Maximal free sequences in a Boolean algebra

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Abstract. We study free sequences and related notions on Boolean algebras. A *free sequence* on a BA A is a sequence $\langle a_\xi : \xi < \alpha\rangle$ of elements of A, with α an ordinal, such that for all $F, G \in [\alpha]^{<\omega}$ with $F < G$ we have $\prod_{\xi\in F} a_\xi \cdot \prod_{\xi\in G} -a_\xi \neq 0$. A free sequence of length α exists iff the Stone space $\mathrm{Ult}(A)$ has a free sequence of length α in the topological sense. A free sequence is *maximal* iff it cannot be extended at the end to a longer free sequence. The main notions studied here are the spectrum function

$$\mathfrak{f}_{\mathrm{sp}}(A) = \{|\alpha| : A \text{ has an infinite maximal free sequence of length } \alpha\}$$

and the associated min-max function

$$\mathfrak{f}(A) = \min(\mathfrak{f}_{\mathrm{sp}}(A)).$$

Among the results are: for infinite cardinals $\kappa \le \lambda$ there is a BA A such that $\mathfrak{f}_{\mathrm{sp}}(A)$ is the collection of all cardinals μ with $\kappa \le \mu \le \lambda$; maximal free sequences in A give rise to towers in homomorphic images of A; a characterization of $\mathfrak{f}_{\mathrm{sp}}(A)$ for A a weak product of free BAs; $\mathfrak{p}(A), \pi\chi_{\mathrm{inf}}(A) \le \mathfrak{f}(A)$ for A atomless; a characterization of infinite BAs whose Stone spaces have an infinite maximal free sequence; a generalization of free sequences to free chains over any linearly ordered set, and the relationship of this generalization to the supremum of lengths of homomorphic images.

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Introduction

A *free sequence* on a BA A is a sequence $\langle a_\xi : \xi < \alpha\rangle$ of elements of A, with α an ordinal, such that for all $F, G \in [\alpha]^{<\omega}$ with $F < G$ we have $\prod_{\xi\in F} a_\xi \cdot \prod_{\xi\in G} -a_\xi \neq 0$. Here $[\alpha]^{<\omega}$ is the collection of all finite subsets of α. We write $F < G$ to mean that $\xi < \eta$ for all $\xi \in F$ and $\eta \in G$. We take empty products to equal 1. Thus if $G = \emptyset$, then our condition just says that $\prod_{\xi\in F} a_\xi \neq 0$. So the elements of a free sequence are nonzero, and have the finite intersection property (abbreviated fip). Also, for $F = \emptyset$ we get $\prod_{\xi\in G} -a_\xi \neq 0$, so that no finite sum of elements of a free sequence is equal to 1; in particular, 1 is not a member of any free sequence.

This notion of free sequence is closely related to the usual notion of a free sequence of points in a topological space, and to the notion of tightness in a topological space. Recall that a sequence $\langle x_\xi : \xi < \alpha\rangle$ of points in a space is free iff for all $\xi < \alpha$ we have $\overline{\{x_\eta : \eta < \xi\}} \cap \overline{\{x_\eta : \xi \leq \eta\}} = \emptyset$. Given a free sequence $\langle a_\xi : \xi < \alpha\rangle$ in the algebraic sense, for each $\xi < \alpha$ let F_ξ be an ultrafilter containing $\{x_\eta : \eta \leq \xi\} \cup \{-x_\eta : \xi < \eta\}$; then $\langle F_\xi : \xi < \alpha\rangle$ is a free sequence in the Stone space. And given a free sequence $\langle F_\xi : \xi < \alpha\rangle$ in the Stone space, for each $\xi < \alpha$ there is an element x_ξ of the Boolean algebra such that $\{F_\eta : \eta < \xi\} \subseteq \{G : G$ is an ultrafilter and $-x_\xi \in G\}$ and $\{F_\eta : \xi \leq \eta\} \subseteq \{G : G$ is an ultrafilter and $x_\xi \in G\}$; then $\langle x_\xi : \xi < \alpha\rangle$ is a free sequence in the algebraic sense. These processes are not inverses of each other, and this gives rise to differences when considering, as we do below, maximal free sequences.

A free sequence as above is *maximal* iff there is no $b \in A$ such that $\langle a_\xi : \xi < \alpha\rangle^\frown\langle b\rangle$ is a free sequence, where $\langle a_\xi : \xi < \alpha\rangle^\frown\langle b\rangle$ is the result of adjoining b at the end of the sequence $\langle a_\xi : \xi < \alpha\rangle$. Now we define

$$\begin{aligned}\mathfrak{f}_{\mathrm{sp}}(A) &= \{|\alpha| : A \text{ has an infinite maximal free sequence of length } \alpha\};\\ \mathfrak{f}(A) &= \min(\mathfrak{f}_{\mathrm{sp}}(A)).\end{aligned}$$

This article studies these two notions, relating them to other functions defined in a similar min-max fashion. Note that maximal free sequences always exist, by Zorn's lemma.

We also consider the topological version. A free sequence $\langle F_\xi : \xi < \alpha\rangle$ of ultrafilters on A is maximal iff there does not exist an ultrafilter G such that $\langle F_\xi : \xi < \alpha\rangle^\frown\langle G\rangle$ is free. Maximal free sequences of ultrafilters do not always exist; those BAs in which they do exist are characterized in Theorem 3.2.

It is natural to generalize the notion of a free sequence by indexing the sequence by any linear order; we call such things *free chains*. Now in the notion of maximal free chains we allow the possibility of inserting elements at any place in the chain. Then the supremum of sizes of free chains in A is equal to the supremum of linearly ordered subsets in homomorphic images of A. $\mathfrak{f}(A)$ is less or equal to the smallest size of a maximal free chain on A.

Notation. For set-theoretical notation we follow Kunen [80]. We follow Koppelberg [89] for Boolean algebraic notation, and Monk [96] for more specialized notation concerning cardinal functions on BAs. For cardinals κ, λ we use $[\kappa, \lambda]_{\mathrm{card}}$ to denote the set of all cardinals μ such that $\kappa \leq \mu \leq \lambda$. Similarly for other intervals, like $[\kappa, \lambda)_{\mathrm{card}}$. $\mathrm{Fr}(\kappa)$ is the free BA on κ generators. Note that if $0 < a < 1$ in $\mathrm{Fr}(\kappa)$, then there is a unique smallest finite nonempty set G of generators such that $a \in \langle G\rangle$; this is called the *support* of a, denoted by $\mathrm{supp}(a)$. We let $\mathrm{supp}(0) = \mathrm{supp}(1) = \emptyset$. $\mathrm{Finco}(\kappa)$ denotes the BA of finite and cofinite subsets of κ. $\overline{A}$ is the completion of A. In several places we use the following construction. Let $\langle A_i : i \in I\rangle$ be a system of BAs, with I infinite. The *weak product* $\prod^{\mathrm{w}}_{i\in I} A_i$ consists of all members x of the full product such that one of the two sets

$$\{i \in I : x_i \neq 0\} \quad \text{or} \quad \{i \in I : x_i \neq 1\}$$

is finite; the corresponding set is then called the *support* of x, and is denoted by $\mathrm{supp}(x)$; x is called *of type* I if $\{i \in I : x_i \neq 0\}$ is finite, *of type* II otherwise.

If A is a BA and $a \in A$, then $\mathcal{S}(a) = \{F \in \mathrm{Ult}(A) : a \in F\}$. Thus $\mathcal{S}$ is the Stone isomorphism from A onto the BA of clopen sets in the Stone space $\mathrm{Ult}(A)$

If L is a linear order, then $\mathrm{Intalg}(L)$ is the interval algebra over L (perhaps after adjoining a first element to L). Any element x of $\mathrm{Intalg}(L)$ has the form $[a_0, b_0) \cup \ldots \cup [a_{m-1}, b_{m-1})$, with $a_0 < b_0 < \cdots < b_{m-1} \leq \infty$. (Here ∞ is not in L.) The intervals $[a_i, b_i)$ are called the *components* of x.

1. $\mathfrak{f}_{\mathrm{sp}}$

Note that if a is an atom of A, then $\langle a\rangle$ is a maximal free sequence. This explains our restriction to infinite α in the definitions of $\mathfrak{f}_{\mathrm{sp}}$ and $\mathfrak{f}$.

Theorem 1.1. (i) *If $\langle a_\xi : x < \alpha\rangle$ is a strictly decreasing sequence of elements of a BA A, with $1 > a_0$, then it is a free sequence.*

(ii) *Any infinite BA has an infinite free sequence.*

PROOF: For (i), suppose that $F, G \subseteq \alpha$ are finite, with $F < G$. If $F = \emptyset \neq G$, then $\prod_{\xi\in F} a_\xi \cdot \prod_{\eta\in G} -a_\eta = -a_\nu \neq 0$, where ν is the least member of G. If $F \neq \emptyset = G$, then $\prod_{\xi\in F} a_\xi \cdot \prod_{\eta\in G} -a_\eta = a_\nu \neq 0$, where ν is the greatest member of F. If $F \neq \emptyset \neq G$, then $\prod_{\xi\in F} a_\xi \cdot \prod_{\eta\in G} -a_\eta = a_\nu \cdot -a_\mu \neq 0$, where ν is the greatest member of F and μ is the least member of G.

(ii) clearly follows from (i). □

Theorem 1.2. *For any infinite cardinal κ, $\mathfrak{f}_{\mathrm{sp}}(\mathrm{Finco}(\kappa)) = \{\omega\}$.*

PROOF: By Theorem 1.1, $\mathrm{Finco}(\kappa)$ has a free sequence of length ω. Now suppose that $\langle a_\xi : \xi < \alpha\rangle$ is a free sequence in $\mathrm{Finco}(\kappa)$, with $\omega_1 \leq \alpha$; we want to get a contradiction.

(1) Each a_ξ, $\xi < \omega_1$, is cofinite.

For, suppose that $\xi < \omega_1$ and a_ξ is finite. Then there exist η, μ with $\xi < \eta < \mu < \omega_1$ and $a_\xi \cap a_\eta = a_\xi \cap a_\mu$. Then $a_\xi \cap a_\eta \cap -a_\mu = 0$, contradiction.

Now by (1), there is a $\Gamma \in [\omega_1]^{\omega_1}$ such that $\langle -a_\xi : \xi \in \Gamma\rangle$ forms a Δ-system, say with kernel b. Take $\xi < \eta < \mu$ all in Γ. Then

$$a_\xi \cap -a_\eta \cap -a_\mu = a_\xi \cap b = 0,$$

contradiction. □

Theorem 1.3. $\mathfrak{f}_{\mathrm{sp}}(\mathrm{Fr}(\kappa)) = \{\kappa\}$.

PROOF: It suffices to show that if $\langle a_\xi : \xi < \alpha\rangle$ is a free sequence with $\alpha < \kappa$, then it can be extended. Let X be a set of free generators of $\mathrm{Fr}(\kappa)$. Choose $x \in X \setminus \bigcup_{\xi<\alpha} \mathrm{supp}(a_\xi)$. Then $\langle a_\xi : \xi < \alpha\rangle^\frown\langle x\rangle$ is still free. □

The following simple proposition will frequently be used in what follows.

Proposition 1.4. *A free sequence $\langle a_\xi : \xi < \alpha\rangle$ of elements of A is maximal iff for every $b \in A$ one of the following conditions holds.*

(i) *There is a finite $F \subseteq \alpha$ such that $\prod_{\xi\in F} a_\xi \cdot b = 0$.*
(ii) *There exist finite $F, G \subseteq \alpha$ such that $F < G$ and $\prod_{\xi\in F} a_\xi \cdot \prod_{\eta\in G} -a_\eta \cdot -b = 0$.*

Proof: Suppose that $\langle a_\xi : \xi < \alpha\rangle$ is maximal, and $b \in A$. Then $\langle a_\xi : \xi < \alpha\rangle^\frown\langle b\rangle$ is no longer free. Let $a_\alpha = b$. Then there exist finite $F, G \subseteq \alpha + 1$ such that $F < G$ and $\prod_{\xi\in F} a_\xi \cdot \prod_{\eta\in G} -a_\eta = 0$. Since $\langle a_\xi : \xi < \alpha\rangle$ itself is free we must have $\alpha \in F \cup G$. If $\alpha \in F$, then α is the largest element of F, $G = \emptyset$, and (i) holds with $F\backslash\{\alpha\}$ in place of F. If $\alpha \in G$, then α is the largest element of G, and (ii) holds with $G\backslash\{\alpha\}$ in place of G.

The converse is clear. □

From results about attainment of tightness in Chapter 12 of Monk [96] we obtain the following upper bound on members of $\mathfrak{f}_{\mathrm{sp}}(A)$. Recall that $\mathrm{t}(A)$ is the *tightness* of A, which is the supremum of the lengths of free sequences in A.

Proposition 1.5. (i) *If A has a free sequence of infinite length α, then there is a $\kappa \in \mathfrak{f}_{\mathrm{sp}}(A)$ such that $|\alpha| \leq \kappa$.*
(ii) *If there is a free sequence $\langle a_\xi : \xi < \alpha\rangle$ such that $|\alpha|$ is the largest size of any free sequence in A, then $|\alpha| \in \mathfrak{f}_{\mathrm{sp}}(A)$.*
(iii) *If $\mathrm{t}(A)$ is a successor cardinal or a limit cardinal of cofinality $> \omega$, then $\mathrm{t}(A) \in \mathfrak{f}_{\mathrm{sp}}(A)$, and in fact $\mathrm{t}(A)$ is the largest member of $\mathfrak{f}_{\mathrm{sp}}(A)$.* □

Proposition 1.6. *Suppose that $\langle a_\xi : \xi < \kappa\rangle$ is a strictly decreasing sequence of elements of a BA A such that $\{a_\xi : \xi < \kappa\}$ generates an ultrafilter on A. Then $\langle a_\xi : \xi < \kappa\rangle$ is a maximal free sequence.*

Proof: $\langle a_\xi : \xi < \kappa\rangle$ is a free sequence by Theorem 1.1. Clearly it is maximal. □

Later we will see that $\mathfrak{p}(A) \leq \mathfrak{f}(A)$ for any atomless BA A. Now Kunen showed in exercise (A10) of VIII in Kunen [80] that there is a model of ZFC with the continuum large and with $\mathscr{P}(\omega)/\mathrm{fin}$ having an ultrafilter generated by a strictly decreasing sequence of length ω_1. Thus by Proposition 1.6 we have $\mathfrak{f}(\mathscr{P}(\omega)/\mathrm{fin}) = \mathfrak{u}(\mathscr{P}(\omega)/\mathrm{fin}) < 2^\omega$ in this model. We do not have any further information about $\mathfrak{f}(\mathscr{P}(\omega)/\mathrm{fin})$.

Proposition 1.7. *Suppose that α and β are infinite ordinals, A has a maximal free sequence of length α, and B has a free sequence of length β. Then $A \times B$ has a maximal free sequence of length $\beta + \alpha$.*

Proof: Let $\langle b_\xi : \xi < \beta\rangle$ be a free sequence in B, and let $\langle a_\xi : \xi < \alpha\rangle$ be a maximal free sequence in A. For any $\xi < \beta + \alpha$ we define

$$c_\xi = \begin{cases} (1, b_\xi) & \text{if } \ \xi < \beta, \\ (1, 0) & \text{if } \ \xi = \beta, \\ (a_\eta, 0) & \text{if } \ \xi = \beta + 1 + \eta. \end{cases}$$

We claim that $\langle c_\xi : \xi < \beta + \alpha \rangle$ is as desired.

To show that $\langle c_\xi : \xi < \beta + \alpha \rangle$ is free, suppose that F and G are finite subsets of $\beta + \alpha$ with $F < G$; we want to show that

$$\prod_{\xi \in F} c_\xi \cdot \prod_{\xi \in G} -c_\xi \neq 0.$$

If $F \subseteq \beta$, this is true via the b_ξ's; otherwise it is true via the a_ξ's.

For maximality, suppose that $(d, e) \in A \times B$; we want to apply 1.4. By 1.4 for A, we have two possibilities.

Case 1. There is a finite $F \subseteq \alpha$ such that $\prod_{\xi \in F} a_\xi \cdot d = 0$. Let $H = \{\beta + 1 + \xi : \xi \in F\}$. Then $\prod_{\xi \in H} c_\xi \cdot (d, e) = (0, 0)$.

Case 2. There are finite $F, G \subseteq \alpha$ such that $F < G$ and $\prod_{\xi \in F} a_\xi \cdot \prod_{\xi \in G} -a_\xi \cdot -d = 0$. Let $H = \{\beta\} \cup \{\beta + 1 + \xi : \xi \in F\}$ and $K = \{\beta + 1 + \xi : \xi \in G\}$. Then $H < K$, and

$$\prod_{\xi \in H} c_\xi \cdot \prod_{\xi \in K} -c_\xi \cdot (-d, -e) = (1, 0) \cdot \prod_{\xi \in F} (a_\xi, 0) \cdot \prod_{\xi \in G} (-a_\xi, 1) \cdot (-d, -e) = (0, 0).$$

□

By $\mathrm{t}'(A)$ we mean the least cardinal greater than the size of each free sequence of A.

Corollary 1.8. *If* $\mathrm{t}'(A) \leq \mathrm{t}'(B)$, *then* $[\mathfrak{f}(A), t'(B))_{\mathrm{card}} \subseteq \mathfrak{f}_{\mathrm{sp}}(A \times B)$. □

Corollary 1.9. $\mathfrak{f}_{\mathrm{sp}}(A) \subseteq \mathfrak{f}_{\mathrm{sp}}(A \times B)$, *for any BA* B. □

Proposition 1.10. $\mathfrak{f}_{\mathrm{sp}}(\prod_{\kappa \in F} \mathrm{Fr}(\kappa)) = [\min F, \max F]_{\mathrm{card}}$ *if* F *is a nonempty finite set of infinite cardinals.*

Proof: By 1.3, 1.8, and 1.9 it suffices to show that $\prod_{\kappa \in F} \mathrm{Fr}(\kappa)$ does not have a maximal free sequence of length less than $\min F$. So, suppose that $\langle a_\xi : \xi < \alpha \rangle$ is a maximal free sequence in $\prod_{\kappa \in F} \mathrm{Fr}(\kappa)$ with $\alpha < \min F$. For each $\kappa \in F$ let x_κ be a free generator of $\mathrm{Fr}(\kappa)$ not in the support of any $a_\xi(\kappa)$. Suppose that $H < G$ are finite subsets of α and $\prod_{\xi \in H} a_\xi \cdot \prod_{\eta \in G} -a_\eta \cdot -x_\kappa = 0$. Then clearly $\prod_{\xi \in H} a_\xi \cdot \prod_{\eta \in G} -a_\eta = 0$, contradiction. Similarly for the other possibility in 1.4. □

In connection with these results, notice that if $\langle x_\alpha : \alpha < \kappa \rangle$ is a system of free generators of A, then this system is a maximal free sequence. In fact, it is clearly a free sequence. To show that it is maximal, suppose that $b \in A$; we check the conditions of 1.4. We may assume that $b \neq 0, 1$. Then there is a finite $M \subseteq \alpha$ and a $\Gamma \subseteq {}^M 2$ such that

$$b = \sum_{\varepsilon \in \Gamma} \prod_{\alpha \in M} x_\alpha^{\varepsilon(\alpha)} \quad \text{and hence} \quad -b = \sum_{\varepsilon \in {}^M 2 \setminus \Gamma} \prod_{\alpha \in M} x_\alpha^{\varepsilon(\alpha)}.$$

If $\forall \varepsilon \in \Gamma\ \exists \xi \in M[\varepsilon(\xi) = 0]$, then 1.4(i) holds with $F = M$. Otherwise, $\forall \varepsilon \in {}^M 2 \setminus \Gamma\ \exists \xi \in M[\varepsilon(\xi) = 0]$, and 1.4(ii) holds with $F = M$ and $G = \emptyset$.

For any α such that $\kappa \leq \alpha < \kappa^+$, we can enumerate the free generators of $\mathrm{Fr}(\kappa)$ in a sequence of length α with no repetitions. Then the argument of the previous paragraph shows that this sequence is maximal free. Thus maximal free sequences can have length a successor ordinal, and lengths with cofinality less than size.

Now we can show that $\mathfrak{f}_{\mathrm{sp}}(A \times B)$ is not in general equal to $\mathfrak{f}_{\mathrm{sp}}(A) \cup \mathfrak{f}_{\mathrm{sp}}(B)$. For example, $\mathfrak{f}_{\mathrm{sp}}(\mathrm{Fr}(\omega) \times \mathrm{Fr}(\omega_2)) = \{\omega, \omega_1, \omega_2\}$, but $\mathfrak{f}_{\mathrm{sp}}(\mathrm{Fr}(\omega)) \cup \mathfrak{f}_{\mathrm{sp}}(\mathrm{Fr}(\omega_2)) = \{\omega, \omega_2\}$.

Proposition 1.11. $\mathfrak{f}_{\mathrm{sp}}(\mathscr{P}(\kappa)) = [\omega, 2^\kappa]_{\mathrm{card}}$ *for any infinite cardinal* κ.

Proof: First we show that $\mathfrak{f}(\mathscr{P}(\omega)) = \omega$. For each $m \in \omega$ let $a_m = \omega \setminus (m+1)$. Thus $\langle a_m : m \in \omega \rangle$ is strictly decreasing, and so by Theorem 1.1 it is a free sequence in $\mathscr{P}(\omega)$. Note that $\{0\} = \omega \setminus a_0$ and $\{m+1\} = a_m \setminus a_{m+1}$ for all $m \in \omega$. Now if $b \in \mathscr{P}(\omega)$ and $b \neq \emptyset$, choose $n \in b$. If $n = 0$, then $-a_0 \cdot -b = 0$, and if $n = m+1$, then $a_m \cdot -a_{m+1} \cdot -b = 0$. It follows that $\langle a_m : m \in \omega \rangle$ is a maximal free sequence. So we have shown that $\mathfrak{f}(\mathscr{P}(\omega)) = \omega$.

Write $\kappa = M \cup N$ with $|M| = \omega$ and $|N| = \kappa$. Then $\mathscr{P}(\kappa) \cong \mathscr{P}(M) \times \mathscr{P}(N)$. Moreover, $\mathscr{P}(N)$ has an independent subset of size 2^κ, and hence a free sequence of that size. So our result follows from Proposition 1.6 and the preceding paragraph. □

Proposition 1.12. *Let L be a linear ordering.*

(i) *If $\langle a_\xi : \xi < \alpha \rangle$ is a strictly increasing sequence with lub b, with α a limit ordinal, then $\langle [a_\xi, b) : \xi < \alpha \rangle$ is a maximal free sequence in* $\mathrm{Intalg}(L)$.

(ii) *If $\langle a_\xi : \xi < \alpha \rangle$ is a strictly decreasing sequence with glb b, with α a limit ordinal, then $\langle [b, a_\xi) : \xi < \alpha \rangle$ is a maximal free sequence in* $\mathrm{Intalg}(L)$.

(iii) *Suppose that $\langle a_\xi : \xi < \alpha \rangle$ is strictly increasing, $\langle b_\xi : \xi < \alpha \rangle$ is strictly decreasing, $\forall \xi < \alpha[a_\xi < b_\xi]$, and there is no element $c \in L$ such that $\forall \xi < \alpha[a_\xi < c < b_\xi]$. Then $\langle [a_\xi, b_\xi) : \xi < \alpha \rangle$ is a maximal free sequence in* $\mathrm{Intalg}(L)$.

Proof: (i): Let x be any nonzero element of $\mathrm{Intalg}(L)$. We consider two cases.

Case 1. For every component $[c,d)$ of x we have $b \le c$ or $d < b$. Clearly then there is a $\xi < \alpha$ such that $[a_\xi, b) \cap [c,d) = \emptyset$ for every component $[c,d)$ of x. Hence $[a_\xi, b) \cap x = \emptyset$, as desired in 1.4.

Case 2. There is a component $[c,d)$ of x such that $c < b \le d$. Then there is a $\xi < \alpha$ such that $[a_\xi, b) \subseteq [c,d) \subseteq x$. So $[a_\xi, b) \cdot -x = \emptyset$, again as desired in 1.4.

The proof of (ii) is similar, but (iii) is more complicated. Clearly $\langle [a_\xi, b_\xi) : \xi < \alpha \rangle$ is a free sequence in $\mathrm{Intalg}(L)$. Now suppose that x is a nonzero element of $\mathrm{Intalg}(L)$. If for every component $[c,d)$ of x there is a $\xi < \alpha$ such that $b_\xi < c$ or $d < a_\xi$, then there is a $\xi < \alpha$ such that $x \cap [a_\xi, b_\xi) = \emptyset$, as desired. So, suppose that there is a component $[c,d)$ of x such that for every $\xi < \alpha$ we have $c \le b_\xi$ and $a_\xi \le d$. Then by the hypothesis of (iii) there is a $\xi < \alpha$ such that $[a_\xi, b_\xi) \subseteq [c,d)$. Hence $[a_\xi, b_\xi) \backslash x = \emptyset$, as desired. □

The following proposition gives a connection between maximal free sequences in a BA A and towers in homomorphic images of A.

Proposition 1.13. *Suppose that $\langle a_\xi : \xi < \alpha \rangle$ is a maximal free sequence in an atomless BA A. For each $\xi \le \alpha$ let F_ξ be an ultrafilter containing the set $\{a_\eta : \eta < \xi\} \cup \{-a_\eta : \xi \le \eta < \alpha\}$. Let $I = \{x \in A : \forall \xi \le \alpha[-x \in F_\xi]\}$. Then I is an ideal in A and $0 < [a_\eta]_I < [a_\xi]_I < 1$ if $\xi < \eta < \alpha$. Moreover, if α is a limit ordinal, then $\prod_{\xi<\alpha} [a_\xi]_I = 0$, while if $\alpha = \beta + 1$ then $[a_\beta]_I$ is an atom of A/I.*

Proof: Clearly I is an ideal on A. Now suppose that $\xi < \eta < \alpha$. If $\nu \le \alpha$ and $a_\eta \cdot -a_\xi \in F_\nu$, then $\eta < \nu$, hence also $\xi < \nu$ and so $a_\xi \in F_\nu$, contradiction. Hence $\forall \nu \le \alpha[-(a_\eta \cdot -a_\xi) \in F_\nu]$, and so $a_\eta \cdot -a_\xi \in I$ and consequently $[a_\eta]_I \le [a_\xi]_I$. Suppose that $[a_\eta]_I = [a_\xi]_I$. Then $a_\xi \cdot -a_\eta \in I$, and so $-a_\xi + a_\eta \in F_{\xi+1}$. Also $a_\xi \in F_{\xi+1}$, so $a_\eta \in F_{\xi+1}$. Since $\xi + 1 \le \eta$, this is a contradiction.

Thus we have shown that $[a_\eta]_I < [a_\xi]_I$ if $\xi < \eta < \alpha$. If $[a_\eta]_I = 0$, then $a_\eta \in I$. But $a_\eta \in F_\alpha$, contradiction. If $[a_\xi]_I = 1$, then $-a_\xi \in I$. But $-a_\xi \in F_0$, contradiction.

Now suppose that $0 < [b]_I < [a_\xi]_I$ for all $\xi < \alpha$. By the maximality of $\langle a_\xi : \xi < \alpha \rangle$ there are then two possibilities. If $\prod_{\xi \in F} a_\xi \cdot b = 0$ for some finite subset F of α, then $[b]_I = 0$, contradiction. Suppose that $\prod_{\xi \in F} a_\xi \cdot \prod_{\eta \in G} -a_\eta \cdot -b = 0$, where $F < G$ are finite subsets of α. If ξ is the greatest member of F and η is the smallest member of G, then $[a_\xi]_I \cdot -[a_\eta]_I \le [b]_I \le [a_\eta]_I$, so that $[a_\xi]_I \cdot -[a_\eta]_I = 0$, contradiction. If ξ is the greatest member of F and $G = \emptyset$, then $[a_\xi]_I \cdot -[b]_I = 0$, hence $[a_\xi]_I \le [b]_I < [a_\xi]_i$, contradiction. If η is the smallest element of G and $F = \emptyset$, then $-[a_\eta]_I \cdot -[b]_I = 0$, so $-[a_\eta]_I \le [b]_I \le [a_\eta]_I$, so that $-[a_\eta]_I = 0$, contradiction. □

Proposition 1.14. *Suppose that $\langle A_i : i \in \omega \rangle$ is a system of infinite BAs.*

(i) *Suppose that $i_0 \in \omega$, and κ is a cardinal such that $\omega \le \kappa < \mathrm{t}'(A_{i_0})$. Then $B \stackrel{\text{def}}{=} \prod^{\mathrm{w}}_{i \in \omega} A_i$ has a maximal free sequence of size κ.*

(ii) $\mathfrak{f}_{\mathrm{sp}}(B) \supseteq [\omega, \lambda)_{\mathrm{card}}$, *where λ is the least cardinal such that no A_i has a free sequence of size λ.*

(iii) *With λ as in (ii), if* $\mathrm{cf}(\lambda) > \omega$ *then* $\mathfrak{f}_{\mathrm{sp}}(B) = [\omega, \lambda)_{\mathrm{card}}$, *while if* $\mathrm{cf}(\lambda) = \omega$ *then* $\mathfrak{f}_{\mathrm{sp}}(B) = [\omega, \lambda]_{\mathrm{card}}$.

(iv) $\mathfrak{f}(B) = \omega$.

PROOF: We may assume that $i_0 = 0$. Let $\langle x_\xi : \xi < \kappa \rangle$ be a free sequence in A_0. We now define a sequence $\langle a_\xi : \xi < \kappa + \omega \rangle$ of elements of B. For $\xi < \kappa$, let

$$a_\xi(i) = \begin{cases} x_\xi & \text{if } i = 0, \\ 1 & \text{if } i > 0. \end{cases}$$

For any $n \in \omega$ define $a_{\kappa+n}$ by

$$a_{\kappa+n}(i) = \begin{cases} 0 & \text{if } i \le n, \\ 1 & \text{if } n < i. \end{cases}$$

Then $\langle a_\xi : \xi < \kappa + \omega \rangle$ is a free sequence. In fact, suppose that $F, G \in [\kappa + \omega]^{<\omega}$ and $F < G$; we want to show that $\prod_{\xi \in F} a_\xi \cdot \prod_{\eta \in G} -a_\eta \neq 0$. If $F \subseteq \kappa$, then $(\prod_{\xi \in F} a_\xi \cdot \prod_{\eta \in G} -a_\eta)(0) \neq 0$ since $\langle x_\xi : \xi < \kappa \rangle$ is a free sequence. If $F \not\subseteq \kappa$, then $(\prod_{\xi \in F} a_\xi \cdot \prod_{\eta \in G} -a_\eta)(n+1) \neq 0$ if $\kappa + n$ is the greatest member of F. To show that $\langle a_\xi : \xi < \kappa + \omega \rangle$ is maximal, let $b \in B$ be given. Choose n greater than each element in the support of b. If b is of type I, then $a_{\kappa+n} \cdot b = 0$. If b is of type II, then $a_{\kappa+n} \cdot -b = 0$.

(ii)–(iv) are immediate from (i). $\square$

Proposition 1.15. *Suppose that ν is an uncountable cardinal, and $\langle \kappa_\xi : \xi < \nu \rangle$ is a system of infinite cardinals. Let $A = \prod^{\mathrm{w}}_{\xi<\nu} \mathrm{Fr}(\kappa_\xi)$. Then*

(i) $[\min_{\xi<\nu} \kappa_\xi, \sup_{\xi<\nu} \kappa_\xi)_{\mathrm{card}} \subseteq \mathfrak{f}_{\mathrm{sp}}(A)$.

(ii) *If* $\min_{\xi<\nu} \kappa_\xi \le \nu$ *and* $\mathrm{cf}(\sup_{\xi<\nu} \kappa_\xi) > \omega$, *then* $\mathfrak{f}_{\mathrm{sp}}(A) = [\min_{\xi<\nu} \kappa_\xi, \sup_{\xi<\nu} \kappa_\xi)_{\mathrm{card}}$.

(iii) *If* $\min_{\xi<\nu} \kappa_\xi \le \nu$ *and* $\mathrm{cf}(\sup_{\xi<\nu} \kappa_\xi) = \omega$, *then* $\mathfrak{f}_{\mathrm{sp}}(A) = \left[\min_{\xi<\nu} \kappa_\xi, \sup_{\xi<\nu} \kappa_\xi\right]_{\mathrm{card}}$.

(iv) *If* $\nu < \min_{\xi<\nu} \kappa_\xi$ *and* $\mathrm{cf}(\sup_{\xi<\nu} \kappa_\xi) > \omega$, *then* $\mathfrak{f}_{\mathrm{sp}}(A) = [\nu, \sup_{\xi<\nu} \kappa_\xi)_{\mathrm{card}}$.

(iv) *If* $\nu < \min_{\xi<\nu} \kappa_\xi$ *and* $\mathrm{cf}(\sup_{\xi<\nu} \kappa_\xi) = \omega$, *then* $\mathfrak{f}_{\mathrm{sp}}(A) = [\nu, \sup_{\xi<\nu} \kappa_\xi]_{\mathrm{card}}$.

(v) *If* $\min_{\xi<\nu} \kappa_\xi \le \nu$, *then* $\mathfrak{f}(A) = \min_{\xi<\nu} \kappa_\xi$.

(vi) *If* $\nu < \min_{\xi<\nu} \kappa_\xi$, *then* $\mathfrak{f}(A) = \nu$.

PROOF: (i) is clear by 1.6 and 1.9. Next we show:

(1) Every maximal free sequence in A has size at least $\min\{\nu, \min_{\xi<\nu} \kappa_\xi\}$.

For, suppose that $\langle f_\xi : \xi < \alpha \rangle$ is a free sequence in A with $\alpha < \min\{\nu, \min_{\xi<\nu} \kappa_\xi\}$. We want to show that it is not maximal. We consider two cases.

Case 1. There is a $\xi < \alpha$ such that f_ξ is of type I. Define g to be of type I, and to have support equal to that of f_ξ, with $g(i)$ a free generator of $\mathrm{Fr}(\kappa_i)$ not in the support of any element $\{f_\eta(i) : \eta < \alpha\}$, for each i in its support. We claim that $\langle f_\xi : \xi < \alpha\rangle^\frown\langle g\rangle$ is still free. If not, there are two possibilities.

Subcase 1.1. There is a finite $F \subseteq \alpha$ such that $\prod_{\eta\in F} f_\eta \cdot g = 0$. We may assume that $\xi \in F$, and this easily gives a contradiction.

Subcase 1.2. There are finite subsets $F < G$ of α such that $\prod_{\eta\in F} f_\eta \cdot \prod_{\eta\in G} -f_\eta \cdot -g = 0$. Then by the choice of g, for every $i \in \mathrm{supp}(g)$ we have $(\prod_{\eta\in F} f_\eta \cdot \prod_{\eta\in G} -f_\eta)(i) = 0$, while for $i \notin \mathrm{supp}(g)$ we have

$$\begin{aligned}\left(\prod_{\eta\in F} f_\eta \cdot \prod_{\eta\in G} -f_\eta\right)(i) &= \left(\prod_{\eta\in F} f_\eta \cdot \prod_{\eta\in G} -f_\eta\right)(i)\cdot 1\\ &= \left(\prod_{\eta\in F} f_\eta \cdot \prod_{\eta\in G} -f_\eta\right)(i)\cdot(-g)(i)\\ &= 0;\end{aligned}$$

hence $\prod_{\eta\in F} f_\eta \cdot \prod_{\eta\in G} -f_\eta = 0$, contradiction.

Case 2. Every f_ξ is of type II. Choose $i \in I\setminus\bigcup_{\xi<\alpha}\mathrm{supp}(f_\xi)$, and let g be such that $g(i)$ is a free generator of $\mathrm{Fr}(\kappa_i)$, with $g(j) = 0$ for all $j \neq i$. Again we claim that $\langle f_\xi : \xi < \alpha\rangle^\frown\langle g\rangle$ is still free. If not, there are two possibilities.

Subcase 2.1. There is a finite $F \subseteq \alpha$ such that $\prod_{\eta\in F} f_\eta \cdot g = 0$ or $\prod_{\eta\in F} f_\eta \cdot -g = 0$. But

$$\left(\prod_{\eta\in F} f_\eta \cdot g\right)(i) = g(i) \neq 0,$$

contradiction; similarly for $\prod_{\eta\in F} f_\eta \cdot -g$.

Subcase 2.2. There are finite subset $F < G$ of α such $\prod_{\eta\in F} f_\eta \cdot \prod_{\eta\in G} -f_\eta \cdot -g = 0$. Then $G \neq \emptyset$ because of the Case 2 condition, and

$$\left(\prod_{\eta\in F} f_\eta \cdot \prod_{\eta\in G} -f_\eta\right)(i) = 0$$

since $G \neq \emptyset$, while for $j \neq i$,

$$\begin{aligned}\left(\prod_{\eta\in F} f_\eta \cdot \prod_{\eta\in G} -f_\eta\right)(j) &= \left(\prod_{\eta\in F} f_\eta \cdot \prod_{\eta\in G} -f_\eta\right)(j)\cdot 1\\ &= \left(\prod_{\eta\in F} f_\eta \cdot \prod_{\eta\in G} -f_\eta\right)(j)\cdot -g(j) = 0;\end{aligned}$$

hence $\prod_{\eta\in F} f_\eta \cdot \prod_{\eta\in G} -f_\eta = 0$, contradiction.

This proves (1).

Now (ii) and (iii) follow by 12.1 and 12.2 of Monk [96].

(2) If $\nu \le \mu < \kappa_0$, then there is a maximal free sequence of length $\nu+\mu+1$.

To prove this, let $\langle x_\xi : \xi < \kappa_0\rangle$ enumerate free generators of $\mathrm{Fr}(\kappa_0)$.

For $\xi, \rho < \nu$ we define

$$f_\xi(\rho) = \begin{cases} x_\xi & \text{if } \rho = 0, \\ 0 & \text{if } \rho = 1+\xi, \\ 1 & \text{otherwise.} \end{cases}$$

For $\eta < \mu$ and $\rho < \nu$ define

$$f_{\nu+\eta}(\rho) = \begin{cases} x_{\nu+\eta} & \text{if } \rho = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Finally, for $\rho < \nu$ define

$$f_{\nu+\mu}(\rho) = \begin{cases} 0 & \text{if } \rho = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly $\langle f_\xi : \xi \le \nu+\mu\rangle$ is a free sequence. Now suppose that $g \in \prod^{\mathrm{w}}_{\xi<\nu} \mathrm{Fr}(\kappa_\xi)$. Let $F = \{\xi < \nu : 1+\xi \in \mathrm{supp}(g)\}$. Choose $\varepsilon \in 2$ so that g^ε is of type I. Then $\prod_{\xi\in F} f_\xi \cdot f_{\nu+\mu} \cdot g^\varepsilon = 0$. So $\langle f_\xi : \xi \le \nu+\mu\rangle$ is maximal.

Thus (2) holds, and (iv) and (v) follow.

(vi) and (vii) are immediate from the preceding conditions. □

This proposition shows that any interval of cardinals can appear as $\mathfrak{f}_{\mathrm{sp}}(A)$ for some atomless BA A, subject only to the cofinality condition indicated. This leaves open the question whether $\mathfrak{f}_{\mathrm{sp}}(A)$ is always an interval of cardinals.

2. $\mathfrak{f}$

We now concentrate on the least member $\mathfrak{f}(A)$ of $\mathfrak{f}_{\mathrm{sp}}(A)$. This is a cardinal similar to many others studied especially for $\mathscr{P}(\omega)/\mathrm{fin}$. Most of our results relate $\mathfrak{f}$ to other functions, namely to $\pi\chi_{\mathrm{inf}}$, $\mathfrak{p}$, $\mathfrak{t}$, $\mathfrak{s}_{\mathrm{mm}}$, and $\mathfrak{i}$, whose definitions we recall below.

By Corollary 1.8 we have

Corollary 2.1. $\mathfrak{f}(A \times B) \leq \min\{\mathfrak{f}(A), \mathfrak{f}(B)\}$. □

A subset X of a BA A is *independent* iff

$$\forall F, G \in [X]^{<\omega} \left[F \cap G = \emptyset \rightarrow \prod_{x \in F} x \cdot \prod_{y \in G} -y \neq 0 \right].$$

As is well-known, X is independent iff it is a set of free generators for the subalgebra which it generates. We define

$$\mathfrak{i}(A) = \min\{|X| : X \text{ is a maximal independent subset of } A\}.$$

Proposition 2.2. *If A and B are atomless and $\mathfrak{f}(A \times B) = \omega$, then $\mathfrak{i}(A) = \omega$ or $\mathfrak{i}(B) = \omega$.*

Proof: Suppose not: $\mathfrak{f}(A \times B) = \omega < \min\{\mathfrak{i}(A), \mathfrak{i}(B)\}$. Let $\langle (a_\xi, b_\xi) : \xi < \alpha \rangle$ be a maximal free sequence in $A \times B$, with α an infinite countable ordinal. Then $\{a_\xi : \xi < \alpha\}$ is contained in a countable atomless subalgebra A' of A. Say that X is an independent set of generators of A'. Then X is not maximal independent in A, so there is a $c \in A \backslash X$ such that $X \cup \{c\}$ is still independent. Hence $c \cdot x \neq 0 \neq -c \cdot x$ for every nonzero element x of A'. Similarly we get a countable atomless subalgebra B' of B and an element $d \in B$ such that $\{b_\xi : \xi < \alpha\} \subseteq B'$ and $d \cdot y \neq 0 \neq -d \cdot y$ for every nonzero $y \in B'$. Now by the maximality of $\langle (a_\xi, b_\xi) : \xi < \alpha \rangle$ we have two cases.

Case 1. $(c, d) \cdot \prod_{\xi \in F} (a_\xi, b_\xi) = (0, 0)$ for some finite subset F of α. By symmetry say that $\prod_{\xi \in F} a_\xi \neq 0$. Then $c \cdot \prod_{\xi \in F} a_\xi = 0$, contradiction.

Case 2. $(c, d) \cdot \prod_{\xi \in F} (a_\xi, b_\xi) \cdot \prod_{\xi \in G} -(a_\xi, b_\xi) = (0, 0)$ for some finite subsets F, G of α with $F < G$. A similar contradiction is reached. □

Proposition 2.3. *Suppose that κ is an uncountable cardinal and I is any nonempty set. Let $A = {}^I \operatorname{Fr}(\kappa)$. Then $\mathfrak{f}(A) = \kappa$.*

Proof: Suppose that $\langle a_\xi : \xi < \alpha \rangle$ is a free sequence in A, with α infinite but with $|\alpha| < \kappa$. For each $i \in I$, let b_i be a free generator of $\operatorname{Fr}(\kappa)$ not in $\bigcup_{\eta < \alpha} \operatorname{supp}(a_\eta(i))$. Clearly $\langle a_\xi : \xi < \alpha \rangle ^\frown \langle b \rangle$ is still free.

Now the Proposition follows by 2.1. □

We now consider the relationship of $\mathfrak{f}$ to other cardinals. See Monk [01] for definitions and background. There are many problems here, so we do not attempt to list all of them, but we formulate some important ones. We restrict ourselves to atomless BAs.

For our first result we need some terminology and notation. A *weak partition* of a BA A is a system of pairwise disjoint elements of A with sum 1. We call it *weak* because we do not assume that all entries are nonzero. A subset X of A is

m-dense, where m is a positive integer, iff for every weak partition $\langle a_i : i < m\rangle$ of A there exist an $x \in X^+$ and an $i < m$ such that $x \leq a_i$. We define

$$\pi\chi_{\inf}(A) = \min\{|X| : X \text{ is } m\text{-dense for every } m \geq 2\}.$$

The notation here comes from a topological equivalent. The π-character of a point x is the smallest size of a collection U of open sets such that each open neighborhood of x contains some element of U. Then $\pi\chi_{\inf}(A)$ is equivalently defined as the least π-character of any ultrafilter on A, thus applying the topological definition to the Stone space of A. The equivalence is proved in Balcar, Simon [91]. Further important facts about $\pi\chi_{\inf}$ can be found in Balcar, Simon [92] and Dow, Steprāns, Watson [96].

Proposition 2.4. *$\pi\chi_{\inf}(A) \leq \mathfrak{f}(A)$ for any atomless BA A.*

PROOF: Suppose that $\langle a_\xi : \xi < \alpha\rangle$ is a maximal free sequence. Suppose that $2 \leq m < \omega$. We claim that

$$\left\{\prod_{\xi\in F} a_\xi : F \in [\alpha]^{<\omega}\right\} \cup \left\{\prod_{\xi\in F} a_\xi \cdot \prod_{\xi\in G} -a_\xi : F, G \in [\alpha]^{<\omega}, F < G\right\}$$

is m-dense. To see this, let $\langle b_i : i < m\rangle$ be a weak partition of A. If there is an $i < m$ such that $\prod_{\xi\in F} a_\xi \cdot \prod_{\xi\in G} -a_\xi \cdot -b_i = 0$ for some finite $F < G$, this is as desired. If for every $i < m$ there is a finite F_i such that $\prod_{\xi\in F_i} a_\xi \cdot b_i = 0$, then with $G = \bigcup_{i<m} F_i$ we have

$$\prod_{\xi\in G} a_\xi = \left(\prod_{\xi\in G} a_\xi\right) \cdot (b_0 + \cdots + b_{m-1}) = 0,$$

contradiction. □

For any BA A, let

$$\mathfrak{p}(A) = \min\{|X| : \sum X = 1 \text{ and } \sum F \neq 1 \text{ for all finite } F \subseteq X\}.$$

Theorem 2.5. *$\mathfrak{p}(A) \leq \mathfrak{f}(A)$ for any atomless BA A.*

PROOF: Let $\langle a_\xi : \xi < \alpha\rangle$ be a maximal free sequence, with α an infinite ordinal. Clearly $\prod_{\xi\in F} a_\xi \neq 0$, for every finite $F \subseteq \alpha$. Suppose that $0 \neq b \leq a_\xi$ for every $\xi < \alpha$. Choose u with $0 < u < b$. First suppose that $\prod_{\xi\in G} a_\xi \cdot u = 0$ for some finite $G \subseteq \alpha$. Now $u < b \leq \prod_{\xi\in G} a_\xi$, so $u = 0$, contradiction. Suppose that $\prod_{\xi\in G} a_\xi \cdot \prod_{\eta\in H} -a_\eta \cdot -u = 0$ with finite $G < H$. If $H \neq \emptyset$, choose $\eta \in H$. Then $u < b \leq a_\eta$, so $-a_\eta < -u$, and it follows that $\prod_{\xi\in G} a_\xi \cdot \prod_{\eta\in H} -a_\eta = \prod_{\xi\in G} a_\xi \cdot \prod_{\eta\in H} -a_\eta \cdot -u = 0$, contradiction. Hence $H = \emptyset$. Hence $\prod_{\xi\in G} a_\xi \leq u < b \leq \prod_{\xi\in G} a_\xi$, contradiction. □

Example 2.6. There is an atomless BA A such that $\mathfrak{f}(A) < \mathfrak{i}(A)$. This is an algebra A of McKenzie, Monk [04]: with $\omega < \kappa < \lambda$ both regular, A has a strictly decreasing sequence of length κ which generates an ultrafilter, while $\mathfrak{i}(A) = \lambda$. See Proposition 1.6.

We define

$$\mathfrak{u}(A) = \min\{|X| : X \text{ generates a nonprincipal ultrafilter on } A\}.$$

Perhaps the most interesting problems concerning $\mathfrak{f}$ are whether there is an atomless BA A such that $\mathfrak{f}(A) < \mathfrak{u}(A)$, or one such that $\mathfrak{u}(A) < \mathfrak{f}(A)$.

A subset X of A is *ideal independent* iff

$$\forall x \in X \forall F \in [X \backslash \{x\}]^{<\omega} \left[x \cdot \prod_{y \in F} -y \neq 0 \right].$$

We define

$$\mathfrak{s}_{\mathrm{mm}}(A) = \min\{|X| : X \text{ is ideal independent in } A\}.$$

For an example with $\mathfrak{f} < \mathfrak{s}_{\mathrm{mm}}$, see Monk [08], proof of Theorem 2.13, and Example 2.6 above. Another interesting problem is whether there is an atomless BA A such that $\mathfrak{s}_{\mathrm{mm}}(A) < \mathfrak{f}(A)$.

A *tower* in a BA A is a subset of $A \backslash \{1\}$ well-ordered by the Boolean ordering, with sum 1.

Example 2.7. There is an atomless BA A such that $\mathfrak{f}(B) < \mathfrak{t}(B)$. Let $A = {}^{\omega_1}\mathrm{Fr}(\kappa)^{\mathrm{w}}$ with $\kappa > \omega_1$, and see Proposition 8(ii) of Monk [01] and Proposition 1.15.

Proposition 2.8. $\mathfrak{f}(A) \leq \mathfrak{t}(A)$ *for any atomless interval* A.

PROOF: By Proposition 1.11, using Proposition 41 of Monk [01]. □

Proposition 2.9. $\mathfrak{f}(A) = \omega$ *for* A *superatomic.*

PROOF: Let $a \in A$ be such that $a/\mathrm{at}(A)$ is an atom. Let $\langle b_\xi : \xi < \kappa \rangle$ enumerate all of the atoms below a. For each $i < \omega$ let $c_i = a \cdot -\sum_{0<j\leq i} b_j$, and let $c_\omega = b_0$. Thus $\langle c_i : i \in \omega \rangle$ is strictly decreasing, and so it is a free sequence. We claim that it is maximal. For, let $d \in A$ be given. Since $c_\omega \cdot d = 0$ or $c_\omega \cdot -d = 0$, maximality follows. □

Since interval algebras do not have uncountable independent subsets, there is no interval algebra A such that $\mathfrak{f}(A) < \mathfrak{i}(A)$. Since a superatomic algebra does not have an infinite independent subset, there is no superatomic algebra A such that $\mathfrak{f}(A) < \mathfrak{i}(A)$.

3. Free sequences of ultrafilters

Recall from the introduction that there is also a topological notion of free sequence, so also a notion of maximal free sequence of ultrafilters. Note that the straightforward method of constructing a maximal free sequence of ultrafilters, namely adding new ultrafilters at the end, one by one, breaks down at limit stages. The results in this section explain why this happens.

Proposition 3.1. *A free sequence $\langle F_\xi : \xi < \alpha\rangle$ of ultrafilters on a BA A is maximal iff $\{F_\xi : \xi < \alpha\}$ is dense in* $\mathrm{Ult}(A)$. ($\mathrm{Ult}(A)$ *is the Stone space of A, and we are dealing here with free sequences in the topological sense.*)

PROOF: Suppose that $\langle F_\xi : \xi < \alpha\rangle$ is a free sequence of ultrafilters. For $\Rightarrow$, suppose that $\{F_\xi : \xi < \alpha\}$ is not dense; let $G \in \mathrm{Ult}(A)\backslash\overline{\{F_\xi : \xi < \alpha\}}$. We claim that $\langle F_\xi : \xi < \alpha\rangle^\frown\langle G\rangle$ is free. Let $F_\alpha = G$. Suppose that $\xi < \alpha + 1$. If $\xi = \alpha$, the desired conclusion is clear. Suppose that $\xi < \alpha$, and $H \in \overline{\{F_\eta : \eta < \xi\}} \cap \overline{\{F_\eta : \xi \leq \eta < \alpha\} \cup \{G\}}$. So $\xi \neq 0$. Since $G \notin \overline{\{F_\eta : \eta < \alpha\}}$, also $G \notin \overline{\{F_\eta : \eta < \xi\}}$, and hence $H \neq G$. Let $a \in H\backslash G$. Then for any $b \in H$ we have $\mathcal{S}(b\cdot a) \cap \{F_\eta : \xi \leq \eta < \alpha\} \neq \emptyset$. Thus $H \in \overline{\{F_\eta : \eta < \xi\}} \cap \overline{\{F_\eta : \xi \leq \eta\}}$, contradiction.

The implication $\Leftarrow$ is clear. □

Theorem 3.2. *For any infinite BA A the following conditions are equivalent:*

(i) *A has a maximal free sequence of ultrafilters;*

(ii) *A is atomic, and there exist an infinite cardinal κ and an isomorphism f of* $\mathrm{Intalg}(\kappa)$ *into A such that $\{f(\{\alpha\}) : \alpha < \kappa\}$ is the set of all atoms of A.*

PROOF: (ii)$\Rightarrow$(i): Assume (ii). For each $\alpha < \kappa$ let F_α be the principal ultrafilter generated by $f(\{\alpha\})$. To show that $\langle F_\alpha : \alpha < \kappa\rangle$ is free, suppose that $\xi < \kappa$. Then $\{F_\eta : \eta < \xi\} \subseteq \mathcal{S}([0, \xi))$ and $\{F_\eta : \xi \leq \eta < \kappa\} \subseteq \mathcal{S}([\xi, \kappa))$. This proves freeness. To prove denseness, for each nonzero $a \in A$, choose $\alpha < \kappa$ such that $f(\{\alpha\}) \leq a$. Then $F_\alpha \in \mathcal{S}(a)$, as desired.

(i)$\Rightarrow$(ii): Let $\langle F_\xi : \xi < \alpha\rangle$ be a maximal free sequence in A, with α an infinite ordinal.

Suppose that A is not atomic. By denseness, there is a smallest $\xi < \alpha$ such that F_ξ has an atomless element a as a member. By freeness, let $y \in A$ be such that $\{F_\eta : \eta \leq \xi\} \subseteq \mathcal{S}(y)$ and $\{F_\eta : \xi < \eta < \alpha\} \subseteq \mathcal{S}(-y)$. Then $a \cdot y \in F_\xi$. Choose b such that $0 < b < a \cdot y$ and $b \in F_\xi$. Then the element $a \cdot y \cdot -b$ is atomless, and since it is $\leq y$, it must be a member of some F_η with $\eta < \xi$, contradiction.

Therefore, A is atomic. By denseness, for each atom a of A there is a $\xi_a < \alpha$ such that $a \in F_{\xi_a}$. So F_{ξ_a} is the principal ultrafilter generated by $\{a\}$. This implies that $\xi_a \neq \xi_b$ for $a \neq b$. Thus ξ is a one-one function, so ξ^{-1} has its natural meaning. Let $\langle \gamma_\eta : \eta < \beta\rangle$ enumerate in increasing order the set $\{\xi_a : a$ an atom of $A\}$. Here β is an infinite ordinal since A is infinite. Now by freeness, for each $\eta < \beta$ choose $h(\eta) \in A$ such that $\{F_\sigma : \sigma < \gamma_\eta\} \subseteq \mathcal{S}(h(\eta))$ and $\{F_\sigma : \gamma_\eta \leq \sigma < \alpha\} \subseteq \mathcal{S}(-h(\eta))$.

(1) If $\eta < \tau$, then $h(\eta) \leq h(\tau)$.

For, suppose to the contrary, and let a be an atom $\leq h(\eta) \cdot -h(\tau)$. Say $\xi_a = \gamma_\sigma$. Now $a \in F_{\xi_a} = F_{\gamma_\sigma}$ and $a \leq h(\eta)$, so $h(\eta) \in F_{\gamma_\sigma}$. It follows that $\gamma_\sigma < \gamma_\eta$, and so $\sigma < \eta$. Hence $\sigma < \tau$. Therefore $F_{\gamma_\sigma} \in \mathcal{S}(h(\tau))$. But $a \leq -h(\tau)$ and $a \in F_{\gamma_\sigma}$, so $-h(\tau) \in F_{\gamma_\sigma}$, contradiction.

(2) $h(0) = 0$.

Suppose not, and let a be an atom $\leq h(0)$. Now $F_{\xi_a} = F_{\gamma_\eta}$ for some $\eta < \beta$. Since $a \in F_{\xi_a}$, it follows that $h(0) \in F_{\gamma_\eta}$. Hence $\gamma_\eta < \gamma_0$, contradiction.

(3) h is one-one.

For, suppose that $\eta < \tau$ and $h(\eta) = h(\tau)$. Then $\gamma_\eta < \gamma_\tau$, and so $F_{\gamma_\eta} \in \mathcal{S}(h(\tau)) = \mathcal{S}(h(\eta))$, hence $\gamma_\eta < \gamma_\eta$, contradiction.

(4) If $\beta = \delta + 1$ for some δ, then $h(\delta) \neq 1$.

For, we have $\{F_\sigma : \gamma_\delta \leq \sigma < \alpha\} \subseteq \mathcal{S}(-h(\delta))$, so $F_{\gamma_\delta} \in \mathcal{S}(-h(\delta))$, hence $-h(\delta) \neq 0$, and (4) holds.

(5) If $\eta + 1 < \beta$, then $\xi^{-1}(\gamma_\eta) \leq h(\eta + 1) \cdot -h(\eta)$.

For, let $a = \xi^{-1}(\gamma_\eta)$. Now $F_{\gamma_\eta} \in \mathcal{S}(h(\eta + 1))$, so $h(\eta + 1) \in F_{\gamma_\eta}$. But also $a \in F_{\xi_a} = F_{\gamma_\eta}$, so $a \cdot h(\eta + 1) \in F_{\gamma_\eta}$. Hence $a \leq h(\eta + 1)$, since a is an atom. Also, $F_{\gamma_\eta} \in \mathcal{S}(-h(\eta))$, so $-h(\eta) \in F_{\gamma_\eta}$ and so $a \leq -h(\eta)$. Thus (5) holds.

The last part of this argument gives

(6) If $\beta = \eta + 1$, then $\xi^{-1}(\gamma_\eta) \leq -h(\eta)$.

(7) If $\eta + 1 < \beta$, then $\xi^{-1}(\gamma_\eta) = h(\eta + 1) \cdot -h(\eta)$.

In fact, let a be an atom $\leq h(\eta + 1) \cdot -h(\eta)$. Say $\xi_a = \gamma_\rho$. Now by (5) and (6) it follows that $\rho = \eta$, so (7) holds.

Similarly, we get

(8) If $\beta = \eta + 1$, then $\xi^{-1}(\gamma_\eta) = -h(\eta)$.

Now we define $f([0, \eta)) = h(\eta)$ for all $\eta < \beta$. Then by the above and Remark 15.2 of Koppelberg [89], f extends to an isomorphism g of Intalg(β) into A. Now let $\eta < \beta$. If $\eta + 1 < \beta$, then

$$g(\{\eta\}) = g([0, \eta + 1)\backslash[0, \eta)) = f([0, \eta + 1)) \cdot -f([0, \eta)) = h(\eta + 1) \cdot -h(\eta),$$

giving an atom of A by (7). Similarly, if $\beta = \eta + 1$, then

$$g(\{\eta\}) = g([0, \infty)\backslash[0, \eta)) = -f(0, \eta) = -h(\eta),$$

again giving an atom.

Clearly every atom of A is obtained in one of these two ways. $\square$

4. Free chains

A natural generalization of the notion of free sequence is as follows. A *free chain* for a BA A is an ordered pair (L, a) such that L is a linear order, $a \in {}^L A$, and for any $F, G \in [L]^{<\omega}$, if $F < G$ then $\prod_{\xi \in F} a_\xi \cdot \prod_{\eta \in G} -a_\eta \neq 0$. We say that (L, a) is a free chain *over* L.

In this section we investigate this notion, and in the next section we consider maximal free chains.

First we define a related topological notion. Let X be a topological space. A *free chain* for X is an ordered pair (L, x) such that L is a linear order, $x \in {}^L X$, and for any $\xi \in L$,

$$\overline{\{x_\eta : \eta < \xi\}} \cap \overline{\{x_\eta : \xi \leq \eta\}} = \emptyset.$$

As in the case of sequences, a BA A has a free chain (L, a) iff $\mathrm{Ult}(A)$ has a free chain (L, x).

For any BA A, we define

$$\mathrm{Length}_{\mathrm{H}+}(A) = \sup\{\mathrm{Length}(B) : B \text{ is a homomorphic image of } A\}.$$

Proposition 4.1. *For any infinite BA A we have*

$$\mathrm{Length}_{\mathrm{H}+}(A) = \sup\{|L| : A \textit{ has a free chain } (L, a)\}.$$

Proof: The proof is just a modification of the proof of Theorem 4.21 of Monk [96]. For $\geq$, suppose that (L, a) is a free chain in A; we will find an ideal I of A such that A/I has a chain of size $|L|$. For each $\xi \in L$ let F_ξ be an ultrafilter on A such that $\{a_\eta : \eta < \xi\} \cup \{-a_\eta : \xi \leq \eta \in L\} \subseteq F_\xi$. Let $Y = \{F_\xi : \xi \in L\}$, and let $I = \{x \in A : Y \subseteq \mathcal{S}(-x)\}$. Clearly I is an ideal in A. We claim that

(1) $\forall \xi, \eta \in L[\xi < \eta \rightarrow a_\eta/I < a_\xi/I]$.

To prove this, suppose that $\xi < \eta$. To show that $a_\eta \cdot -a_\xi \in I$, take any $\rho \in L$. If $\eta < \rho$, then also $\xi < \rho$ and so $a_\xi \in F_\rho$, and it follows that $-a_\eta + a_\xi \in F_\rho$, so that $F_\rho \in \mathcal{S}(-a_\eta + a_\xi)$. If $\rho \leq \eta$, then $-a_\eta \in F_\rho$ and again $F_\rho \in \mathcal{S}(-a_\eta + a_\xi)$. So $a_\eta \cdot -a_\xi \in I$. Thus $a_\eta/I < a_\xi/I]$. Also, $a_\xi \in F_\eta$ and $-a_\eta \in F_\eta$, so it follows that $a_\eta/I \neq a_\xi/I$. Thus (1) holds.

Conversely, suppose that I is an ideal in A and $\langle a_\alpha/I : \alpha \in L\rangle$ is a chain in A/I. Let $\alpha <_L \beta$ iff $a_\alpha/I < a_\beta/I$. This makes L into a linear order. We may assume that $a_0/I \neq 0$ and no a_α/I is equal to 1. We claim then that $\langle -a_\alpha : \alpha \in L\rangle$ is a free chain. For, suppose that $F, G \in [L]^{<\omega}$ with $F < G$. Then if both F and G are nonempty, we have

$$\left(\prod_{\alpha \in F} -a_\alpha \cdot \prod_{\beta \in G} a_\beta\right)/I = \prod_{\alpha \in F}(-(a_\alpha/I)) \cdot \prod_{\beta \in G}(a_\beta/I) = -(a_\alpha/I) \cdot (a_\beta/I),$$

where α is the largest element of F and β is the smallest element of G. So $-(a_\alpha/I)\cdot(a_\beta/I)\neq 0$, and hence $\prod_{\alpha\in F}-a_\alpha\cdot\prod_{\beta\in G}a_\beta\neq 0$. The case when one of F,G is empty is treated similarly. $\square$

Note that in $\mathrm{Intalg}(\mathbb{R})$ every infinite free sequence is countable, while there are uncountable free chains.

5. Maximal free chains

Zorn's lemma can be applied to obtain maximal free chains, for example by considering linear orders on subsets of $|A|^+$. We now define

$$\mathrm{fchn}_{\mathrm{spect}}(A)=\{|L| : A \text{ has an infinite maximal free chain over } L\};$$
$$\mathrm{fchn}_{\mathrm{mm}}(A)=\min(\mathrm{fchn}_{\mathrm{spect}}(A)).$$

Proposition 5.1. $\mathrm{fchn}_{\mathrm{spect}}(\mathrm{Finco}(\kappa))=\{\omega\}$ *for any infinite cardinal* κ.

Proof: This holds by Proposition 5.1 and Corollary 5.29 of Rosenstein [82]. $\square$

Proposition 5.2. $\mathrm{fchn}_{\mathrm{spect}}(\mathrm{Fr}(\kappa))=\{\kappa\}$ *for any infinite cardinal* κ. $\square$

Proposition 5.3. *Suppose that (I,a) is a free chain in A. Then the following conditions are equivalent.*

(i) *(I,a) is maximal.*

(ii) *For all $b\in A$ and all $M,N\subseteq I$, if $M<N$ and $M\cup N=I$, then there exist finite $F\subseteq M$ and $G\subseteq N$ such that one of the following conditions holds:*

(a) $\prod_{\xi\in F}a_\xi\cdot b\cdot\prod_{\eta\in G}-a_\eta=0$;

(b) $\prod_{\xi\in F}a_\xi\cdot -b\cdot\prod_{\eta\in G}-a_\eta=0$. $\square$

Note here that one of M,N,F,G can be empty.

Proposition 5.4. *Suppose that (L,a) is an infinite maximal free chain in A, and (M,b) is an infinite maximal free chain in B. Assume that $M\cap L=\emptyset$, and let m be a set not in $M\cup L$. Order $M\cup\{m\}\cup L$ in the natural order $M<m<L$. Then $A\times B$ has a maximal free chain of the form $(M\cup\{m\}\cup L,c)$.*

Proof: Define

$$c_\xi=\begin{cases}(1,b_\xi) & \text{if } \xi\in M,\\ (1,0) & \text{if } \xi=m,\\ (a_\xi,0) & \text{if } \xi\in L.\end{cases}$$

Then $\langle c_\xi : \xi\in M\cup\{m\}\cup L\rangle$ is a free chain. In fact, suppose that $F,G\in[M\cup\{m\}\cup L]^{<\omega}$ with $F<G$. If $F\subseteq M$, then $\prod_{\xi\in F}c_\xi\cdot\prod_{\eta\in G}-c_\eta\neq 0$ because of the b_ξ's, and if $F\not\subseteq M$, then $\prod_{\xi\in F}c_\xi\cdot\prod_{\eta\in G}-c_\eta\neq 0$ because of the a_ξ's.

Now for maximality, suppose that $C < D$ with $C \cup D = M \cup \{m\} \cup L$, and $(e_0, e_1) \in A \times B$.

Case 1. $C \subseteq M$. Apply maximal freeness of the b_ξ's to the pair $(C, D \cap M)$ to obtain finite $F \subseteq C$ and $G \subseteq D \cap M$ such that $e_1 \cdot \prod_{\xi\in F} b_\xi \cdot \prod_{\eta\in G} -b_\eta = 0$ or $-e_1 \cdot \prod_{\xi\in F} b_\xi \cdot \prod_{\eta\in G} -b_\eta = 0$. Then $(e_0, e_1) \cdot \prod_{\xi\in F} c_\xi \cdot \prod_{\eta\in G} -c_\xi \cdot (0,1) = (0,0)$ or $-(e_0, e_1) \cdot \prod_{\xi\in F} c_\xi \cdot \prod_{\eta\in G} -c_\xi \cdot (0,1) = (0,0)$.

Case 2. $C \not\subseteq M$. Here one can use the maximal freeness of the a_ξ's similarly. □

Proposition 5.5. $\mathfrak{f}(A) \leq \mathrm{fchn}_{\mathrm{mm}}(A)$ *for any infinite BA* A. □

There are many problems concerning $\mathrm{fchn}_{\mathrm{mm}}$.

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References

Balcar B., Simon P. [91], *On minimal π-character of points in extremally disconnected compact spaces*, Topology Appl. **41** (1991), 133–145.

Balcar B., Simon P. [92], *Reaping number and π-character of Boolean algebras*, Discrete Math. **108** (1992), 5–12.

Blass A. [10], *Combinatorial cardinal characteristics of the continuum*, Handbook of Set Theory (Foreman, Kanamori, eds), vol. 1, pp. 395–490.

Dow A., Steprāns J., Watson S. [96], *Reaping numbers of Boolean algebras*, Bull. London Math. Soc. **28** (1996), no. 6, 591–599.

Hausdorff F. [1908], *Grundzüge einer Theorie der geordneten Mengen*, Math. Ann. **65** (1908), 435–482.

Koppelberg S. [89], *The General Theory of Boolean Algebras*, Handbook on Boolean algebras, vol. 1, North-Holland, Amsterdam, 1989, xix+312 pp.

Kunen K. [80], *Set Theory*, North Holland, Amsterdam-New York, 1980, xvi+313 pp.

McKenzie R., Monk J.D. [04], *On some small cardinals for Boolean algebras*, J. Symbolic Logic **69** (2004), no. 3, 674–682.

Monk J.D. [96], *Cardinal Invariants on Boolean Algebras*, Birkhäuser, Basel, 1996, 298 pp.

Monk J.D. [96b], *Minimum sized infinite partitions of Boolean algebras*, Math. Logic Quart. **42** (1996), no. 4, 537–550.

Monk J.D. [01], *Continuum cardinals generalized to Boolean algebras*, J. Symbolic Logic **66** (2001), no. 4, 1928–1958.

Monk J.D. [02], *An atomless interval Boolean algebra A such that* $\mathfrak{a}(A) < \mathfrak{t}(A)$, Algebra Universalis **47** (2002), no. 4, 495–500.

Monk J.D. [08], *Maximal irredundance and maximal ideal independence in Boolean algebras*, J. Symbolic Logic **73** (2008), no. 1, 261–275.

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