Commutative zeropotent semigroups with few prime ideals

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Abstract. We construct an infinite commutative zeropotent semigroup with only two prime ideals.

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The following remarkable problem has been standing open for some time: Does there exist an infinite commutative semigroup with only finitely many endomorphisms? We conjecture that if there is an example, then it can be found among commutative zeropotent semigroups. In this paper we construct a commutative zeropotent semigroup with only two prime ideals. Although this does not solve the problem, we hope that an example of an infinite commutative semigroup with only two endomorphisms could be possibly obtained by means of a similar, more complicated construction.

We adopt the additive notation for commutative semigroups. By a commutative zeropotent semigroup, shortly *czp-semigroup*, we mean a commutative semigroup A satisfying $x+x=y+y+y$ for all $x,y \in A$. Then $x+x=y+y$ for all $x,y \in A$, the element $x+x$ (for any $x \in A$) is denoted by o_A (or just by o) and $x+x=o$, $x+o=o$ for all $x \in A$.

Natural examples of czp-semigroups can be obtained in the following way: Take an arbitrary set X and let A be the set of all subsets of X; for $a,b \in A$ put $a+b=a\cup b$ if a,b are nonempty and disjoint; in all other cases put $a+b=\emptyset$. Subsemigroups embeddable into such semigroups A are called *representable*.

By an *ideal* of a czp-semigroup A we mean a subset I of A such that $o \in I$ and $x+y \in I$ whenever $x \in I$ and $y \in A$. By a *prime ideal* of A we mean an ideal I of A such that whenever $x+y \in I$ then either $x+y=o$ or $x \in I$ or $y \in I$.

If I is a prime ideal of a czp-semigroup A then the mapping $\phi_I : A \to A$ defined by $\phi_I(x)=o$ for $x \in I$ and $\phi_I(x)=x$ for $x \notin I$, is an endomorphism of A. Thus if A has only finitely many endomorphisms then it has only finitely many prime ideals. It is easy to see that an infinite representable czp-semigroup has always infinitely many endomorphisms.

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The aim of this note is to construct an infinite czp-semigroup with only two prime ideals. The problem whether there is an infinite czp-semigroup with only finitely many endomorphisms, remains open.

Denote by X the absolutely free algebra with four unary operations $\alpha_1, \beta_1, \alpha_2, \beta_2$ and one binary operation γ, over an infinite countable set of variables. Elements of X will be called *terms*. Finite (non necessarily nonempty) sequences of elements of $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ will be called *words*. For a term x, the terms wx (where w is any word) are called *x-based*. Every x-based term other than x can be uniquely expressed as νy for some x-based term y and some $\nu \in \{\alpha_1, \beta_1, \alpha_2, \beta_2\}$; the term y is called the *chief subterm* of νy.

Denote by T the free czp-semigroup over (the underlying set of) X; its elements are all finite subsets of X (we identify elements x of X with $\{x\}$), $o = \emptyset$, and

$$u + v = \begin{cases} u \cup v \text{ if } u, v \text{ are nonempty and disjoint} \\ o \text{ otherwise} \end{cases}$$

Denote by R_1 the set of the pairs $\langle x, \alpha_1 x + \beta_1 x\rangle$, by R_2 the set of the pairs $\langle x, \alpha_2 x + \beta_2 x\rangle$, and by R_3 the set of the pairs $\langle \alpha_1 x + \alpha_2 x, y + \gamma(x, y)\rangle$ for $x, y \in X$, $x \neq y$.

For $u, v \in T \setminus \{o\}$ and $j = 1, 2, 3$ write $u \to_j v$ if there is a pair $\langle p, q\rangle \in R_j$ such that $p \subseteq u$, q is disjoint with u and $v = (u \setminus p) \cup q$. Write $u \equiv_j v$ if either $u \to_j v$ or $v \to_j u$. Thus $\equiv_j$ is a symmetric relation on $T \setminus \{o\}$. Clearly, $u \equiv_j v$ if and only if there is a pair $\langle p, q\rangle \in R_j \cup R_j^{-1}$ such that $p \subseteq u$, q is disjoint with u and $v = (u \setminus p) \cup q$.

By a *derivation* we mean a finite sequence $u_0, \dots, u_n$ ($n \geq 0$) of elements of $T \setminus \{o\}$ such that for any $i = 1, \dots, n$, $u_{i-1} \equiv_j u_i$ for some $j \in \{1, 2, 3\}$. By a derivation from u to v we mean a derivation, the first member of which is u and the last member of which is v. Clearly, if $u_0, \dots, u_n$ is a derivation then $u_n, u_{n-1}, \dots, u_0$ is also a derivation.

Denote by U_0 the set of the elements u of $T \setminus \{o\}$ for which there are $j \in \{1, 2, 3\}$ and a pair $\langle p, q\rangle \in R_j \cup R_j^{-1}$ such that $p \subseteq u$ and q is not disjoint with u. Denote by U the set of the elements $u \in T \setminus \{o\}$ such that there exists a derivation from u to an element of U_0. Thus if one member of a derivation belongs to U then all members belong to U.

Define a binary relation $\sim$ on T as follows: $u \sim v$ if and only if either $u, v \in U \cup \{o\}$ or there is a derivation from u to v. It is easy to check that $\sim$ is an equivalence on T.

Lemma 1. *Let $u, v \in T \setminus \{o\}$, $x \in X$ and $j \in \{1, 2, 3\}$. If $u \equiv_j v$ then either $u + x \equiv_j v + x$ or both $u + x$ and $v + x$ belong to $U \cup \{o\}$. If $u \in U$ then $u + x \in U \cup \{o\}$.*

Proof: Let $u \equiv_j v$. We have $v = (u \setminus p) \cup q$ for some $\langle p, q\rangle \in R_j \cup R_j^{-1}$ with $p \subseteq u$ and $q \cap u = \emptyset$. If $x \notin u \cup q$ then evidently $p \subseteq u \cup \{x\}$, $q \cap (v \cup \{x\}) = \emptyset$ and $v \cup \{x\} = ((u \cup \{x\}) \setminus p) \cup q$, so that $u + x = u \cup \{x\} \equiv_j v \cup \{x\} = v + x$. If

$x \in u \setminus p$ then $u+x=v+x=o$. If $x \in p$ then $u+x=o$ and $v+x=v \cup \{x\} \in U$. If $x \in q$ then $u+x=u \cup \{x\} \in U$ and $v+x=o$. The second statement is also easy to see. $\square$

Lemma 2. *$\sim$ is the congruence of T generated by $R_1 \cup R_2 \cup R_3$.*

PROOF: Using Lemma 1 one can easily check that $\sim$ is a congruence. Clearly, $R_1 \cup R_2 \cup R_3$ is contained in $\sim$ and if a congruence contains $R_1 \cup R_2 \cup R_3$ then it contains $\sim$. $\square$

By a simple derivation we mean a derivation $u_0, \dots, u_n$ such that $u_0 \in X$ and for all $i \in \{1, \dots, n\}$ either $u_{i-1} \rightarrow_1 u_i$ or $u_{i-1} \rightarrow_2 u_i$. Clearly, $u_1, \dots, u_n$ are then sets of at least two u_0-based terms different from u_0.

Lemma 3. *Let $u_0, \dots, u_n$ be a simple derivation; let $\{x, wx\} \subseteq u_i$ for some $i \in \{0, \dots, n\}$, some term x and some word w. Then w is empty.*

PROOF: Suppose that some u_i contains both x and $w\nu x$ where w is a word and $\nu \in \{\alpha_1, \beta_1, \alpha_2, \beta_2\}$, and take the least index i with this property. We have $i > 0$, since u_0 contains only one term. By the minimality of i, either x or $w\nu x$ does not belong to u_{i-1}. Since u_i results from u_{i-1} by removing one term and adding two other terms of the same length, precisely one of the terms x and $w\nu x$ does not belong to u_{i-1}.

Case 1: $x \notin u_{i-1}$ and $w\nu x \in u_{i-1}$. Since x belongs to u_i but not to u_{i-1}, the chief subterm of x belongs to u_{i-1}. But then u_{i-1} contains both this chief subterm and its proper extension $w\nu x$, a contradiction with the minimality of i.

Case 2: $x \in u_{i-1}$ and $w\nu x \notin u_{i-1}$. Since $w\nu x$ belongs to u_i but not to u_{i-1}, the chief subterm of $w\nu x$ belongs to u_{i-1}. By the minimality of i, w is empty and the chief subterm is x. Thus $x \in u_{i-1}$, a contradiction. $\square$

Lemma 4. *Let $u_0, \dots, u_n$ be a simple derivation and $i \in \{0, \dots, n\}$. Then neither $\{w\alpha_1 x, w'\alpha_2 x\} \subseteq u_i$ nor $\{w\beta_1 x, w'\beta_2 x\} \subseteq u_i$ for any $x \in X$ and any words w, w'.*

PROOF: Suppose that i is the least index for which this is not true. It is sufficient to consider the case when $\{w\alpha_1 x, w'\alpha_2 x\} \subseteq u_i$. At least one of these two elements does not belong to u_{i-1}. Without loss of generality, $w'\alpha_2 x \notin u_{i-1}$. Since u_i results from u_{i-1} by removing one term and adding two other ones, the removed term is the chief subterm of $w'\alpha_2 x$. The other added element cannot be $w\alpha_1 x$, so $w\alpha_1 x \in u_{i-1}$. Thus if w' is nonempty then u_{i-1} contains two terms contradicting the minimality of i. We get that w' is empty. Thus u_{i-1} contains the terms x and $w\alpha_1 x$, a contradiction with Lemma 3. $\square$

Lemma 5. *No member of a simple derivation belongs to U_0.*

PROOF: Let $u_0, \dots, u_n$ be a simple derivation and suppose that $u_n \in U_0$. There are $j \in \{1, 2, 3\}$ and $\langle p, q \rangle \in R_j \cup R_j^{-1}$ such that $p \subseteq u_n$ and q has a common element with u_n. If $p = \{\alpha_1 x, \alpha_2 x\}$, we get a contradiction by Lemma 4. We cannot have $p = \{y, \gamma(x, y)\}$, since u_n does not contain a term starting with γ

(unless $n = 0$, but this is not the case since u_0 contains only one term). Thus $j \neq 3$. If $p = \{x\}$ and $q = \{\alpha_j x, \beta_j x\}$ then u_n contains x and one of the terms $\alpha_j x, \beta_j x$, a contradiction by Lemma 3. Finally, if $p = \{\alpha_j x, \beta_j x\}$ and $q = \{x\}$ then u_n contains all these three terms, again a contradiction by Lemma 3. $\square$

Lemma 6. *Let $u_0, \dots, u_n$ be a derivation such that $u_0 \in X$. Then $u_n \notin U_0$ and if u_n is a singleton then $u_n = u_0$.*

Proof: Suppose that there is a derivation contradicting this assertion, and let $u_0, \dots, u_n$ be one with the least possible n. Clearly, $n > 0$. Let i be the largest index such that $u_0, \dots, u_i$ is a simple derivation.

Suppose that $i = n$, so that $u_0, \dots, u_n$ is a simple derivation. If $u_n \in U_0$, we get a contradiction by Lemma 5. Clearly, u_n cannot be a singleton if $n > 0$, and for $n = 0$ we have $u_n = u_0$. Thus $i < n$.

If $u_i \to_3 u_{i+1}$ then u_i contains both $\alpha_1 x$ and $\alpha_2 x$ for some $x \in X$, a contradiction with Lemma 4.

If $u_{i+1} \to_3 u_i$ then u_i has more than one element and contains a term starting with γ, which is evidently not possible since $u_0, \dots, u_i$ is simple.

If $u_i \to_j u_{i+1}$ for some $j \in \{1, 2\}$ then $u_0, \dots, u_{i+1}$ is a simple derivation, a contradiction with the maximality of i.

Thus $u_{i+1} \to_j u_i$ for some $j \in \{1, 2\}$. We have $u_i = (u_{i+1} \setminus \{x\}) \cup \{\alpha_j x, \beta_j x\}$ for some $x \in u_{i+1}$ and $\{\alpha_j x, \beta_j x\} \cap u_{i+1} = \emptyset$. Let k be the least index such that either $\alpha_j x$ or $\beta_j x$ belongs to u_k; thus $k \leq i$. Clearly, $k > 0$. It follows that $x \in u_{k-1}$ and both $\alpha_j x$ and $\beta_j x$ belong to u_k. Now it is easy to see that the sequence

$$u_0, \dots, u_{k-1}, v_{k+1}, \dots, v_i, u_{i+2}, \dots, u_n$$

where $v_l = (u_l \setminus \{\alpha_j x, \beta_j x\}) \cup \{x\}$ for $l = k+1, \dots, i$ is a derivation from u_0 to u_n, a contradiction with the minimality of n. $\square$

Theorem. *$T/\sim$ is an infinite czp-semigroup with only two prime ideals. (The two prime ideals are $T/\sim$ and $\{o_{T/\sim}\}$).*

Proof: It follows easily from Lemma 6 that the elements $x/\sim$ of $T/\sim$, with x running over X, are pairwise different and different from $o' = o_{T/\sim}$. (We have $o' = U \cup \{o\}$.) Let I be a prime ideal of $T/\sim$ different from $T/\sim$. Clearly, there is an element x of X such that $x/\sim \notin I$. Take any element y of X different from x. Since $x \sim \alpha_1 x + \beta_1 x$, we have $\alpha_1 x/\sim \notin I$. Since $x \sim \alpha_2 x + \beta_2 x$, we have $\alpha_2 x/\sim \notin I$. Thus $(\alpha_1 x + \alpha_2 x)/\sim \notin I$. Since $\alpha_1 x + \alpha_2 x \sim y + \gamma(x, y)$, we have $y/\sim \notin I$. Thus the complement of I contains all elements $z/\sim$ with $z \in X$ and $I = \{o'\}$. $\square$

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