Hyperplane section $\mathbb{OP}_0^2$ of the complex Cayley plane as the homogeneous space $\mathrm{F}_4/\mathrm{P}_4$

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Abstract. We prove that the exceptional complex Lie group F_4 has a transitive action on the hyperplane section of the complex Cayley plane $\mathbb{OP}^2$. Although the result itself is not new, our proof is elementary and constructive. We use an explicit realization of the vector and spin actions of $\mathrm{Spin}(9,\mathbb{C}) \leq \mathrm{F}_4$. Moreover, we identify the stabilizer of the F_4-action as a parabolic subgroup P_4 (with Levi factor $\mathrm{B}_3\mathrm{T}_1$) of the complex Lie group F_4. In the real case we obtain an analogous realization of $\mathrm{F}_4{}^{(-20)}/\mathrm{P}_4$.

Keywords: Cayley plane, octonionic contact structure, twistor fibration, parabolic geometry, Severi varieties, hyperplane section, exceptional geometry

Classification: Primary 32M12; Secondary 14M17

1. Introduction

The real octonionic projective plane $\mathbb{OP}^2_{\mathbb{R}}$, also called Cayley plane or octave plane, has been thoroughly treated in the literature. It appears in numerous contexts. It is a projective plane where the Desargues axiom does not hold. It was firstly considered by Ruth Moufang [21], who found a relation of the so called little Desargues axiom and the alternativity of the coordinate ring. It is well known that $\mathbb{OP}^2_{\mathbb{R}}$ is a Riemanian symmetric manifold $\mathrm{F}_4/\mathrm{Spin}(9)$. Due to its relation to the exceptional Jordan algebra $\mathcal{J}_3(\mathbb{O})$, there is also a connection of this plane to a model of quantum mechanics considered by Neumann, Jordan and Wigner [14]. More recently, the authors of [7] show that the Cayley plane consists of normalized solutions of a Dirac equation. For more details and connections with physics we refer to the article by Baez [3].

It is possible to mimic the construction of classical projective plane $\mathbb{RP}^2$ via equivalence classes of triples (see [11]) also in the case of $\mathbb{OP}^2_{\mathbb{R}}$, but usually Freudenthal's approach via the exceptional Jordan algebra $\mathcal{J}_3(\mathbb{O})$ is used. The idea is that lines in space correspond to projectors with one-dimensional image. Hence the Cayley plane can be defined as elements of (real) projectivization of $\mathcal{J}_3(\mathbb{O})$ of rank

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one. Now the rank for octonionic matrices is a bit tricky due to the nonassociativity and requires the definition of Jordan cross product of these matrices. For details we refer to Jacobson's monograph [13]. There one can also find a classification of orbits of the automorphism group of $\mathcal{J}_3(\mathbb{O})$ (which is F_4) from which it follows that $\mathbb{OP}^2_{\mathbb{R}}$ is a homogeneous space. (The isotropy subgroup is determined for example in [10], [22].)

In fact, Jacobson's book [13] treats octonionic algebras over general field and hence we get the definition of the complex Cayley plane $\mathbb{OP}^2$ as well. This space is also of geometric interest, as it is an exceptional member of the Severi varieties — the unique extremal varieties for secant defects. For details see [18], [19].

Now, let us consider the intersection of the complex Cayley plane $\mathbb{OP}^2$ with the hyperplane given by traceless matrices $\mathcal{J}_0 := \{A \in \mathcal{J}_3(\mathbb{O}_{\mathbb{C}}) \mid \operatorname{Tr} A = 0\}$. The resulting space is studied in [18], [19], where the authors call it the generic hyperplane section and denote it by $\mathbb{OP}^2_0$. It is a total space of a certain twistor fibration over the real Cayley plane (see [2], [8]). Because $\mathbb{OP}^2_0$ is a complex projective variety, the stabilizer is a parabolic subgroup of F_4. The authors of [18] state that the isomorphism $\mathbb{OP}^2_0 = \mathrm{F}_4/\mathrm{P}_4$ is suggested by 'geometric folding'. A rigorous proof of this isomorphism can be gleaned from [13]. This proof however requires a lot of the theory of nonassociative algebras, most notably the Jordan coordinization theorem. Quite a short proof can be given using the Borel fixed point theorem. In a hope to make $\mathbb{OP}^2_0$ more accessible to geometrically inclined audience, we present a *constructive* proof of the transitivity of the action of F_4 on $\mathbb{OP}^2_0$ based on the representation theory of complex spin groups. From the theory of nonassociative algebras only Artin's theorem is needed. Following the approach of [10], we explicitly realize the spin groups $\mathrm{Spin}(9,\mathbb{C})$ and $\mathrm{Spin}(8,\mathbb{C})$ as subgroups of F_4 and we use the description of their actions to find the reduction of an arbitrary element to a previously chosen one.

It is well known that the Cartan geometry modeled on the pair $(\mathrm{F}_4, \mathrm{P}_4)$ is rigid, i.e. any regular normal Cartan geometry of this type is locally isomorphic to the homogeneous model. The real version of this pair corresponding to the group $\mathrm{F}_4^{(-20)}$ appears as a conformal infinity of the Einstein space $\mathbb{O}\mathrm{H}^2$ [4]. The geometry obtained is called 'octonionic-contact', because there is a naturally defined eight-dimensional maximally nonintegrable subbundle of the tangent bundle. The contact geometry in the classical sense (studied for example in [15], [16]) is also present among the homogeneous spaces of the group F_4 — namely the one whose isotropy group is the parabolic subgroup corresponding to the other 'outer' simple root of the Lie algebra of $\mathfrak{f}_4$.

After some necessary definitions in Section 2, we describe explicitly the presentations of $\mathrm{Spin}(9,\mathbb{C})$ and $\mathrm{Spin}(8,\mathbb{C})$ inside of $\mathrm{End}(\mathbb{O}^2) \otimes_{\mathbb{R}} \mathbb{C}$ in Section 3. We also explicitly describe vector and spinor representations of $\mathrm{Spin}(9,\mathbb{C})$ in such a way that their image is inside F_4. Section 3 continues with the proof of the transitivity of the action of F_4 on $\mathbb{OP}^2_0$. We conclude by dealing with the real case. In the last section we compute the stabilizer of a point.

2. Notations and definitions

2.1 Complexified octonions and the hyperplane section. For a comprehensive reference on octonionic algebras over any field we refer to [22]. We denote by $\mathbb{O}$ the octonionic algebra over the field of complex numbers. The complex-valued 'norm' on $\mathbb{O}$ is denoted by N. The algebra $\mathbb{O}$ is normed ($N(ab) = N(a)N(b)$) but it fails to be a division ring, since N is isotropic. This algebra is not associative. Nevertheless, it is alternative, which means that the trilinear form (called the associator) $[u, v, w] \mapsto (uv)w - u(vw)$ is completely skew-symmetric. Later on we will use the so called *Artin's theorem* which states that any subalgebra of an alternative algebra generated by two elements is associative. It follows that products involving only two elements can be written without parenthesis unambiguously.

The symbol L_u denotes the operator of left multiplication by u, i.e. $L_u(v) := uv$ for any $v \in \mathbb{O}$. Note that $L_u L_v \neq L_{uv}$ in general due to the nonassociativity of octonionic algebras.

Since there is up to isomorphism only one octonionic algebra over $\mathbb{C}$ we can think of $\mathbb{O}$ in the following way: $\mathbb{O} = \mathbb{O}_{\mathbb{R}} \otimes \mathbb{C} = \mathbb{O}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, where $\mathbb{O}_{\mathbb{R}}$ is the classical real algebra of octonions ([3]). The multiplication on this tensor product is canonically defined by

$$(o_1 \otimes z_1)(o_2 \otimes z_2) := o_1 o_2 \otimes z_1 z_2 \text{ for } o_1,\, o_2 \in \mathbb{O}, z_1\,, z_2 \in \mathbb{C}$$

and conjugation is given by $\overline{o \otimes z} := \bar{o} \otimes z$.

The multiplication of an arbitrary element $o \otimes z \in \mathbb{O}$ by a complex number w is understood in the sense of multiplication by element $1 \otimes w$, i.e. $w(o \otimes z) := o \otimes (wz)$. We identify the elements of $\mathbb{R} \otimes \mathbb{C}$ with complex numbers under the canonical isomorphism $r \otimes w \mapsto rw$, for $r \in \mathbb{R}$, $w \in \mathbb{C}$. The real and imaginary parts of $o \otimes z$ are defined to be $(\Re\, o) \otimes z$ and $(\Im\, o) \otimes z$, where $\Re\, o$ and $\Im\, o$ are the real and purely imaginary part of o respectively.

The mentioned complex valued quadratic form N is given by

$$N(o \otimes z) := o\bar{o}zz, \qquad o \in \mathbb{O}, z \in \mathbb{C}.$$

Following Springer [22], we denote by $\langle \cdot, \cdot \rangle$ the double of the bilinear form associated to N, $\langle x, y \rangle = N(x+y) - N(x) - N(y)$. An octonion $u \in \mathbb{O}$ is pure imaginary if and only if $\langle u, 1 \rangle = 0$.

For later use, we will record here several useful identities which hold in any octonionic algebra and whose proof can also be found in [22]

$$\langle xy, z \rangle = \langle y, \bar{x}z \rangle$$

$$x(\bar{x}y) = N(x)y \tag{1a}$$

$$u(\bar{x}y) + x(\bar{u}y) = \langle u, x \rangle y \tag{1b}$$

$$u(\bar{x}(uy)) = ((u\bar{x})u)y. \tag{1c}$$

Due to the nonassociativity of the algebras involved we need to make clear distinction between associative algebras of $\mathbb{C}$-linear endomorphisms, which we denote by End, and the possibly nonassociative algebras of $n \times n$ matrices with entries in some algebra $\mathbb{F}$ which are denoted by $M(n, \mathbb{F})$.

The conjugation on $\mathbb{O}$ naturally defines the conjugation on $M(n, \mathbb{O})$. The conjugate of an element $A \in M(n, \mathbb{O})$ is denoted by $\bar{A}$. The symbol $\mathrm{Herm}(n, \mathbb{O})$ stands for the set of $n \times n$ hermitian matrices over $\mathbb{O}$, i.e.

$$\mathrm{Herm}(n, \mathbb{O}) = \{A \in M(n, \mathbb{O}) | \bar{A}^T = A\}.$$

We denote the subspace of trace-free matrices by lower index $\mathrm{Herm}_0(n, \mathbb{O})$. All tensor products in this article are taken over the real numbers.

The complex exceptional Jordan algebra $\mathcal{J}_3(\mathbb{O})$ is the vector space $\mathrm{Herm}(3, \mathbb{O})$ endowed with the symmetric product $\circ : \mathrm{Herm}(3, \mathbb{O}) \times \mathrm{Herm}(3, \mathbb{O}) \to \mathrm{Herm}(3, \mathbb{O})$ defined by $A \circ B := \frac{1}{2}(AB + BA)$.

Now we define the basic object of our interest.

Definition 2.1.1. The hyperplane section of the complex Cayley plane $\mathbb{OP}_0^2$ is the projectivization over $\mathbb{C}$ of the following subset of $\mathcal{J}_3(\mathbb{O})$

$$\widehat{\mathbb{OP}}_0^2 := \left\{A \in \mathrm{Herm}(3,\, \mathbb{O}) \middle|\; A^2 = 0,\ \mathrm{tr} A = 0,\ A \neq 0\right\}.$$

2.2 The spin groups. For an n-dimensional complex vector space $\mathbb{V}$ and a nondegenerate quadratic form N on $\mathbb{V}$, we denote the corresponding Clifford algebra by $C\ell(\mathbb{V}, N)$ (our convention is $vv = -N(v)$). The spin group of $C\ell(\mathbb{V}, N)$ is denoted by $\mathrm{Spin}(\mathbb{V}, N)$. It is generated inside $C\ell(\mathbb{V}, N)$ by products uv, $u, v \in \mathbb{V}$ where $N(u) = N(v) = 1$. By $\mathrm{Spin}(n, \mathbb{C})$ we denote the spin group associated to the standard quadratic form $\sum_{i=1}^n z_i^2$ on $\mathbb{C}^n$.

For $w \in \mathbb{C}$ we define *the generalized complex sphere*

$$S^{n-1}(w) = \{0 \neq z \in \mathbb{V} \,|\, N(z) = w^2\}.$$

As a consequence of Witt's theorem we have

Lemma 2.2.1. *The group* $\mathrm{Spin}(n, \mathbb{C})$ *acts transitively via the vector representation on the generalized complex spheres.*

2.3 Complex Lie algebra $\mathfrak{f}_4$. The complex exceptional Lie group F_4 can be defined as the automorphism group of the complex exceptional Jordan algebra $(\mathcal{J}_3(\mathbb{O}), \circ)$ (see [22]). In other words F_4 is the subgroup of $\mathrm{GL}(27, \mathbb{C})$ such that $g \in \mathrm{F}_4$ if and only if $g(A \circ B) = gA \circ gB$ for every $A, B \in \mathrm{Herm}(3, \mathbb{O})$.

The action of F_4 preserves the trace on $\mathrm{Herm}(3, \mathbb{O})$. This can be easily seen from the equality

$$\mathrm{Tr}\, A = \frac{1}{3} \mathrm{Tr}\, (B \mapsto A \circ B).$$

It is easy to verify that the action of $\mathrm{O}(3,\mathbb{C})$ on $\mathrm{Herm}(3,\mathbb{O})$ given by

$$\mathrm{O}(3,\,\mathbb{C}) \ni g \longmapsto (A \mapsto gAg^T), \quad A \in \mathrm{Herm}(3,\mathbb{O})$$

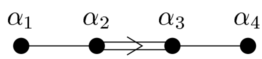
defines an injective group homomorphism $\mathrm{O}(3,\mathbb{C}) \hookrightarrow \mathrm{F}_4$.

Now we present basic facts about the complex simple Lie algebra $\mathfrak{f}_4$ of the group F_4. We shall use these facts as well as the properties of the root system of the Lie algebra $\mathfrak{f}_4$ in the last section of this text. Details can be found in [5].

There exist a choice of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{f}_4$, an orthonormal (with respect to the Killing form of $\mathfrak{f}_4$) basis $\{\epsilon_i\}_{i=1}^4$ of $\mathfrak{h}^*$ and a choice of simple roots

$$\Delta = \left\{\alpha_1 = \epsilon_2 - \epsilon_3,\ \alpha_2 = \epsilon_3 - \epsilon_4,\ \alpha_3 = \epsilon_4,\ \alpha_4 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)\right\}.$$

In this convention the Dynkin diagram is



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The set Δ determines the set of positive roots Φ^+. For any root α, we define the coroot $H_\alpha \in \mathfrak{h}$ by $\lambda(H_\alpha) = 2\langle\lambda,\alpha\rangle/2\langle\alpha,\alpha\rangle$, where $\langle\ ,\ \rangle$ is the Killing form.

The fundamental weights $\{\varpi_i\}_{i=1}^4$ are defined as the dual basis to the simple coroots. We denote the irreducible representation of $\mathfrak{f}_4$ with the highest weight λ by ϱ_λ.

3. Action of F_4 on $\widehat{\mathbb{OP}}_0^2$

In this section we explicitly describe the group $\mathrm{Spin}(9,\mathbb{C})$ as a multiplicative subgroup of $\mathrm{End}(\mathbb{O}^2) \otimes \mathbb{C}$ and construct its representation on $\mathrm{Herm}(3,\mathbb{O})$. Using this representation, we prove that F_4 acts transitively on the hyperplane section $\widehat{\mathbb{OP}}_0^2$. The scalar multiplication on the algebra $\mathrm{End}(\mathbb{O}^2) \otimes \mathbb{C}$ acts only on the first part of the tensor product, i.e. $w \cdot (A \otimes z) = (wA) \otimes z$ for $w, z \in \mathbb{C}$, $A \in \mathrm{End}(\mathbb{O}^2)$.

3.1 Realisation of $\mathrm{Spin}(9,\mathbb{C})$**.** First we need an auxiliary result concerning the Clifford algebra $C\ell(\mathbb{O}, N)$.

Lemma 3.1.1. *The map* $\mu : \mathbb{O} \to \mathrm{End}(\mathbb{O}^2)$ *given by*

$$u \longmapsto \begin{pmatrix} 0 & L_u \\ -L_{\bar{u}} & 0 \end{pmatrix}$$

can be uniquely extended to the isomorphism of complex associative algebras $C\ell(\mathbb{O}, N) \simeq \mathrm{End}(\mathbb{O}^2)$.

Proof: Easy calculation and (1a) shows that $\mu(u)\mu(u) = -N(u)\,\mathrm{Id}$. Using the universal property of Clifford algebras and the fact that the algebra $C\ell(8,\mathbb{C})$ is simple (see [9]), we immediately get the result. $\square$

Let $\mathbb{V}_9$ be the complex vector space $\mathbb{C} \oplus \mathbb{O}$. We define the quadratic form N' by $(r,u) \mapsto r^2 + N(u)$. Let $\kappa : \mathbb{V}_9 \to \mathrm{End}(\mathbb{O}^2) \otimes \mathbb{C}$ be the homomorphism of vector spaces given by

$$\kappa : (r,u) \longmapsto \begin{pmatrix} r & L_u \\ L_{\bar{u}} & -r \end{pmatrix} \otimes \imath,$$

where $\imath$ denotes the imaginary unit in $\mathbb{C}$.

Proposition 3.1.2. *The Clifford algebra $C\ell(\mathbb{V}_9, N')$ is isomorphic (as an associative algebra) to $\mathrm{End}(\mathbb{O}^2) \otimes \mathbb{C}$.*

PROOF: It is known (see e.g. [9]) that $C\ell(\mathbb{V}_9, N') \simeq M(16,\mathbb{C}) \oplus M(16,\mathbb{C})$. Calculation and (1a) shows that $\kappa(r,u)\kappa(r,u) = -N'(r,u)\,\mathrm{Id}$. The universal mapping property of Clifford algebras gives us the following commutative diagram

$$\begin{tikzcd} \mathbb{V}_9 \arrow[r, "i"] \arrow[dr, "\kappa"'] & M(16,\mathbb{C}) \oplus M(16,\mathbb{C}) \arrow[d, "f"] \\ & \mathrm{End}(\mathbb{O}^2) \otimes \mathbb{C}\ . \end{tikzcd}$$

Because $\kappa(-1,0)\kappa(0,u) = \mu(u) \otimes 1$, we see that the image of f generates the subalgebra $\mathrm{End}(\mathbb{O}^2) \otimes 1$. The equality

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \otimes \imath = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \imath \cdot \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} \otimes 1$$

implies that the image of f generates the whole algebra $\mathrm{End}(\mathbb{O}^2) \otimes \mathbb{C}$. Since the dimensions of the considered algebras are the same, it follows that f is an isomorphism. $\square$

Lemma 3.1.3. *The spin group $\mathrm{Spin}(\mathbb{V}_9, N')$ is generated (inside $\mathrm{End}(\mathbb{O}^2) \otimes \mathbb{C}$) by elements of the form*

$$g_{r,u} := \begin{pmatrix} r & -L_u \\ L_{\bar{u}} & r \end{pmatrix} \otimes 1, \quad r \in \mathbb{C},\ u \in \mathbb{O}, \quad r^2 + u\bar{u} = 1.$$

PROOF: The spin group is by definition generated by products of the form $\kappa(r,u)\kappa(s,v)$, where $N'(r,u) = N'(s,v) = 1$. Since $g_{r,u} = \kappa(r,u)\kappa(-1,0)$ and $\kappa(r,u)\kappa(s,v) = g_{r,u}g_{-s,v}$, the lemma follows. $\square$

For brevity we will identify $A \otimes 1 \in \mathrm{End}(\mathbb{O}^2) \otimes \mathbb{C}$ with $A \in \mathrm{End}(\mathbb{O}^2)$ from now on; i.e.

$$g_{r,u} = \begin{pmatrix} r & -L_u \\ L_{\bar{u}} & r \end{pmatrix}\ .$$

3.2 Representations of $\mathrm{Spin}(\mathbb{V}_9, N')$**.** We will use the following decomposition of $\mathrm{Herm}(3,\mathbb{O})$

$$\begin{pmatrix} r_1 & \bar{x}_1 & \bar{x}_2 \\ x_1 & r_2 & x_3 \\ x_2 & \bar{x}_3 & r_3 \end{pmatrix} = \begin{pmatrix} r_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \bar{x}_1 & \bar{x}_2 \\ x_1 & 0 & 0 \\ x_2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & s & x_3 \\ 0 & \bar{x}_3 & -s \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix}$$

in order to define the action of $\mathrm{Spin}(\mathbb{V}_9, N')$ on it. In other words — we take the $\mathbb{C}$-linear isomorphism $\mathrm{Herm}(3,\mathbb{O}) \to \mathbb{C} \oplus \mathbb{O}^2 \oplus \mathrm{Herm}_0(2,\mathbb{O}) \oplus \mathbb{C}$ and we endow each of the spaces in the decomposition with an action of $\mathrm{Spin}(\mathbb{V}_9, N')$. The $\mathbb{O}^2$ summand will be *the spinor part* and we will call the $\mathrm{Herm}(2,\mathbb{O})_0$ summand *the vector part*.

Lemma 3.2.1. *Let Φ be the linear isomorphism between the space of trace-free hermitian matrices* $\mathrm{Herm}_0(2,\mathbb{O})$ *and* $\kappa(\mathbb{V}_9)$ *defined by*

$$\Phi : \begin{pmatrix} s & x \\ \bar{x} & -s \end{pmatrix} \mapsto \begin{pmatrix} s & L_x \\ L_{\bar{x}} & -s \end{pmatrix} \otimes \imath$$

and let ϱ_V be the vector representation of $\mathrm{Spin}(\mathbb{V}_9, N')$.

If we define the representation of $\mathrm{Spin}(\mathbb{V}_9, N')$ *on* $\mathrm{Herm}_0(2,\mathbb{O})$ *by* $\xi_V(g)a := \Phi^{-1}(\varrho_V(g)\Phi(a))$, *the following formula holds for the generators* $g_{r,u}$ *of* $\mathrm{Spin}(\mathbb{V}_9, N')$

$$\xi_V(g_{r,u}) \begin{pmatrix} s & x \\ \bar{x} & -s \end{pmatrix} = \left[\begin{pmatrix} r & -u \\ \bar{u} & r \end{pmatrix} \begin{pmatrix} s & x \\ \bar{x} & -s \end{pmatrix} \right] \begin{pmatrix} r & u \\ -\bar{u} & r \end{pmatrix}$$

$$= \begin{pmatrix} s\big(r^2 - N(u)\big) - r\langle x, u\rangle & 2rsu + r^2 x - u\bar{x}u \\ 2rs\bar{u} + r^2\bar{x} - \bar{u}x\bar{u} & -s\big(r^2 - N(u)\big) + r\langle \bar{x}, \bar{u}\rangle \end{pmatrix}. \tag{2}$$

Proof: The vector representation of $\mathrm{Spin}(\mathbb{V}_9, N')$ is given by $v \mapsto gvg^{-1}$ where v is an element of $\kappa(\mathbb{V}_9)$ and $g \in \mathrm{Spin}(\mathbb{V}_9, N')$. For $g_{r,u} = \kappa(r,u)\kappa(-1,0)$ we get $g_{r,u}^{-1} = g_{r,-u}$.

Thus we have the following formula for $\rho_V(g_{r,u})$ evaluated on $v = \left(\begin{smallmatrix} s & L_x \\ L_{\bar{x}} & -s \end{smallmatrix}\right) \otimes \imath$

$$\begin{pmatrix} s\big(r^2 - N(u)\big) - r(L_u L_{\bar{x}} + L_x L_{\bar{u}}) & 2rsL_u + r^2 L_x - L_u L_{\bar{x}} L_u \\ 2rsL_{\bar{u}} + r^2 L_{\bar{x}} - L_{\bar{u}} L_x L_{\bar{u}} & -s\big(r^2 - N(u)\big) + r(L_{\bar{u}} L_x + L_{\bar{x}} L_u) \end{pmatrix} \otimes \imath.$$

From (1b) we have $L_u L_{\bar{x}} + L_x L_{\bar{u}} = L_{\langle x,u\rangle}$. With the help of the first Moufang identity (1c) we may substitute $L_u L_{\bar{x}} L_u = L_{(u\bar{x})u}$. Applying the isomorphism Φ to the result gives the expression for $\xi_V(g_{r,u})\Phi^{-1}(v)$ which agrees with (2). □

The spinor representation of $\mathrm{Spin}(\mathbb{V}_9, N')$ acts on $\mathbb{O}^2$ by (see Chapter 6 of [9] for details)

$$\xi_S(g_{r,u})(x_1, x_2) = \begin{pmatrix} r & -L_u \\ L_{\bar{u}} & r \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} rx_1 - ux_2 \\ \bar{u}x_1 + rx_2 \end{pmatrix}.$$

We let the $\mathrm{Spin}(\mathbb{V}_9, N')$ act on the rest of the summands of $\mathrm{Herm}(3,\mathbb{O})$ trivially and denote the resulting action by ξ.

Proposition 3.2.2. *The representation ξ is faithful and preserves the Jordan product. In other words* $\mathrm{Spin}(\mathbb{V}_9, N') \simeq \mathrm{Im}(\xi)$ *is a subgroup of* F_4.

Proof: Since the spinor representation ξ_S is faithful, the representation ξ is faithful as well. In order to prove that this action preserves the Jordan product we introduce the following three by three hermitian matrix

$$G_{r,u} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & -u \\ 0 & \bar{u} & r \end{pmatrix} \in \mathrm{Herm}(3, \mathbb{O}),$$

where $(r, u) \in \mathbb{V}_9$ is of unit norm. Straightforward calculations reveal that $G_{r,u}^{-1} = G_{r,-u}$ and that $G_{r,u} A G_{r,u}^{-1}$ gives the expression for the action of $\xi(g_{r,u})$ on A. Moreover the expression $G_{r,u} A G_{r,u}^{-1}$ is unambiguous for any $A \in \mathrm{Herm}(3, \mathbb{O})$.

Put $g = g_{r,u}$, $G = G_{r,u}$ for simplicity. For each $A \in \mathrm{Herm}(3, \mathbb{O})$ we have

$$(\xi(g)A)(\xi(g)A) = (GAG^{-1})(GAG^{-1}).$$

Let us suppose for a moment that $(GAG^{-1})(GAG^{-1}) = G(A(G^{-1}G)A)G^{-1}$. Then we would have

$$(\xi(g)A)(\xi(g)A) = \xi(g)(A^2) \tag{3}$$

for any $A \in \mathrm{Herm}(3, \mathbb{O})$. Using this equality for $A + B$ instead of A we would get on the left hand side

$$\begin{aligned} (\xi(g)(A+B))\,(\xi(g)(A+B)) &= \big(\xi(g)A + \xi(g)B\big)\big(\xi(g)A + \xi(g)B\big) \\ &= (\xi(g)A)^2 + (\xi(g)A)(\xi(g)B) \\ &\quad + (\xi(g)B)(\xi(g)A) + (\xi(g)B)^2, \end{aligned}$$

while the right hand side would equal

$$\xi(g)\left((A+B)^2\right) = \xi(g)(A^2) + \xi(g)(AB) + \xi(g)(BA) + \xi(g)(B^2).$$

Using (3) for $\xi(g)(A^2)$ and $\xi(g)(B^2)$ we would get that

$$(\xi(g)A)(\xi(g)B) + (\xi(g)B)(\xi(g)A) = \xi(g)(AB + BA).$$

So we only need to prove that we can rearrange the brackets in the expression $(GAG^{-1})(GAG^{-1})$. From the Artin's theorem it follows that

$$(u_1 a u_2)(u_3 a u_4) = u_1(a(u_2 u_3)a)u_4,$$

where u_i are elements of the linear span of $\{r, u, \bar{u}\}$ and $a \in \mathbb{O}$ is arbitrary. Using the same trick as above and writing this equality for $a + b$ instead of a we get

$$(u_1 a u_2)(u_3 b u_4) + (u_1 b u_2)(u_3 a u_4) = u_1(a(u_2 u_3)b)u_4 + u_1(b(u_2 u_3)a)u_4.$$

The equation

$$
\begin{aligned}
&((GAG^{-1})(GAG^{-1}))_{a,b}\\
&\quad = \frac{1}{2}\sum_{i,j,\dots,m} (G_{a,i}A_{i,j}G^{-1}_{j,k})(G_{k,l}A_{l,m}G^{-1}_{m,b}) + (G_{a,l}A_{l,m}G^{-1}_{m,k})(G_{k,i}A_{i,j}G^{-1}_{j,b})
\end{aligned}
$$

and the fact that $G_{i,j}$ are from the linear span of $\{r, u, \bar{u}\}$ imply

$$(GAG^{-1})(GAG^{-1}) = G(A(G^{-1}G)A)G^{-1} = GA^2G^{-1}.$$

□

Remark 3.2.3. One could define the representation ξ directly using the matrix $G_{r,u}$. It is however not clear that the expression $G_{r,u}AG^{-1}_{r,u}$ defines a representation due to the nonassociativity of the product of $\mathrm{Herm}(3,\mathbb{O})$.

3.3 The subgroup $\mathrm{Spin}(8,\mathbb{C})$**.** The usual presentation of spin groups gives (see Lemma 3.1.1) the following set of generators of $\mathrm{Spin}(\mathbb{O}, N)$

$$\left\{ \begin{pmatrix} -L_uL_{\bar{v}} & 0 \\ 0 & -L_{\bar{u}}L_v \end{pmatrix} \,\middle|\, u, v \in \mathbb{O},\ N(u) = N(v) = 1 \right\}.$$

One can obtain matrices of this form as products $g_{0,u}g_{0,v}$ which means that these generators are in fact elements of $\mathrm{Spin}(\mathbb{V}_9, N')$. The formula for the restriction of ξ_V to the subgroup $\mathrm{Spin}(\mathbb{O}, N)$

$$\xi_V\left(\begin{pmatrix} L_uL_{\bar{v}} & 0 \\ 0 & L_{\bar{u}}L_v \end{pmatrix}\right)\begin{pmatrix} s & x_3 \\ \bar{x}_3 & -s \end{pmatrix} = \begin{pmatrix} s & u(\bar{v}x_3\bar{v})u \\ \bar{u}(v\bar{x}_3v)\bar{u} & -s \end{pmatrix} \tag{4}$$

is easily proved using (2).

Analogously, the action of $\mathrm{Spin}(\mathbb{O}, N)$ on $\mathbb{O}^2$ is given by

$$\xi_S\left(\begin{pmatrix} L_uL_{\bar{v}} & 0 \\ 0 & L_{\bar{u}}L_v \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u(\bar{v}x_1) \\ \bar{u}(vx_2) \end{pmatrix}, \tag{5}$$

which is the direct sum of two inequivalent spinor representations of $\mathrm{Spin}(\mathbb{O}, N)$. Please note that the quadratic form N is invariant with respect to all the three inequivalent actions of $\mathrm{Spin}(\mathbb{O}, N)$ on the vector space $\mathbb{O}$.

3.4 Transitivity of the F_4 **action on** $\widehat{\mathbb{OP}}_0^2$**.**

Lemma 3.4.1. *Let*

$$A = \begin{pmatrix} -2t & \bar{x}_1 & \bar{x}_2 \\ x_1 & t+s & \bar{x}_3 \\ x_2 & x_3 & t-s \end{pmatrix}$$

be an element of $\widehat{\mathbb{OP}}_0^2$*. Then the vector part of* A *is isotropic (i.e.* $s^2 + N(x_3) = 0$*) if and only if* $N(x_1) = N(x_2) = 0$ *and if and only if* $t = 0$*.*

Proof: The statement is a straightforward consequence of the fact that diagonal elements of A^2 must equal zero. □

Theorem 3.4.2. *The group* F_4 *acts transitively on* $\widehat{\mathbb{OP}}^2_0$. *For every* $A \in \widehat{\mathbb{OP}}^2_0$ *there exists* $g \in \mathrm{F}_4$ *such that*

$$g \cdot A = \begin{pmatrix} \imath & 1 & 0 \\ 1 & -\imath & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

Proof: First we suppose that $A \in \widehat{\mathbb{OP}}^2_0$ has nonisotropic vector part. In such case we can use Lemma 2.2.1 to prove that there exists an element $h_1 \in \mathrm{Spin}(\mathbb{V}_9, N')$ such that

$$\xi(h_1)A = \begin{pmatrix} r_1 & \bar{x}_1 & \bar{x}_2 \\ x_1 & r_2 & 0 \\ x_2 & 0 & r_3 \end{pmatrix}, \text{ with } r_1, r_2, r_3 \in \mathbb{C},\ x_1, x_2 \in \mathbb{O}.$$

Let us denote $\xi(h_1) =: g_1 \in \mathrm{F}_4$. The matrix $(g_1 \cdot A)^2$ has the form

$$\begin{pmatrix} r_1^2 + N(x_1) + N(x_2) & \bar{x}_1(r_1 + r_2) & \bar{x}_2(r_1 + r_3) \\ x_1(r_1 + r_2) & r_2^2 + N(x_2) & x_1\bar{x}_2 \\ x_2(r_1 + r_3) & x_2\bar{x}_1 & r_3^2 + N(x_2) \end{pmatrix}. \tag{7}$$

This is a zero matrix, in particular $N(x_1)N(x_2) = N(x_1\bar{x}_2) = 0$, so x_1 and x_2 cannot be both non-isotropic. On the other hand, they cannot be both isotropic because of Lemma 3.4.1.

Assume first that $N(x_1) \neq 0$ and $N(x_2) = 0$. The action of $\mathrm{Spin}(\mathbb{O}, N)$ preserves the vector part $\left(\begin{smallmatrix} r_2 & 0 \\ 0 & r_3 \end{smallmatrix}\right)$ of $g_1 \cdot A$ because of (4). Let

$$h_2 := \kappa(0, -1)\kappa(0, \frac{x_1}{\sqrt{N(x_1)}}) \in \mathrm{Spin}(\mathbb{O}, N)$$

and $\xi(h_2) =: g_2 \in \mathrm{F}_4$. By (5), g_2 sends the spinor part $x_1 \oplus x_2$ of $g_1 \cdot A$ to $x_1' \oplus x_2'$ where $x_1' = \sqrt{N(x_1)} \in \mathbb{C}$ and $x_2' = \frac{1}{\sqrt{N(x_1)}} x_1 x_2$. The matrix $(g_2 g_1 \cdot A)^2$ has the same form as (7) with x_1 and x_2 substituted by x_1' and x_2'. It is still a zero matrix and its $(2,3)$-position $0 = x_1'\bar{x}_2'$ implies $x_2' = 0$ (x_1' is a nonzero complex number). The other positions of this matrix imply $0 = r_3^2 + N(x_2')$, so $r_3 = 0$, and $r_1^2 + N(x_1') = r_1^2 + (x_1')^2 = 0$, so

$$g_2 g_1 \cdot A = \begin{pmatrix} \pm\imath w & w & 0 \\ w & \mp\imath w & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some $0 \neq w \in \mathbb{C}$.

The case $N(x_1) = 0$, $N(x_2) \neq 0$ leads in a similar way to a matrix of the form $\left(\begin{smallmatrix} \pm\imath w & 0 & w \\ 0 & 0 & 0 \\ w & 0 & \mp\imath w \end{smallmatrix}\right)$, $0 \neq w \in \mathbb{C}$, which can be transformed by the orthogonal matrix $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)$ to the previous one. One can get rid of the sign ambiguity with $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ and the matrix $\left(\begin{smallmatrix} \imath w & w & 0 \\ w & -\imath w & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ can be transformed to the canonical form (6) by conjugating by the orthogonal matrix

$$\begin{pmatrix} \frac{1}{\sqrt{w}} & 0 & \frac{-\imath\sqrt{1-w}}{\sqrt{w}} \\ \frac{-\imath(1-w)}{\sqrt{w}} & \sqrt{w} & -\frac{\sqrt{1-w}}{\sqrt{w}} \\ \imath\sqrt{1-w} & \sqrt{1-w} & 1 \end{pmatrix}.$$

So, $g_3 g_2 g_1 \cdot A$ has the canonical form (6), where g_3 is some element in the image of the embedding $O(3,\mathbb{C}) \hookrightarrow \mathrm{F}_4$ defined in Section 2.3.

If A has isotropic but nonzero vector part, then the preceding lemma implies that the topleft element of A is 0. Using Lemma 2.2.1 we can find an element $g' \in \xi(\mathrm{Spin}(\mathbb{V}_9, N')) \leq \mathrm{F}_4$ such that $g' \cdot A = \left(\begin{smallmatrix} 0 & \bar{x}_1 & \bar{x}_2 \\ x_1 & \imath w & w \\ x_2 & w & -\imath w \end{smallmatrix}\right)$ where $w \neq 0$. Conjugation by $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ leads to a matrix whose top left element is $\imath w \neq 0$. By the previous lemma, such a matrix has nonisotropic vector part and we have reduced this case to the already solved one.

Finally, suppose that A has zero vector part, $A = \left(\begin{smallmatrix} 0 & \bar{x}_1 & \bar{x}_2 \\ x_1 & 0 & 0 \\ x_2 & 0 & 0 \end{smallmatrix}\right)$. This matrix is nonzero by definition. If $x_2 \neq 0$, then the action of $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ transforms it to a matrix with nonzero vector part. The case $x_1 \neq 0$ is treated similarly. $\square$

Remark 3.4.3. We see from the proof that in order to prove transitivity of F_4 on $\mathbb{OP}_0^2$, it is sufficient to consider only discrete subgroup of $\mathrm{O}(3,\mathbb{C})$ isomorphic to S_3 — a permutation group on three letters. This is a manifestation of the triality principle.

Now we prove that the cone $\widehat{\mathbb{OP}}_0^2$ over $\mathbb{OP}_0^2$ is a smooth manifold.

Proposition 3.4.4. *The space $\widehat{\mathbb{OP}}_0^2$ is a smooth manifold of dimension* 32.

Proof: Let as define the smooth map $f : \mathrm{Herm}(3,\mathbb{O})_0 \to \mathrm{Herm}(3,\mathbb{O})_0$ by $f(A) := A^2$. We use the implicit function theorem to show that $\widehat{\mathbb{OP}}_0^2 = f^{-1}(0)\backslash\{0\}$ is a smooth manifold. The differential of f at A is easily proved to be $B \mapsto 2A \circ B$. We already know that F_4 acts transitively on $f^{-1}(0) \setminus \{0\} = \widehat{\mathbb{OP}}_0^2$ and so we have

$$\dim\ker(B \mapsto A \circ B) = \dim\ker\left(B \mapsto g\cdot(A \circ (g^{-1}\cdot B))\right) = \dim\ker\left(B \mapsto (g\cdot A)\circ B\right)$$

for any $g \in \mathrm{F}_4$. So, the differential df of f has constant rank on the set $f^{-1}(0)\backslash\{0\}$ and $\widehat{\mathbb{OP}}_0^2$ is a smooth manifold.

The kernel of the differential of f at the canonical point (6) equals

$$\left\{ \begin{pmatrix} \imath\Re(x_1) & x_1 & x_2 \\ \bar{x}_1 & -\imath\Re(x_1) & -\imath x_2 \\ \bar{x}_2 & -\imath\bar{x}_2 & 2\Re(x_1) \end{pmatrix} \middle| \, x_1, x_2 \in \mathbb{O} \right\}$$

and is isomorphic to the tangent space of $\widehat{\mathbb{OP}}^2_0$ at that point. □

3.5 The real case. By choosing an appropriate involution on $\mathcal{J}_3(\mathbb{O}_\mathbb{C})$ we get a model for $\mathrm{F}_4^{(-20)}/\mathrm{P}_4$ — i.e. the conformal infinity of the Einstein space $\mathbb{O}H^2$. According to Yokota [23] the following real subalgebra of $\mathcal{J}_3(\mathbb{O}_\mathbb{C})$

$$\left\{ A \in \mathcal{J}_3(\mathbb{O}_\mathbb{C}) : I_1 \overline{A}^T I_1 = A, I_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$
$$= \left\{ \begin{pmatrix} r_1 & x_1 & x_2 \\ -\overline{x}_1 & r_2 & x_3 \\ -\overline{x}_2 & \overline{x}_3 & r_3 \end{pmatrix} : x_i \in \mathbb{O}_\mathbb{R}, r_i \in \mathbb{R} \right\}$$

has $\mathrm{F}_4^{(-20)}$ as its automorphism group. By restricting the map κ to $\mathbb{R} \oplus \mathbb{O}_\mathbb{R}$ we get presentation of $\mathrm{Spin}(9, \mathbb{R})$ and the restriction of our representation ξ maps $\mathrm{Spin}(9, \mathbb{R})$ into $\mathrm{F}_4^{(-20)}$. Instead of $\mathrm{O}(3, \mathbb{C})$ we have the compact orthogonal group $\mathrm{O}(3, \mathbb{R})$.

The model of $\mathrm{F}_4^{(-20)}/\mathrm{P}_4$ is given by the same equations as in the complex case. Since there are no isotropic elements in the vector part, the proof of transitivity is now much simpler. By transitivity of $\mathrm{SO}(9, \mathbb{R})$ on spheres we can map any element of our model to a matrix of the form $\begin{pmatrix} -2t & x_1 & x_2 \\ -\overline{x}_1 & t+s & 0 \\ -\overline{x}_2 & 0 & t-s \end{pmatrix}$. The square of this matrix has to be zero by definition which for diagonal elements gives three equations that yield easily $t^2 - s^2 = 0$. The case $t = -s$ leads to $x_1 = 0$ and can be reduced to the case of $t = s$ by conjugation with $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

The case $t = s$ gives $x_2 = 0$ and we can easily find an action of $\mathrm{Spin}(8, \mathbb{R})$ that maps x_1 to a positive real number which gives us a matrix in the form $\begin{pmatrix} -r & x & 0 \\ -x & r & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where all the entries are real and $r^2 = x^2$. We can reduce the case $r = -x$ to the case $r = x$ by conjugation with $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Thus we can map an arbitrary element A from our real Jordan algebra, such that $\mathrm{Tr}\, A = 0$ and $A^2 = 0$, to a matrix of the form $\begin{pmatrix} -x & x & 0 \\ -x & x & 0 \\ 0 & 0 & 0 \end{pmatrix}$ where x is a positive real number. This shows that $\mathrm{F}_4^{(-20)}$ has transitive action on the real projectivization of the appropriate set.

4. Description of the stabilizer of the F_4 action

In this section we will identify the stabilizer of $\mathbb{OP}_0^2$ as a concrete parabolic subgroup of F_4.

Lemma 4.0.1. *There exists up to isomorphism only one irreducible representation ϱ of the group F_4 such that*

$$1 < \dim_{\mathbb{C}} \varrho \leq 26.$$

The highest weight of this representation is $\varpi_4 = \epsilon_1$.

PROOF: Let $\lambda, \mu \in \mathfrak{h}^*$ be two integral dominant weights, $\mu \neq 0$. By a direct application of the Weyl dimensional formula (see Goodman, Wallach [9]), we obtain that $\dim \varrho_{\lambda+\mu} > \dim \varrho_\lambda$. Using the program LiE [20], we get $\dim \rho_{\varpi_1} = 52$, $\dim \varrho_{\varpi_2} = 1274$, $\dim \varrho_{\varpi_3} = 273$ and $\dim \varrho_{\varpi_4} = 26$. By the previous inequality, we see that there is only one irreducible 26-dimensional representations of the Lie algebra $\mathfrak{f}_4$. $\square$

Since $\dim \mathcal{J}_0 = 26$ and all finite dimensional representation of the simple Lie group F_4 are completely reducible, we obtain immediately the following.

Proposition 4.0.2. *The restriction to the defining representation of F_4 on $\mathcal{J}_0 = \mathrm{Herm}(3, \mathbb{O})_0$ is isomorphic to the 26-dimensional irreducible representation ϱ_{ϵ_1}.*

It is clear from definition that $\mathbb{OP}_0^2$ is a projective variety. According to Humphreys [12] this implies that the stabilizer group of any point is a parabolic subgroup of F_4. Since any parabolic subgroup contains Borel subgroup, it follows that the points of the variety are lines spanned by highest weight vectors.

For a fixed choice of the Cartan subalgebra $\mathfrak{h}$ and simple roots Δ there is a $1-1$ correspondence between isomorphism classes of parabolic subalgebras $\mathfrak{p} \subseteq \mathfrak{g}$ and subsets $\Sigma \subseteq \Delta$ of the set Δ of simple roots described e.g. in [6, Chapter 3]. We will denote the parabolic subalgebra corresponding to $\Sigma = \{\alpha_i\}$ by $\mathfrak{p}_i$.

Because the highest weight of $\mathcal{J}_0$ is ϵ_1, the following theorem follows directly from [6, Theorem 3.2.5]. Its proof is not difficult — it is based on the fact that for each $X \in \mathfrak{g}_\alpha$ one can find $Y \in \mathfrak{g}_{-\alpha}$ such that $[Y, X] = H_\alpha$, where $H_\alpha(\lambda) = \langle \lambda, \alpha \rangle$ and the fact that the set of weights is invariant under the action of Weyl group.

Theorem 4.0.3. *Let P be the stabilizer of a point $p \in \mathbb{OP}_0^2$ with respect to the action of the group F_4. Then the Lie algebra $\mathfrak{p}$ of the group P is isomorphic to $\mathfrak{p}_4$.*

Remark 4.0.4. We see that $\widehat{\mathbb{OP}}_0^2$ is the F_4-orbit of the highest weight vector in $\mathcal{J}_0$. Points in $\widehat{\mathbb{OP}}_0^2$ are exactly all possible highest weight vectors for this representation, corresponding to different choices of $\mathfrak{h}$ and Φ^+. The real case can be treated in similar manner with analogous results. See [6] for details.

Remark 4.0.5. From the computation of the harmonic curvature (as done for example in [17], also see [6]) one can prove that the homogeneous space does not admit curved deformations in the sense of regular normal Cartan geometries.

However, if one relaxes the regularity condition there are some deformations of this structure [1].

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