_\beta, \beta)$ is a Hom-Akivis algebra.

Example 4.7. Let $(\mathcal{A}, \cdot)$ be the two-dimensional algebra with basis $\{e_1, e_2\}$ and multiplication given by

$$e_1 \cdot e_2 = e_1,\ e_2 \cdot e_2 = e_1$$

and all missing products are 0. Then $(\mathcal{A},\cdot)$ is nonassociative since, e.g., $(e_1 \cdot e_2) \cdot e_2 = e_1 \neq 0 = e_1 \cdot (e_2 \cdot e_2)$. Theorem 2.1 implies that the Akivis algebra $(\mathcal{A},[-,-],[-,-,-])$ associated to $(\mathcal{A},\cdot)$ has the following multiplication table:

$$\begin{aligned}
[e_1,e_2] &= e_1,\\
[e_1,e_2,e_2] &= e_1,\\
[e_2,e_2,e_2] &= e_1.
\end{aligned}$$

Next, if we define a linear map $\beta : \mathcal{A} \mapsto \mathcal{A}$ by

$$\beta(e_1) = 2e_1,\ \beta(e_2) = e_1 + e_2,$$

then β is an endomorphism of $(\mathcal{A},[-,-],[-,-,-])$ and, defining on $\mathcal{A}$ the operations $[-,-]_\beta$ and $[-,-,-]_\beta$ by

$$\begin{aligned}
[e_1,e_2]_\beta &= 2e_1,\\
[e_1,e_2,e_2]_\beta &= 4e_1,\\
[e_2,e_2,e_2]_\beta &= 4e_1,
\end{aligned}$$

we get, by Corollary 4.5, that $(\mathcal{A},[-,-]_\beta,[-,-,-]_\beta,\beta)$ is a Hom-Akivis algebra.

Exploiting the idea behind Theorem 4.4, we conclude this section by constructing recursively in the following theorem a sequence of Hom-Akivis algebras starting from a given Akivis algebra (which is seen as a Hom-Akivis algebra with the identity map as the twisting map).

Theorem 4.8. *Let $(\mathcal{A},[-,-],[-,-,-])$ be an Akivis algebra and β an endomorphism of $(\mathcal{A},[-,-],[-,-,-])$. Let $\beta^0 = \mathrm{id}_{\mathcal{A}}$ and, for any integer $n \geq 1$, $\beta^n = \beta \circ \beta^{n-1}$. Define on $\mathcal{A}$ a bilinear operation $[-,-]_{\beta^n}$ and a trilinear operation $[-,-,-]_{\beta^n}$ by*

$$\begin{aligned}
[x,y]_{\beta^n} &:= \beta([x,y]_{\beta^{n-1}}),\\
[x,y,z]_{\beta^n} &:= \beta^2([x,y,z]_{\beta^{n-1}}),
\end{aligned}$$

for all $x,y,z \in \mathcal{A}$. Then $\mathcal{A}_{\beta^n} := (\mathcal{A},[-,-]_{\beta^n},[-,-,-]_{\beta^n},\beta^n)$ is a multiplicative Hom-Akivis algebra.

Proof: If $n = 1$ then $\mathcal{A}_\beta := (\mathcal{A},[-,-]_\beta,[-,-,-]_\beta,\beta)$ is a multiplicative Hom-Akivis algebra by Corollary 4.5.

Suppose now that, up to $n-1$, $\mathcal{A}_{\beta^{n-1}}$ are multiplicative Hom-Akivis algebras. Then

$$\begin{aligned}
&\circlearrowleft_{(x,y,z)}[[x,y]_{\beta^n},\beta^n(z)]_{\beta^n} = \circlearrowleft_{(x,y,z)}\beta^2([[x,y]_{\beta^{n-1}},\beta^{n-1}(z)]_{\beta^{n-1}})\\
&= \beta^2(\circlearrowleft_{(x,y,z)}[[x,y]_{\beta^{n-1}},\beta^{n-1}(z)]_{\beta^{n-1}})\\
&= \beta^2(\circlearrowleft_{(x,y,z)}[x,y,z]_{\beta^{n-1}} - \circlearrowleft_{(x,y,z)}[y,x,z]_{\beta^{n-1}})
\end{aligned}$$

$$= \circlearrowleft_{(x,y,z)}\beta^2([x,y,z]_{\beta^{n-1}}) - \circlearrowleft_{(x,y,z)}\beta^2([y,x,z]_{\beta^{n-1}})$$
$$= \circlearrowleft_{(x,y,z)}[x,y,z]_{\beta^n} - \circlearrowleft_{(x,y,z)}[y,x,z]_{\beta^n}$$

(observe that we used the Hom-Akivis identity in $\mathcal{A}_{\beta^{n-1}}$) so that $\mathcal{A}_{\beta^n}$ is a Hom-Akivis algebra. The multiplicativity of $\mathcal{A}_{\beta^n}$ follows from the fact that β is an endomorphism of $(\mathcal{A}, [-,-], [-,-,-])$. □

5. Hom-Malcev algebras from Hom-Akivis algebras

As for Akivis algebras, the notion of a Hom-Akivis algebra seems to be too wide in order to develop interesting specific results. For this purpose, it would be natural to consider some additional conditions and properties on Hom-Akivis algebras.

In this section, we consider Hom-alternativity and Hom-flexibility in Hom-Akivis algebras. The main result here (see Theorem 5.5) is that the Hom-Akivis algebra associated with a Hom-alternative algebra has a Hom-Malcev structure (this could be seen as another version of Theorem 3.8 in [22]).

Since only the ternary operation of an Akivis algebra is involved in its alternativity or flexibility [3], we report these notions to Hom-Akivis algebras in the following

Definition 5.1. A Hom-Akivis algebra $\mathcal{A}_\alpha := (\mathcal{A}, [-,-], [-,-,-], \alpha)$ is said to be:

(i) *Hom-flexible*, if $[x,y,x] = 0$, for all $x, y \in \mathcal{A}$;
(ii) *Hom-alternative*, if $[-,-,-]$ is alternating (i.e. $[-,-,-]$ vanishes whenever any pair of variables are equal).

Remark. By linearization, as for associators in nonassociative algebras, one checks that

(1) the Hom-flexible law $[x,y,x] = 0$ in $\mathcal{A}_\alpha$ is equivalent to $[x,y,z] = -[z,y,x]$;
(2) the Hom-alternativity of $\mathcal{A}_\alpha$ is equivalent to its *left Hom-alternativity* ($[x,x,y] = 0$) and *right Hom-alternativity* ($[y,x,x] = 0$) for all $x, y \in \mathcal{A}$.

The following result is an immediate consequence of Theorem 4.2 and Definition 5.1.

Theorem 5.2. *Let* $(\mathcal{A}, \mu, \alpha)$ *be a non-Hom-associative algebra and* $(\mathcal{A}, [-,-], \mathfrak{as}(-,-,-), \alpha)$ *its associate Hom-Akivis algebra.*

(i) *If* $(\mathcal{A}, \mu, \alpha)$ *is Hom-flexible, then* $(\mathcal{A}, [-,-], \mathfrak{as}(-,-,-), \alpha)$ *is Hom-flexible.*
(ii) *If* $(\mathcal{A}, \mu, \alpha)$ *is Hom-alternative, then so is* $(\mathcal{A}, [-,-], \mathfrak{as}(-,-,-), \alpha)$.

□

We have the following characterization of Hom-Lie algebras in terms of Hom-Akivis algebras.

Proposition 5.3. *Let $\mathcal{A}_\alpha := (\mathcal{A}, [-,-], [-,-,-], \alpha)$ be a Hom-flexible Hom-Akivis algebra. Then $\mathcal{A}_\alpha$ is a Hom-Lie algebra if and only if $\circlearrowleft_{(x,y,z)}[x,y,z] = 0$, for all $x, y \in \mathcal{A}$.*

Proof: The Hom-Akivis identity (4.1) and the Hom-flexibility in $\mathcal{A}_\alpha$ imply

$$\circlearrowleft_{(x,y,z)}[[x,y],\alpha(z)] = 2 \circlearrowleft_{(x,y,z)}[x,y,z]$$

so that $\circlearrowleft_{(x,y,z)}[[x,y],\alpha(z)] = 0$ if and only if $\circlearrowleft_{(x,y,z)}[x,y,z] = 0$ (recall that the ground field $\mathbb{K}$ is of characteristic 0). □

The following result is a slight generalization of Proposition 3.17 in [22], which in turn generalizes a similar well-known result in alternative rings:

Proposition 5.4. *Let $\mathcal{A}_\alpha := (\mathcal{A}, [-,-], [-,-,-], \alpha)$ be a Hom-alternative Hom-Akivis algebra. Then*

$$\circlearrowleft_{(x,y,z)}[[x,y],\alpha(z)] = 6 \circlearrowleft_{(x,y,z)}[x,y,z] \tag{5.1}$$

for all $x, y, z \in \mathcal{A}$.

Proof: The application to (4.1) of the Hom-alternativity in $\mathcal{A}_\alpha$ gives the proof. □

We now come to the main result of this section, which is Theorem 3.8 in [22] but from a point of view of Hom-Akivis algebras.

In [22] D. Yau introduced the notion of a Hom-Malcev algebra: a *Hom-Malcev algebra* is a Hom-algebra $(\mathcal{A}, [-,-], \alpha)$ such that the binary operation $[-,-]$ is skew-symmetric and that the identity

$$\circlearrowleft_{(\alpha(x),\alpha(y),z_x)}[[\alpha(x),\alpha(y)],\alpha(z_x)] = [\circlearrowleft_{(x,y,z)}[[x,y],\alpha(z)],\alpha^2(x)] \tag{5.2}$$

holds for all $x, y, z \in \mathcal{A}$, where $z_x := [x,z]$ and $\circlearrowleft_{(\alpha(x),\alpha(y),z_x)}$ denotes the sum over cyclic permutation of $\alpha(x), \alpha(y)$, and z_x. The identity (5.2) is called the *Hom-Malcev identity*.

Observe that when $\alpha = \mathrm{id}$ then, by the skew-symmetry of $[-,-]$, the Hom-Malcev identity reduces to the Malcev identity ([16], [18]).

The alternativity in Akivis algebras leads to Malcev algebras [3]. The Hom-version of this result is the following

Theorem 5.5. *Let $(\mathcal{A}, \cdot, \alpha)$ be a Hom-alternative Hom-algebra and $(\mathcal{A}, [-,-], \mathfrak{as}(-,-,-), \alpha)$ its associate Hom-Akivis algebra, where $[x,y] = x \cdot y - y \cdot x$ for all $x, y \in \mathcal{A}$. Then $(\mathcal{A}, [-,-], \mathfrak{as}(-,-,-), \alpha)$ reduces to a Hom-Malcev algebra.*

Proof: From Theorem 4.2 we get that $(\mathcal{A}, [-,-], \mathfrak{as}(-,-,-), \alpha)$ is Hom-alternative so that (5.1) implies

$$\circlearrowleft_{(\alpha(x),\alpha(y),z)}[[\alpha(x),\alpha(y)],\alpha(z)] = 6\ \mathfrak{as}(\alpha(x),\alpha(y),z) \tag{5.3}$$

for all $x, y, z \in \mathcal{A}$. Now, in (5.3) replace z with $z_x := [x, z]$ to get

$$\circlearrowleft_{(\alpha(x),\alpha(y),z_x)}[[\alpha(x),\alpha(y)],\alpha(z_x)] = 6\ \mathfrak{as}(\alpha(x),\alpha(y),z_x). \tag{5.4}$$

But $\mathfrak{as}(\alpha(x),\alpha(y),z_x) = [\mathfrak{as}(x,y,z),\alpha^2(x)]$ in $(\mathcal{A},\cdot,\alpha)$ (see [21, Corollary 3.15]) so that (5.4) reads

$$\circlearrowleft_{(\alpha(x),\alpha(y),z_x)}[[\alpha(x),\alpha(y)],\alpha(z_x)] = 6\ [\mathfrak{as}(x,y,z),\alpha^2(x)]$$

i.e., by (5.1) (viewing $[x,y,z]$ as $\mathfrak{as}(x,y,z)$),

$$\circlearrowleft_{(\alpha(x),\alpha(y),z_x)}[[\alpha(x),\alpha(y)],\alpha(z_x)] = [(\circlearrowleft_{(x,y,z)}[[x,y],\alpha(z)]),\alpha^2(x)]$$

and one recognizes the Hom-Malcev identity (5.2). Therefore, we get that $(\mathcal{A},[-,-],\mathfrak{as}(-,-,-),\alpha)$ has a Hom-Malcev algebra structure. □

Remark. The procedure described in the proof of Theorem 5.5 somewhat repeats the one given by A.I. Maltsev in [16] when constructing Moufang-Lie algebras (now called Malcev algebras) from alternative algebras.

Example 5.6. By Theorem 4.2, the associated Hom-Akivis algebra of the Hom-flexible Hom-algebra of Example 3.4 is Hom-flexible (see Theorem 5.2). Moreover, Example 3.5 and Theorem 5.2 (see also Theorem 3.8 in [22]) imply that such a Hom-Akivis algebra is also a Hom-Malcev algebra.

Acknowledgment. I thank A. Makhlouf and D. Yau for their relevant comments and suggestions after reading the first draft of this paper, and for communicating with me about its specific topics. I also thank the referee for valuable suggestions which helped to improve some parts of this paper.

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(*Received* June 9, 2010, *revised* September 2, 2011)