C^1-smoothness of Nemytskii operators on Sobolev-type spaces of periodic functions

I. Kmit

Abstract. We consider a class of Nemytskii superposition operators that covers the nonlinear part of traveling wave models from laser dynamics, population dynamics, and chemical kinetics. Our main result is the C^1-continuity property of these operators over Sobolev-type spaces of periodic functions.

Keywords: Nemytskii operators, Sobolev-type spaces of periodic functions, C^1-smoothness

Classification: 47H99, 46E30

1. Motivation and main result

Development of a bifurcation theory for hyperbolic PDEs encounters significant difficulties caused by the fact that hyperbolic operators have worse regularity properties in comparison to ODEs and parabolic PDEs. Such a theory has to cover one- and multi-parameter bifurcations (both local and global), stability of bifurcating solutions, and periodic synchronizations. For hyperbolic problems all these topics currently remain challenging research directions. In each of them, investigation of smoothness properties of Nemytskii superposition operators plays an important role.

Not losing potential applicability to the aforementioned topics, here we consider Nemytskii operators in the context of the traveling wave models from laser dynamics [14], [17], [18]. The models describe the dynamics of multisection semiconductor lasers. They include a semilinear first-order one-dimensional hyperbolic system.

As an additional source of motivation, note that some problems of population dynamics [7], [8], [9], [15], chemical kinetics [2], [3], [19], [20], [21], and kinetic gas dynamics [6], [10], [16] have the same hyperbolic operator. Thus, our analysis applies to those problems as well, even when they have a different type of boundary conditions.

In the case of the traveling wave models, we deal with periodic-Dirichlet problems and our overall goal is to provide a bifurcation analysis for them. The basic idea is to apply techniques based on the Implicit Function Theorem in Banach spaces and the Lyapunov-Schmidt reduction (see, e.g., [5], [11]). The first problem

supported by a Humboldt Research Fellowship

to solve on this way is to establish the Fredholm solvability of the corresponding linearized problems, what is done in [12], [13]. To make the linearization procedure correct and to solve the so-called "range" equation (obtained after a Lyapunov-Schmidt reduction) via Implicit Function Theorem, we would need appropriate smoothness properties of the Nemytskii superposition operators with respect to the function spaces used in [12], [13]. The results obtained in this paper are sufficient to achieve this goal.

Due to the great importance of Nemytskii operators in the theory of nonlinear equations, their smoothness properties in different function spaces were extensively studied (see, e.g., [4]). Here we involve into consideration new function spaces important for solving nonlinear hyperbolic PDEs.

To state our main result, let us introduce the function spaces we are working with: For $\gamma \geq 0$ we denote by W^γ the vector Banach space of all locally integrable functions $u : [0,1]\times\mathbb{R} \to \mathbb{R}^n$ such that $u(x,t) = u(x,t+2\pi)$ for almost all $x \in (0,1)$ and $t \in \mathbb{R}$ and that

$$\|u\|^2_{W^\gamma} = \sum_{s\in\mathbb{Z}} (1+s^2)^\gamma \int_0^1 \left\| \int_0^{2\pi} u(x,t)e^{-ist}\,dt \right\|^2 dx < \infty. \tag{1}$$

Here and throughout $\|\cdot\|$ is the Hermitian norm in $\mathbb{C}^n$. In other words, W^γ is the anisotropic Sobolev space of all measurable functions $u : [0,1] \times \mathbb{R} \to \mathbb{R}^n$ such that $u(x,t) = u(x,t+2\pi)$ for almost all $x \in (0,1)$ and $t \in \mathbb{R}$ and that the distributional partial derivatives of u with respect to t up to the order γ are locally quadratically integrable. Furthermore, given $a \in L^\infty((0,1);\mathbb{R}^n)$ with $\operatorname{ess\,inf} |a_j| > 0$ for all $j \leq n$, we introduce the function spaces

$$V^\gamma = \left\{ u \in W^\gamma : \partial_x u \in W^{\gamma-1}, [\partial_t u_j + a_j \partial_x u_j]_{j=1}^n \in W^\gamma \right\}$$

endowed with the norms

$$\|u\|^2_{V^\gamma} = \|u\|^2_{W^\gamma} + \left\| [\partial_t u_j + a_j \partial_x u_j]_{j=1}^n \right\|^2_{W^\gamma}. \tag{2}$$

In the notation V^γ we drop the dependence of this space on a. It should be stressed that our results hold true for each a. Note that the space V^γ is larger than the space of all $u \in W^\gamma$ with $\partial_t u \in W^\gamma$ and $\partial_x u \in W^\gamma$.

We will focus on the pair of function spaces (V^2, W^2), for which we prove our main result given by Theorem 1. It is important that V^2 is embedded into the algebra of (continuous) functions with pointwise multiplication (see assertion (ii) of Lemma 2 and the embedding (5) below). This will allow us to use pointwise nonlinearities for the description of our Nemytskii operators.

Given a function $f(x,u) : (0,1) \times \mathbb{R}^n \to \mathbb{R}$ defined for almost all $x \in (0,1)$ and all $y \in \mathbb{R}^n$, let

$$[F(u)]\,(x,t) = f(x,u(x,t)). \tag{3}$$

We will show that F is a C^1-smooth superposition operator from V^2 into W^2.

For the sake of technical simplicity and without loss of generality we will suppose that $n = 1$.

Theorem 1. *Suppose that $f(\cdot,\cdot) \in L^\infty(0,1; C^4[-M,M])$ for each $M > 0$. Then $F(u) \in C^1(V^2, W^2)$.*

It should be emphasized here that, by physical reasons, the function f can have discontinuities with respect to the first argument, and the assumption of the theorem covers such cases.

Note also that under additional regularity assumptions on f, we can extend Theorem 1 to any desired smoothness of the operator F and to the pair of spaces (V^γ, W^γ) for any integer $\gamma \geq 2$.

2. Properties of the used function spaces

As usual, by $H^1(0,1)$ we denote the Sobolev space of all functions $u \in L^2(0,1)$ such that the weak derivative u' belongs to $L^2(0,1)$. The norm in $H^1(0,1)$ is defined by

$$\|u\|^2_{H^1(0,1)} = \sum_{j=0}^{1} \int_0^1 |u^{(j)}(x)|^2 dx.$$

Similarly, by $H^1((0,1) \times (0,2\pi))$ we denote the Sobolev space of all functions $u \in L^2((0,1) \times (0,2\pi))$ such that for every multiindex $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ with $|\alpha| \leq 1$, the weak partial derivative $D^\alpha u$ belongs to $L^2((0,1) \times (0,2\pi))$. The norm in $H^1((0,1) \times (0,2\pi))$ is given by

$$\|u\|^2_{H^1((0,1)\times(0,2\pi))} = \sum_{|\alpha|\leq 1} \int_0^1 \int_0^{2\pi} |D^\alpha u(x,t)|^2 \, dx \, dt.$$

Moreover, by $H^1((0,2\pi); H^1(0,1))$ we denote the abstract Sobolev space of all locally quadratically Bochner integrable maps $u : (0,2\pi) \to H^1(0,1)$ such that the distributional derivative u' is also locally quadratically Bochner integrable, with the norm

$$\|u\|^2_{H^1((0,2\pi);H^1(0,1))} = \sum_{j=0}^{1} \int_0^{2\pi} \|u^{(j)}(t)\|^2_{H^1(0,1)} \, dt.$$

Note that the space $H^1((0,2\pi); H^1(0,1))$ is smaller than the classical Sobolev space $H^1((0,1) \times (0,2\pi))$, and we have the continuous embeddings

$$H^1\left((0,1) \times (0,2\pi)\right) \hookrightarrow L^p\left((0,1) \times (0,2\pi)\right) \text{ for all } p \in [2,\infty), \tag{4}$$

$$H^1\left((0,2\pi); H^1(0,1)\right) \hookrightarrow C\left([0,1] \times [0,2\pi]\right), \tag{5}$$

see [1, Theorem 5.4].

We now establish some properties of the function spaces V^1 and V^2 introduced in Section 1, which are needed for proving Theorem 1.

Lemma 2. *We have the following continuous embeddings:*
(i) $V^1 \hookrightarrow H^1((0,1)\times(0,2\pi))$;
(ii) $V^2 \hookrightarrow H^1((0,2\pi);H^1(0,1))$.

PROOF: Notice the continuous embedding

$$V^\gamma \hookrightarrow W^\gamma \hookrightarrow W^{\gamma-1}, \quad \gamma \geq 1, \tag{6}$$

that is a straightforward consequence of the definitions of the spaces V^γ and W^γ.

(i) Take $u \in V^1$. Then $u \in W^1$ and, therefore, $\partial_t u \in W^0$ with

$$\|\partial_t u\|_{W^0}^2 \leq \|u\|_{W^0}^2 + \|\partial_t u\|_{W^0}^2 = \|u\|_{W^1}^2 \leq \|u\|_{V^1}^2. \tag{7}$$

Moreover, by the definition of V^1, we have $\partial_t u + a\partial_x u \in W^1$. On the account of the embedding (6),

$$\|\partial_t u + a\partial_x u\|_{W^0}^2 \leq \|\partial_t u + a\partial_x u\|_{W^1}^2 + \|u\|_{W^1}^2 = \|u\|_{V^1}^2. \tag{8}$$

By triangle inequality

$$\|a\partial_x u\|_{W^0}^2 - \|\partial_t u\|_{W^0}^2 \leq \|\partial_t u + a\partial_x u\|_{W^0}^2. \tag{9}$$

Since $a \in L^\infty(0,1)$ with ess inf $|a| > 0$, it follows by (7)–(9), that

$$\|\partial_x u\|_{W^0} \leq c\|u\|_{V^1},$$

where the constant c does not depend on u. Therefore

$$\|u\|_{W^0} + \|\partial_x u\|_{W^0} + \|\partial_t u\|_{W^0} \leq (2+c)\|u\|_{V^1}.$$

To finish the proof of this part, it remains to note that $W^0 = L^2((0,1)\times(0,2\pi))$.

(ii) We proceed similarly: Take $u \in V^2$. Then $u \in W^2$, and we have u as well as $\partial_t u$ and $\partial_x u$ in W^1. Moreover, $\|u\|_{W^1} \leq \|u\|_{W^2} \leq \|u\|_{V^2}$ and $\|\partial_t u\|_{W^1} \leq \|u\|_{W^2} \leq \|u\|_{V^2}$. This implies that $\|\partial_x u\|_{W^1} \leq c\|u\|_{V^2}$, where the constant c does not depend on u. Claim (ii) readily follows from these estimates. □

The following fact is similar to the density result for Sobolev spaces (see [1, Section III]) and proved by the same method.

Lemma 3. *The subspace $C^\infty \cap V^2$ is dense in V^2.*

PROOF: Set $\Pi = (0,1)\times(0,2\pi)$. By periodicity, speaking of a function in V^2, we can assume its restriction to $\overline{\Pi}$. We will use this convention in the course of the proof of the lemma.

Let φ be a non-negative $C^\infty(\mathbb{R}^2)$-function that vanishes outside a unit disk and satisfies the condition $\int \varphi(x)\,dx = 1$. Take $u \in V^2$ and consider its regularization defined by

$$u_\varepsilon(x,t) = \frac{1}{\varepsilon^2}\int_\Pi u(\xi,\tau)\varphi\left(\frac{x-\xi}{\varepsilon}, \frac{t-\tau}{\varepsilon}\right)\,d\xi d\tau$$

for $\varepsilon < \operatorname{dist}((x,t),\partial\Pi)$. Due to the properties of the convolutions, for any strict subdomain $\Pi' \subset \Pi$ it holds $\partial_t^\alpha u_\varepsilon \to \partial_t^\alpha u$ and $\partial_t^\beta[\partial_x u_\varepsilon] \to \partial_t^\beta[\partial_x u]$ in $L^2(\Pi')$ as $\varepsilon \to 0$ for $\alpha = 0,1,2$ and $\beta = 0,1$ (see [1, Section III] for details). This implies, in particular, that $v_\varepsilon \to v$ in $L^2(\Pi')$ as $\varepsilon \to 0$, where $v = \partial_t u + a(x)\partial_x u$ and $v_\varepsilon = \partial_t u_\varepsilon + a(x)\partial_x u_\varepsilon$. Now we intend to prove that $u_\varepsilon \to u$ in V^2 on Π' as $\varepsilon \to 0$. It suffices to show that $\partial_t^\alpha v_\varepsilon \to \partial_t^\alpha v$ in $L^2(\Pi')$ as $\varepsilon \to 0$ for $\alpha = 1,2$. Fix $\varepsilon_0 < \operatorname{dist}(\Pi',\partial\Pi)$ and consider $\varepsilon < \varepsilon_0$. Then for any $\psi \in C_0^\infty(\Pi')$ we have

$$\begin{aligned}
&\int_{\Pi'} \left(\partial_t u_\varepsilon(x,t) + a(x)\partial_x u_\varepsilon(x,t)\right)\partial_t^\alpha \psi(x,t)\,dx\,dt \\
&\quad = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\partial_t u + a\partial_x u\right](x-\xi,t-\tau)\varphi\left(\frac{\xi}{\varepsilon},\frac{\tau}{\varepsilon}\right)\partial_t^\alpha\psi(x,t)\,d\xi d\tau dx\,dt \\
&\quad = \frac{(-1)^\alpha}{\varepsilon^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\partial_t^\alpha\left[\partial_t u + a\partial_x u\right](x-\xi,t-\tau)\varphi\left(\frac{\xi}{\varepsilon},\frac{\tau}{\varepsilon}\right)\psi(x,t)\,d\xi\,d\tau\,dx\,dt \\
&\quad = (-1)^\alpha\int_{\Pi'}\partial_t^\alpha\,(v)_\varepsilon\,(x,t)\psi(x,t)\,dx\,dt.
\end{aligned}$$

Therefore, $\partial_t^\alpha(v_\varepsilon)(x,t) = (\partial_t^\alpha v)_\varepsilon(x,t)$ in the sense of distributions on Π'. Since $\partial_t^\alpha v \in L^2(\Pi)$ for $\alpha = 1,2$,

$$\lim_{\varepsilon\to 0}\|\partial_t^\alpha v_\varepsilon - \partial_t^\alpha v\|_{L^2(\Pi')} = \lim_{\varepsilon\to 0}\|(\partial_t^\alpha v)_\varepsilon - \partial_t^\alpha v\|_{L^2(\Pi')} = 0,$$

as desired.

Consider now the following locally finite open covering of Π:

$$\begin{aligned}
\Pi_1 &= \left\{(x,t)\in\Pi \,:\, \operatorname{dist}\left((x,t),\partial\Pi\right) > \frac{1}{2}\right\}, \\
\Pi_j &= \left\{(x,t)\in\Pi \,:\, \frac{1}{j+1} < \operatorname{dist}\left((x,t),\partial\Pi\right) < \frac{1}{j-1}\right\}, \quad j \ge 2.
\end{aligned}$$

Let $\eta_1, \eta_2, \dots$ be a partition of unity subordinate to the covering $\{\Pi_{j+1}\setminus\Pi_{j-1}\}$. Then, given $j \ge 1$, the product $\eta_j u$ is in V^2 and has support contained in Π_j. Consider now the mollification $(\eta_j u)_\varepsilon$. Given $\varepsilon_0 > 0$, we can choose a sequence ε_j such that

$$\varepsilon_j < \operatorname{dist}\left(\Pi_{j+1},\partial\Pi_{j+3}\right) \quad \text{and} \quad \|(\eta_j u)_{\varepsilon_j} - \eta_j u\|_{V^2} \le \frac{\varepsilon_0}{2^{j+1}}.$$

Let $w = \sum_{j=1}^\infty (\eta_j u)_{\varepsilon_j}$. It follows from the definition of the partition of unity that at each $x \in \Pi$ only finitely many terms in the sum are nonzero. Since each term is smooth, this implies $w \in C^\infty(\Pi)$. Moreover, using the triangle inequality, we have

$$\|w-u\|_{V_n^2} \le \sum_{j=1}^{n+2} \|(\eta_j u)_{\varepsilon_j} - \eta_j u\|_{V_n^2} \le \sum_{j=1}^{\infty} \varepsilon_0 2^{-j} = \varepsilon_0,$$

where $\|\cdot\|_{V_n^2}$ is defined by (2) with the integral over

$$\Pi_{1/n} = \left\{ (x,t) \in \Pi \,:\, \operatorname{dist}\left((x,t), \partial\Pi\right) > \frac{1}{n} \right\}$$

in place of the integral over Π. This yields

$$\|w-u\|_{V^2} = \sup_{n\ge 1} \|w-u\|_{V_n^2} \le \varepsilon_0.$$

Since $\varepsilon_0 > 0$ is arbitrary, the set $\sum_{j=1}^{n} (\eta_j u)_{\varepsilon_j}$, $n \ge 3$, is the desired dense set from $C^\infty \cap V^2$. □

3. C¹-smoothness of the Nemytskii operator from V^2 into W^2 (proof of Theorem 1)

We split the proof into two lemmas.

Lemma 4. *The superposition operator F given by the formula* (3) *maps V^2 into W^2.*

Proof: For any function $u \in V^2$, denote by $F'(u)$ and $F''(u)$ the superposition operators by putting, for almost all $x \in (0,1)$,

$$\begin{aligned} \left[F'(u)\right](x,t) &= \left(\partial_u f\right)(x, u(x,t)), \\ \left[F''(u)\right](x,t) &= \left(\partial_u^2 f\right)(x, u(x,t)). \end{aligned}$$

As $V^2 \hookrightarrow C([0,1]\times[0,2\pi])$ continuously (see Lemma 2(ii) and the embedding (5)), we can identify any $u \in V^2$ with a uniformly continuous and 2π-periodic in t function on $[0,1] \times \mathbb{R}$. Furthermore, we have the inequality

$$\|u\|_{C([0,1]\times[0,2\pi])} \le C_0 \|u\|_{V^2} \quad \text{for all} \quad u \in V^2, \tag{10}$$

the constant C_0 being independent of u. Combining this with the smoothness assumptions on f, we conclude that, given $u \in V^2$, the functions $[F(u)](x,t)$, $[F'(u)](x,t)$, and $[F''(u)](x,t)$ belong to $L^\infty((0,1)\times(0,2\pi))$.

Claim 1. *$F(u)$ maps V^2 into W^1.* Fix an arbitrary $u \in V^2$, set

$$K = \|u\|_{C([0,1]\times[0,2\pi])}, \tag{11}$$

and consider $(u^m)_{m\in\mathbb{Z}}$ to be a sequence in $C^\infty \cap V^2$ converging to u in V^2. By (10), we have this convergence also in $C([0,1]\times[0,2\pi])$. For almost all $x \in (0,1)$ and all $t \in \mathbb{R}$ we have

$$[\partial_t F(u^m)](x,t) = [F'(u^m)](x,t)\partial_t u^m(x,t). \tag{12}$$

Let us show that

$$F'(u^m)\partial_t u^m \to F'(u)\partial_t u \text{ in } L^2\left((0,1)\times(0,2\pi)\right) \quad \text{as} \quad m \to \infty. \tag{13}$$

Indeed,

$$\begin{aligned}
&\int_0^1 \int_0^{2\pi} \left|F'(u^m)\partial_t u^m - F'(u)\partial_t u\right|^2 \, dx\, dt \\
&\le 2\int_0^1 \int_0^{2\pi} \left|F'(u^m) - F'(u)\right|^2 \left|\partial_t u^m\right|^2 \, dx\, dt \qquad (14)\\
&\quad + 2\int_0^1 \int_0^{2\pi} \left|F'(u)\right|^2 \left|\partial_t u^m - \partial_t u\right|^2 \, dx\, dt \\
&\le 2\int_0^1 \int_0^{2\pi} \left|\int_0^1 (\partial_u^2 f)(x, \sigma u^m + (1-\sigma)u)\, d\sigma\right|^2 \left|u^m - u\right|^2 \left|\partial_t u^m\right|^2 \, dx\, dt \\
&\quad + 2\int_0^1 \int_0^{2\pi} \left|(\partial_u f)(x,u)\right|^2 \left|\partial_t u^m - \partial_t u\right|^2 \, dx\, dt \\
&\le 2\left\|u^m - u\right\|^2_{C([0,1]\times[0,2\pi])} \left\|\partial_u^2 f\right\|^2_{L^\infty((0,1)\times(-3K;3K))} \left\|\partial_t u^m\right\|^2_{W^0} \qquad (15)\\
&\quad + 2\left\|\partial_u f\right\|^2_{L^\infty((0,1)\times(-K;K))} \left\|\partial_t u^m - \partial_t u\right\|^2_{W^0}.
\end{aligned}$$

The latter inequality is true for all sufficiently large $m \in \mathbb{N}$. Since $(u^m)_{m\in\mathbb{N}}$ converges to u in V^2 and $V^2 \hookrightarrow L^2(0,1;H^1(0,2\pi))$, the sequence $(\partial_t u^m)_{m\in\mathbb{N}}$ is bounded in $L^2((0,1)\times(0,2\pi))$ and converges to $\partial_t u$ in $L^2((0,1)\times(0,2\pi))$. This shows the convergence (13). It follows by Hölder's inequality that for any $\varphi \in \mathcal{D}((0,1)\times(0,2\pi))$

$$\begin{aligned}
&\int_0^1 \int_0^{2\pi} \left(F(u)\partial_t\varphi + F'(u)\partial_t u\varphi\right)\, dx\, dt \\
&\qquad = \lim_{m\to\infty} \left[\int_0^1 \int_0^{2\pi} \left(F(u^m)\partial_t\varphi + F'(u^m)\partial_t u^m\varphi\right)\, dx\, dt\right].
\end{aligned} \tag{16}$$

By (12), the expression under the limit sign is equal to zero. Hence (16) implies

$$\int_0^1 \int_0^{2\pi} \left(F(u)\partial_t\varphi + F'(u)\partial_t u\varphi\right)\, dx\, dt = 0$$

for any $\varphi \in \mathcal{D}((0,1)\times(0,2\pi))$. This means that $F(u)$ has a weak partial derivative in t given by the formula

$$\partial_t F(u) = F'(u)\partial_t u.$$

Recall that $[F'(u)](x,t) \in L^\infty((0,1)\times(0,2\pi))$ and $\partial_t u \in L^2((0,1)\times(0,2\pi))$. It is immediate that $[\partial_t F(u)](x,t) \in L^2((0,1)\times(0,2\pi))$ and therefore $[F(u)](x,t) \in W^1$. Since $u \in V^2$ is arbitrary, the desired assertion is therewith proved.

Claim 2. *$F(u)$ maps V^2 into W^2.* As above, fix an arbitrary $u \in V^2$ and choose $(u^m)_{m\in\mathbb{Z}}$ as in Claim 1. Similarly to the proof of Claim 1, one can show the convergence

$$F''(u^m)\left(\partial_t u^m\right)^2 + F'(u^m)\partial_t^2 u^m \to F''(u)\left(\partial_t u\right)^2 + F'(u)\partial_t^2 u \quad \text{in } L^2\left((0,1)\times(0,2\pi)\right) \text{ as } m\to\infty \tag{17}$$

and that

$$\partial_t^2 F(u) = F''(u)\left(\partial_t u\right)^2 + F'(u)\partial_t^2 u. \tag{18}$$

The only difference appearing here concerns the estimation of the following integral:

$$\begin{aligned}
&\int_0^1\int_0^{2\pi}\left|F''(u^m)\left(\partial_t u^m\right)^2 - F''(u)\left(\partial_t u\right)^2\right|^2 dx\,dt\\
&\le 2\int_0^1\int_0^{2\pi}\left|(\partial_u^2 f)(x,u^m) - (\partial_u^2 f)(x,u)\right|^2\left|\partial_t u^m\right|^4 dx\,dt\\
&\quad + 2\int_0^1\int_0^{2\pi}\left|(\partial_u^2 f)(x,u)\right|^2\left|\left(\partial_t u^m\right)^2 - \left(\partial_t u\right)^2\right|^2 dx\,dt\\
&\le 2\int_0^1\int_0^{2\pi}\left|\int_0^1\left(\partial_u^3 f\right)\left(x,\sigma u^m + (1-\sigma)u\right)d\sigma\right|^2\left|u^m-u\right|^2\left|\partial_t u^m\right|^4 dx\,dt\\
&\quad + 2\left\|\partial_u^2 f\right\|^2_{L^\infty((0,1)\times(-K;K))}\\
&\quad \times\int_0^1\left\|\partial_t u^m(x,\cdot) - \partial_t u(x,\cdot)\right\|^2_{L^\infty(0,2\pi)} dx\\
&\quad \times\int_0^{2\pi}\left\|\partial_t u^m(\cdot,t) + \partial_t u(\cdot,t)\right\|^2_{L^\infty(0,1)} dt\\
&\le 2\left\|\partial_u^3 f\right\|^2_{L^\infty((0,1)\times(-3K;3K))}\left\|u^m-u\right\|_C^2\left\|\partial_t u^m\right\|^4_{L^4}\\
&\quad + 2\left\|\partial_u^2 f\right\|^2_{L^\infty((0,1)\times(-K;K))}\int_0^1\left\|\partial_t u^m(x,\cdot) - \partial_t u(x,\cdot)\right\|^2_{L^\infty(0,2\pi)} dx\\
&\quad \times\int_0^{2\pi}\left\|\partial_t u^m(\cdot,t) + \partial_t u(\cdot,t)\right\|^2_{L^\infty(0,1)} dt,
\end{aligned} \tag{19}$$

where the constant K is defined by the formula (11). The right hand side tends to zero by Lemma 2, the embedding (4), and the embedding

$$V^2 \hookrightarrow W^2 \hookrightarrow L^2\left(0,1;C^1[0,2\pi]\right).$$

Turning back to (18), we obtain $[\partial_t^2 F(u)](x,t) \in L^2((0,1)\times(0,2\pi))$. Hence $[F(u)](x,t) \in W^2$ as desired. $\square$

Lemma 5. *The mapping $u \in V^2 \to F(u) \in W^2$ is C^1-smooth and for all $u, v \in V^2$ it holds*

$$[F'(u)v]\,(x,t) = (\partial_u f)(x, u(x,t))v(x,t). \tag{20}$$

Proof: We now prefer to work with the following norm in W^2:

$$\|w\|_{W^2}^2 = \|\partial_t^2 w\|_{W^0}^2. \tag{21}$$

Note that it is equivalent to the W^2-norm introduced by (1).

To prove the continuity of the mapping $u \in V^2 \to F(u) \in W^2$, fix an arbitrary $u \in V^2$. On the account of the expression (18) for $\partial_t^2 F(u)$ and the estimates (14) and (19) with u^m replaced by $u+v$, we derive the following inequality for all $v \in V^2$ with $\|v\|_{V^2} \le K/C_0$, where the constant C_0 is fixed to satisfy (10) and K is determined by (11):

$$\begin{aligned}
&\frac{1}{2}\|\partial_t^2 F(u+v)(x,t) - \partial_t^2 F(u)(x,t)\|_{W^0}^2 \\
&\quad\le \|\partial_u^3 f\|^2_{L^\infty((0,1)\times(-3K,3K))}\|\partial_t(u+v)\|^2_{L^2(0,1;L^\infty(0,2\pi))} \\
&\qquad \times \|\partial_t(u+v)\|^2_{L^2(0,2\pi;L^\infty(0,1))}\|v\|^2_{C([0,1]\times[0,2\pi])} \\
&\qquad + \|\partial_u^2 f\|^2_{L^\infty((0,1)\times(-K,K))}\|\partial_t(2u+v)\|^2_{L^2(0,2\pi;L^\infty(0,1))}\|\partial_t v\|^2_{L^2(0,1;L^\infty(0,2\pi))} \\
&\qquad + \|\partial_u^2 f\|^2_{L^\infty((0,1)\times(-3K,3K))}\|\partial_t^2(u+v)\|^2_{W^0}\|v\|^2_{C([0,1]\times[0,2\pi])} \\
&\qquad + \|\partial_u f\|^2_{L^\infty((0,1)\times(-K,K))}\|\partial_t^2 v\|^2_{W^0} \le C\|v\|^2_{V^2},
\end{aligned}$$

the constant C being dependent on f and u, but not on v. We conclude that

$$\|\partial_t^2 F(u+v)(x,t) - \partial_t^2 F(u)(x,t)\|_{W^0}^2 = O(\|v\|^2_{V^2})$$

as $\|v\|_{V^2} \to 0$. The continuity of F is therefore proved.

Let us now show that the operator $u \to F(u)$ is continuously differentiable. Fix $u \in V^2$ and introduce the bounded linear operator $G : V^2 \to W^2$ defined by the formula

$$[G(u)v](x,t) = (\partial_u f)(x, u(x,t))v(x,t).$$

From the smoothness assumptions on f and the proof of Lemma 4 it follows that $(\partial_u f)(x, u(x,t)) \in W^2$. Since $V^2 \hookrightarrow W^2$ continuously, W^2 is an algebra of functions, and $v \in V^2$, the correctness of the definition of the operator G is straightforward.

Our next concern is to show that F is differentiable in u and that $F'(u) = G(u)$. Similarly to the above, fix $u \in V^2$ and consider $w \in V^2$ with $\|w\|_{V^2} \le K/C_0$, where C_0 is a certain constant satisfying (10) and K is specified by (11). It follows by (10) that $\|w\|_{C([0,1]\times[0,2\pi])} \le K$. The desired assertion now follows from the following estimate:

$$
\begin{aligned}
&\|F(u+w)(x,t) - F(u)(x,t) - [G(u)w](x,t)\|_{W^2}\\
&\quad = \|f(x,u+w) - f(x,u) - (\partial_u f)(x,u)w\|_{W^2}\\
&\quad = \left\| w\int_0^1 \left[(\partial_u f)(x,u+\sigma w) - (\partial_u f)(x,u)\right]\, d\sigma\right\|_{W^2}\\
&\quad = \left\| w^2\int_0^1\int_0^1 \sigma(\partial_u^2 f)(x,u+\sigma\sigma_1 w)\, d\sigma d\sigma_1\right\|_{W^2}\\
&\quad = \left\| \partial_t^2\left[w^2\int_0^1\int_0^1 \sigma(\partial_u^2 f)(x,u+\sigma\sigma_1 w)\, d\sigma d\sigma_1\right]\right\|_{W^0}\\
&\quad = \Bigg\| 2(w\partial_t^2 w + (\partial_t w)^2)\int_0^1\int_0^1 \sigma(\partial_u^2 f)(x,u+\sigma\sigma_1 w)\, d\sigma d\sigma_1\\
&\qquad + w^2\int_0^1\int_0^1 \sigma(\partial_u^4 f)(x,u+\sigma\sigma_1 w)\left[\partial_t u + \sigma\sigma_1\partial_t w\right]^2\, d\sigma d\sigma_1\\
&\qquad + w^2\int_0^1\int_0^1 \sigma(\partial_u^3 f)(x,u+\sigma\sigma_1 w)\left[\partial_t^2 u + \sigma\sigma_1\partial_t^2 w\right]\, d\sigma d\sigma_1\\
&\qquad + 2w\partial_t w\int_0^1\int_0^1 \sigma(\partial_u^3 f)(x,u+\sigma\sigma_1 w)\left[\partial_t u + \sigma\sigma_1\partial_t w\right]\, d\sigma d\sigma_1\Bigg\|_{W^0}\\
&\quad \le 4\left(\|w\|_{L^4}^2 + \|\partial_t w\|_{L^4}^2 + \|w\|_{V^2}\|w\|_C\right)\|f(\cdot,\cdot)\|_{L^\infty((0,1),C^4(-3K,3K))}\\
&\qquad \times\left(1 + \|u\|_{W^2} + \|\partial_t u\|^2_{L^4((0,1)\times(0,2\pi))} + \|w\|_{W^2} + \|w\|^2_{L^4((0,1)\times(0,2\pi))}\right).
\end{aligned}
$$

In the last inequality we again used Lemma 2 and the embedding (4). The continuous differentiability of F is proved, which completes the proof of the lemma. ☐

References

[1] Adams R.A., *Sobolev spaces*, Academic Press, New York, 1975.

[2] Akramov T.A., *On the behavior of solutions to a certain hyperbolic problem*, Siberian Math. J. **39** (1998), no. 1, 1–17.

[3] Akramov T.A., Belonosov V.S., Zelenyak T.I., Lavrent'ev M.M., Jr., Slin'ko M.G., Sheplev V.S., *Mathematical Foundations of Modeling of Catalytic Processes: A Review*, Theoretical Foundations of Chemical Engineering **34** (2000), no. 3, 295–306.

[4] Appell J., Zabrejko P., *Nonlinear Superposition Operators*, Cambridge University Press, Cambridge, UK, 1990.

[5] Chow S.-N., Hale J.K., *Methods of Bifurcation Theory*, Grundlehren der Math. Wissenschaften, 251, Springer, New York-Berlin, 1982.

[6] Conner H.E., *Some general properties of a class of semilinear hyperbolic systems analogous to the differential-integral equations of gas dynamics*, J. Differential Equations **10** (1971), 188–203.

[7] Hillen T., *Existence theory for correlated random walks on bounded domains*, Can. Appl. Math. Q. **18** (2010), no. 1, 1–40.

[8] Hillen T., Hadeler K.P., *Hyperbolic systems and transport equations in mathematical biology*, in Analysis and Numerics for Conservation Laws, G. Warnecke, Springer, Berlin, 2005, pp. 257–279.

[9] Horsthemke W., *Spatial instabilities in reaction random walks with direction-independent kinetics*, Phys. Rev. E **60** (1999), 2651–2663.

[10] Illner R, Reed M., *Decay to equilibrium for the Carleman model in a box*, SIAM J. Appl. Math. **44** (1984), 1067–1075.

[11] Kielhöfer H., *Bifurcation Theory. An Introduction with Applications to PDEs*, Appl. Math. Sciences, 156, Springer, New York-Berlin, 2004.

[12] Kmit I., Recke L.,, *Fredholm alternative for periodic-Dirichlet problems for linear hyperbolic systems*, J. Math. Anal. Appl. **335** (2007), 355–370.

[13] Kmit I., Recke L., *Fredholmness and smooth dependence for linear hyperbolic periodic-Dirichlet problems*, J. Differential Equations, to appear.

[14] Lichtner M., Radziunas M., Recke L., *Well-posedness, smooth dependence and center manifold reduction for a semilinear hyperbolic system from laser dynamics*, Math. Methods Appl. Sci. **30** (2007), 931–960.

[15] Lutscher F., Stevens A., *Emerging patterns in a hyperbolic model for locally interacting cell systems*, J. Nonlinear Sci. **12** (2002), no 6, 619–640.

[16] Platkowski T., Illner R., *Discrete velocity models of the Boltzmann equation: A survey on the mathematical aspects of the theory*, SIAM Review **30** (1988), 213–255.

[17] Radziunas M., *Numerical bifurcation analysis of the traveling wave model of multisection semiconductor lasers*, Phys. D **213** (2006), 575–613.

[18] Radziunas M., Wünsche H.-J., *Dynamics of multisection DFB semiconductor lasers: traveling wave and mode approximation models*, in Optoelectronic Devices – Advanced Simulation and Analysis, ed. by J. Piprek, Springer, Berlin, 2005, pp. 121–150.

[19] Slin'ko M.G., *History of the development of mathematical modeling of catalytic processes and reactors*, Theoretical Foundations of Chemical Engineering **41** (2007), no. 1, 13–29.

[20] Zelenyak T.I., *On stationary solutions of mixed problems relating to the study of certain chemical processes*, Differ. Equations **2** (1966), 98–102.

[21] Zelenyak T.I., *The stability of solutions of mixed problems for a particular quasi- linear equation*, Differ. Equations **3** (1967), 9–13.

Institute for Applied Problems of Mechanics and Mathematics, Ukrainian Academy of Sciences, Naukova St. 3b, 79060 Lviv, Ukraine

E-mail: kmit@informatik.hu-berlin.de

(*Received* January 24, 2011, *revised* July 22, 2011)