Comment.Math.Univ.Carolin. 52,4 (2011) 611–622

A continuous operator extending fuzzy ultrametrics

I. Stasyuk, E.D. Tymchatyn

Abstract. We consider the problem of simultaneous extension of fuzzy ultrametrics defined on closed subsets of a complete fuzzy ultrametric space. We construct an extension operator that preserves the operation of pointwise minimum of fuzzy ultrametrics with common domain and an operation which is an analogue of multiplication by a constant defined for fuzzy ultrametrics. We prove that the restriction of the extension operator onto the set of continuous, partial fuzzy ultrametrics is continuous with respect to the Hausdorff metric topology.

Keywords: fuzzy ultrametric, continuous extension operator, Hausdorff metric

Classification: 54A40, 54C20, 54E70

1. Introduction

The theory of extensions of metric structures started with Hausdorff's result [7] stating that every compatible metric on a closed subset of a metrizable topological space admits an extension to a compatible metric on the whole space. Hausdorff's theorem has been generalized and rediscovered by many authors and results on extensions of metrics that are analogues of those on extending functions have been obtained. Also, extensions for special classes of metrics such as complete, convex, Lipschitz and ultrametrics have been constructed. The problem of constructing continuous, linear operators extending the cone of continuous metrics defined on a closed subset of a metric space was posed and solved for some special cases by C. Bessaga (see [2]). T. Banakh [1] was first to give its complete solution (see also [11] and [19]). The obtained results are analogues of the Dugundji extension theorem for continuous functions. Further generalizations in this direction are done in constructions of simultaneous extensions of metric structures, i.e., in the case of variable domains. The second named author and M. Zarichnyi in [17] obtained a counterpart for metrics of the result of H.P. Künzi and L. Shapiro [8] on simultaneous, linear, continuous extensions of continuous functions defined on compact subsets of a metric space (see also [16]). The same authors in [18] constructed a continuous operator simultaneously extending continuous ultrametrics defined on closed subsets of a compact, zero-dimensional metric space (see [15] for its generalizations to the non-compact case).

The first named author was supported by Ontario MRI Postdoctoral fellowship at Nipissing University for advanced study and research in Mathematics. The second named author was supported in part by NSERC grant No. OGP 0005616.

In the current note we consider the problem of simultaneous extensions of fuzzy ultrametrics. There are several approaches to defining fuzzy metrics and fuzzy ultrametrics and we will use those introduced by A. George and P. Veeramani in [4] and D. Miheţ in [9]. It is known that for a fuzzy ultrametric space there exists an ultrametric compatible with its topology and so the space is zero-dimensional (see e.g. [5]). A. Savchenko and M. Zarichnyi in [14] proved that every fuzzy ultrametric compatible with the topology of a closed subspace of a fuzzy, separable, zero-dimensional, metrizable space can be extended to a compatible fuzzy ultrametric on the whole space. A. Savchenko in [13] constructed a continuous extension operator for stationary fuzzy metrics defined on closed subsets of a compact metrizable space. We construct an operator extending fuzzy ultrametrics defined on variable, closed subsets of a complete fuzzy ultrametric space. Our extension operator preserves minima of fuzzy ultrametrics with common domains and also preserves the analogue of the operation of multiplication by a constant for fuzzy (ultra)metrics described in [14]. We prove that the restriction of the extension operator onto the set of continuous partial fuzzy ultrametrics is continuous in the sense that if graphs of fuzzy ultrametrics converge in the Hausdorff metric then their extensions converge uniformly on compact sets.

2. Preliminaries

We are going to recall some basic definitions and facts from the theory of fuzzy metric spaces.

Definition 1. A binary operation $*\colon [0,1]\times[0,1]\to[0,1]$ is said to be a t-norm if $*$ is associative, commutative, $a*1=a$ for every $a\in[0,1]$ and $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for any $a,b,c,d\in[0,1]$.

If the map $*$ is also continuous then it is called a continuous t-norm.

Definition 2. A triple $(X,M,*)$ is called a fuzzy metric space if X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X\times X\times(0,\infty)$ with the following properties for every $x,y,z\in X$ and $t,s>0$:

1) $0<M(x,y,t)\le 1$;
2) $M(x,y,t)=1$ if and only if $x=y$;
3) $M(x,y,t)=M(y,x,t)$;
4) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$;
5) $M(x,y,\cdot)\colon(0,\infty)\to(0,1]$ is continuous.

Definition 3. Let $(X,M*)$ be a fuzzy metric space. For $x\in X$, $r\in(0,1)$ and $t>0$ the set $B(x,r,t)=\{y\in X : M(x,y,t)>1-r\}$ is called the open ball of radius r centered at x with respect to t.

It is known that the family of all open balls in a fuzzy metric space forms a base of a topology which is always metrizable.

Definition 4. Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is called Cauchy if for every $\varepsilon \in (0,1)$ and every $t > 0$ there exists a positive integer n_0 such that $M(x_n, x_m, t) > 1 - \varepsilon$ whenever $n, m > n_0$.

Definition 5. A fuzzy metric space is called complete if every Cauchy sequence in it is convergent.

Definition 6. A fuzzy metric space $(X, M, *)$ is called stationary if M does not depend on t that is $M(x, y, t) = M(x, y, s)$ for every $x, y \in X$ and $t, s > 0$. In this case M is said to be a stationary fuzzy metric.

Definition 7. A map $M\colon X \times X \times (0, \infty)$ is called a fuzzy pseudometric if it satisfies conditions 1), 3), 4) and 5) of Definition 2 and additionally

2′) $M(x, y, t) = 1$ whenever $x = y$ for every $t > 0$.

One can give examples of several standard continuous t-norms such as the minimum $\wedge$, the product $\cdot$ or the Lukasiewicz t-norm $\mathcal{L}$ defined by $x\mathcal{L}y = \max\{0, x + y - 1\}$.

Definition 8. A triple $(X, M, *)$ is called a fuzzy ultrametric space if X is a nonempty set, $*$ is the minimum $\wedge$ and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying conditions 1), 2), 3) and 5) of Definition 2 and moreover

4′) $M(x, y, t) * M(y, z, s) \leq M(x, z, \max\{t, s\})$.

It is known (see [9]) that 4′) is equivalent to the condition

$$M(x, y, t) * M(y, z, t) \leq M(x, z, t)$$

for every $x, y, z \in X$ and $t > 0$. Further throughout this paper we assume that the t-norm $*$ is the minimum $\wedge$.

Suppose that $(X, M, \wedge)$ is a complete fuzzy ultrametric space. Then the map $d_M\colon X \times X \to [0, \infty)$ defined by $d_M(x, y) = 1 - \inf_{t>0} M(x, y, t)$ for every $x, y \in X$ is known to be an ultrametric on X (see [5, Corollary 27]). Recall that this means that d_M satisfies the strong triangle inequality $d_M(x, y) \leq \max\{d_M(x, z), d_M(z, y)\}$ for every $x, y, z \in X$. So (X, d_M) is a bounded ultrametric space. Denote by $\exp(X)$ the set of all closed, nonempty subsets of X endowed with the Hausdorff metric topology generated by d_M. The Hausdorff distance generated by d_M between $A, B \in \exp(X)$ is defined by the formula

$$H_{d_M}(A, B) = \max\left\{\sup_{a \in A} d_M(a, B), \sup_{b \in B} d_M(b, A)\right\}.$$

It is known that the Hausdorff distance generated by an ultrametric is an ultrametric itself. Therefore, $(\exp(X), H_{d_M})$ is a bounded ultrametric space.

For every $A \in \exp(X)$ let $\mathcal{F}(A)$ stand for the set of all fuzzy ultrametrics on A. Note that the set $\mathcal{F}(A)$ is closed under the operation $\wedge$ applied pointwise to its elements. Also for every $c \in (0, 1]$ and $N \in \mathcal{F}(A)$ the map $c \odot N\colon A \times A \times (0, \infty)$ defined by $c \odot N(x, y, t) = 1 - c + cN(x, y, t)$ for $x, y \in A$ and $t > 0$ is a fuzzy ultrametric on A (see [14, Lemma 3.7]). We will use the symbol $\odot$ later to indicate the described above operation applied to any real function, not necessarily a fuzzy ultrametric. We write $\operatorname{dom} N = A$ if $N \in \mathcal{F}(A)$. Let $\mathcal{F} = \bigcup\{\mathcal{F}(A) : A \in \exp(X),\ |A| \geq 2\}$ be the family of all partial fuzzy ultrametrics defined on closed, non-degenerate subsets of X. Here we assume that $|\operatorname{dom} N| \geq 2$ for every $N \in \mathcal{F}$ to avoid trivialities. For every $N \in \mathcal{F}$ let

$$\alpha_N = \inf\{N(x,y,t) : (x,y,t) \in \operatorname{dom} N \times \operatorname{dom} N \times (0,\infty)\}.$$

Note that $\alpha_N \in [0,1)$ for every $N \in \mathcal{F}$ because there are at least two distinct points in $\operatorname{dom} N$ for every $N \in \mathcal{F}$. We consider the problem of simultaneous extension of the elements of $\mathcal{F}$ over X.

Let $K = \{(x,A) \in X \times \exp(X) : x \in A\}$ and let σ be the ultrametric on the set $X \times \exp(X)$ defined as $\sigma((x,A),(y,B)) = \max\{d_M(x,y), H_{d_M}(A,B)\}$ for all $(x,A),(y,B) \in X \times \exp(X)$. Thus, K is a closed subset of $X \times \exp(X)$. Denote by $S((x,A),r)$ the open ball of radius $r > 0$ centered at (x,A) in the metric space $(X \times \exp(X), \sigma)$. Let $\mathbb{N}_+$ stand for the set of all positive integers and let $\mathbb{R}^{X\times X\times(0,\infty)}$ denote the set of all real-valued functions on $X \times X \times (0,\infty)$.

3. Extension of fuzzy ultrametrics

In order to construct an extension operator for fuzzy ultrametrics we will need the following result from [15]:

Lemma. *For a complete ultrametric space Y let $\exp Y$ be the set of its closed and bounded subsets with the Hausdorff metric. Then there exists a uniformly continuous function $f\colon Y \times \exp Y \to Y$ such that $f(y,B) \in B$ for every $y \in Y$, $B \in \exp Y$ and $f(y,B) = y$ whenever $y \in B$.*

The following theorem extends in several ways the result of A. Savchenko and M. Zarichnyi (see [14, Theorem 3.15]).

Theorem 1. *Let $(X, M, \wedge)$ be a complete fuzzy ultrametric space. There exists an operator $u\colon \mathcal{F} \to \mathcal{F}(X)$ with the following properties for every $N, P \in \mathcal{F}$ and $c \in (0,1]$:*

(a) *$u(N)$ extends N over X that is $u(N)(x,y,t) = N(x,y,t)$ for every $x, y \in \operatorname{dom} N$ and $t > 0$;*
(b) *$u(c \odot N) = c \odot u(N)$;*
(c) *$u(N \wedge P) = u(N) \wedge u(P)$ whenever $\operatorname{dom} N = \operatorname{dom} P$;*
(d) *$\alpha_{u(N)} = \alpha_N$.*

Proof: It is easy to check that the ultrametric space (X, d_M) is complete since the corresponding fuzzy ultrametric space $(X, M, \wedge)$ is. Since (X, d_M) is bounded we can apply the above lemma to find a uniformly continuous function $f\colon X \times \exp(X) \to X$ such that $f(x,A) \in A$ for every $x \in X$, $A \in \exp(X)$ and $f(x,A) = x$ whenever $x \in A$.

For every $i \in \mathbb{N}_+$ let

$$\mathcal{V}_i = \{S((x,A),1/i) : (x,A) \in X \times \exp(X)\}.$$

Since in every ultrametric space two balls of the same radius either coincide or have empty intersection, the members of $\mathcal{V}_i$ are pairwise disjoint. Recall that $K = \{(x,A) \in X \times \exp(X) : x \in A\}$.

For every $i \in \mathbb{N}_+$ let $V_i = \bigcup\{U \in \mathcal{V}_i : U \cap K \neq \emptyset\}$. So V_i is both open and closed in $X \times \exp(X)$ and $K \subset V_{i+1} \subset V_i$ for every $i \in \mathbb{N}_+$. Then

$$\mathcal{W}_i = \{V_i\} \cup \{V \in \mathcal{V}_i : V \cap K = \emptyset\}$$

is a pairwise disjoint clopen cover of $X \times \exp(X)$ and $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$. For each $i \in \mathbb{N}_+$ define a map $w_i \colon \exp(X) \to \mathbb{R}^{X\times X}$ as follows

$$w_i(A)(x,y) = \begin{cases} \frac{1}{2} & \text{if } (x,A) \text{ and } (y,A) \text{ lie in distinct elements of } \mathcal{W}_i; \\ 1 & \text{if } (x,A) \text{ and } (y,A) \text{ lie in the same element of } \mathcal{W}_i \end{cases}$$

for every $A \in \exp(X)$ and $x, y \in X$. Let $w \colon \exp(X) \to \mathbb{R}^{X\times X}$ be defined by the formula

$$w(A)(x,y) = \min_{i\in\mathbb{N}_+} \left\{ \frac{1}{i} \odot w_i(A)(x,y) \right\}$$

for $x, y \in X$ and $A \in \exp(X)$. We now verify that the map w is well-defined. If $A \in \exp(X)$ and $x, y \in X$ with $x = y$ or $x, y \in A$ then $w_i(A)(x,y) = 1$ for every $i \in \mathbb{N}_+$. Then

$$\frac{1}{i} \odot w_i(A)(x,y) = 1 - \frac{1}{i} + \frac{1}{i} w_i(A)(x,y) = 1$$

for every $i \in \mathbb{N}_+$ and we obtain $w(A)(x,y) = 1$.

Now suppose that $x \neq y$ and x, y are not both in A. Then there exists $i_0 \in \mathbb{N}_+$ such that (x,A) and (y,A) belong to different elements of the cover $\mathcal{W}_{i_0}$ and thus, $w_{i_0}(A)(x,y) = 1/2$. This means that $w_i(A)(x,y) = 1/2$ for every $i > i_0$. We obtain for every $i > i_0$

$$\begin{aligned} \frac{1}{i_0} \odot w_{i_0}(A)(x,y) &= 1 - \frac{1}{i_0} + \frac{1}{i_0} w_{i_0}(A)(x,y) \\ &= 1 - \frac{1}{i_0} + \frac{1}{2i_0} = 1 - \frac{1}{2i_0} < 1 - \frac{1}{2i} = 1 - \frac{1}{i} + \frac{1}{2i} \\ &= 1 - \frac{1}{i} + \frac{1}{i} w_i(A)(x,y) = \frac{1}{i} \odot w_i(A)(x,y). \end{aligned}$$

Therefore,

$$w(A)(x,y) = \min_{1\le i\le i_0} \left\{ \frac{1}{i} \odot w_i(A)(x,y) \right\}$$

and the map w is well-defined. Note that the map $w(A)$ is continuous with respect to x and y for every $A \in \exp(X)$ due to the properties of the functions $w_i(A)$ and continuity of the operations min and $\odot$.

Since $\alpha_N \in [0,1)$ we get $(1-\alpha_N) \in (0,1]$ for every $N \in \mathcal{F}$.

Define a map $u\colon \mathcal{F} \to \mathbb{R}^{X\times X\times(0,\infty)}$ by the formula

$$u(N)(x,y,t) = \min\{N(f(x,\operatorname{dom} N), f(y,\operatorname{dom} N), t), (1-\alpha_N)\odot w(\operatorname{dom} N)(x,y)\}$$

for $N \in \mathcal{F}$, $x,y \in X$ and $t>0$.

Now we are going to show that $u(N)$ is a fuzzy ultrametric on X for every $N \in \mathcal{F}$. Let $N \in \mathcal{F}$ be fixed.

It is clear that the function $N'\colon X\times X\times(0,\infty)$ defined by

$$N'(x,y,t) = N(f(x,\operatorname{dom} N), f(y,\operatorname{dom} N), t)$$

is a fuzzy ultrapseudometric on X.

Each map $w_i(\operatorname{dom} N)$ is a stationary fuzzy ultrapseudometric on X because if $(x,\operatorname{dom} N)$ and $(y,\operatorname{dom} N)$ belong to different elements of the cover $\mathcal{W}_i$, that is $w_i(\operatorname{dom} N)(x,y) = 1/2$ then for arbitrary $z \in X$ we have either $w_i(\operatorname{dom} N)(x,z) = 1/2$ or $w_i(\operatorname{dom} N)(y,z) = 1/2$. Thus,

$$w_i(\operatorname{dom} N)(x,y) \wedge w_i(\operatorname{dom} N)(y,z) \le w_i(\operatorname{dom} N)(x,z).$$

So we conclude that $\frac{1}{i}\odot w_i(\operatorname{dom} N)$ is a stationary fuzzy ultrapseudometric on X for every $i \in \mathbb{N}_+$ and, therefore, so are each of $w(\operatorname{dom} N)$ and $(1-\alpha_N)\odot w(\operatorname{dom} N)$.

This means that $u(N)$ is a fuzzy ultrapseudometric on X as the minimum of two fuzzy ultrapseudometrics on X. To show that $u(N)$ is actually a fuzzy ultrametric choose any distinct $x,y \in X$. If $x,y \in \operatorname{dom} N$ then by the properties of the map f we have $f(x,\operatorname{dom} N) = x \ne y = f(y,\operatorname{dom} N)$. So $N(f(x,\operatorname{dom} N), f(y,\operatorname{dom} N), t) = N(x,y,t) < 1$ and therefore, $u(N)(x,y,t) < 1$ for every $t>0$. If x and y are not both in $\operatorname{dom} N$ then as we noted before there exists $i_0 \in \mathbb{N}_+$ such that $(x,\operatorname{dom} N)$ and $(y,\operatorname{dom} N)$ belong to different elements of the cover $\mathcal{W}_{i_0}$ and thus, $w_{i_0}(\operatorname{dom} N)(x,y) = 1/2$ and

$$\frac{1}{i_0}\odot w_{i_0}(\operatorname{dom} N)(x,y) = 1 - \frac{1}{i_0} + \frac{1}{2i_0} = 1 - \frac{1}{2i_0}.$$

We obtain

$$\begin{aligned}(1-\alpha_N)\odot w(\operatorname{dom} N)(x,y) &= \alpha_N + (1-\alpha_N)w(\operatorname{dom} N)(x,y)\\ &\le \alpha_N + (1-\alpha_N)\left(\frac{1}{i_0}\odot w_{i_0}(\operatorname{dom} N)(x,y)\right)\\ &= \alpha_N + (1-\alpha_N)\left(1-\frac{1}{2i_0}\right) < 1.\end{aligned}$$

As before we get $u(N)(x,y,t) < 1$. Therefore, $u(N)(x,y,t) = 1$ if and only if $x = y$ and we see that $u(N)$ is a fuzzy ultrametric on X.

To show that $u(N)$ extends N over X for every $N \in \mathcal{F}$, take any $x, y \in \operatorname{dom} N$ and $t > 0$. By the properties of the maps f and $\{w_i\}_{i \in \mathbb{N}_+}$ we obtain $f(x, \operatorname{dom} N) = x$, $f(y, \operatorname{dom} N) = y$ and $w_i(\operatorname{dom} N)(x, y) = 1$ for every $i \in \mathbb{N}_+$. Then $\frac{1}{i} \odot w_i(\operatorname{dom} N)(x, y) = 1 - 1/i + 1/i = 1$ for every $i \in \mathbb{N}_+$ and we obtain $w(\operatorname{dom} N)(x, y) = 1$. Also, since

$$(1 - \alpha_N) \odot w(\operatorname{dom} N)(x, y) = \alpha_N + (1 - \alpha_N) = 1$$

we get $u(N)(x, y, t) = \min\{N(x, y, t), 1\} = N(x, y, t)$. So u is an extension operator.

Now let us show that $u(c \odot N) = c \odot u(N)$ for every $c \in (0, 1]$. Fix $N \in F$ and $c \in (0, 1]$. Let $B = \operatorname{dom} N = \operatorname{dom}(c \odot N)$. Then $w(\operatorname{dom} N) = w(\operatorname{dom}(c \odot N)) = w(B)$.

Note that if $c_1, c_2 \in (0, 1]$ then $(c_1 c_2) \odot w(B) = c_1 \odot (c_2 \odot w(B))$. Indeed,

$$\begin{aligned}(c_1 c_2) \odot w(B) &= 1 - c_1 c_2 + c_1 c_2 w(B) = 1 - c_1 + c_1(1 - c_2 + c_2 w(B)) \\ &= c_1 \odot (1 - c_2 + c_2 w(B)) = c_1 \odot (c_2 \odot w(B)).\end{aligned}$$

Also,

$$\begin{aligned}\alpha_{c \odot N} &= \alpha_{1 - c + cN} = \inf\{1 - c + cN(x, y, t) : x, y \in B, t > 0\} \\ &= 1 - c + c \inf\{N(x, y, t) : x, y \in B, t > 0\} = 1 - c + c\alpha_N.\end{aligned}$$

Therefore,

$$\begin{aligned}(1 - \alpha_{c \odot N}) \odot w(B) &= (1 - 1 + c - c\alpha_N) \odot w(B) = (c(1 - \alpha_N)) \odot w(B) \\ &= c \odot ((1 - \alpha_N) \odot w(B)) = 1 - c + c \cdot ((1 - \alpha_N) \odot w(B)).\end{aligned}$$

For every $x, y \in X$ and $t > 0$ we obtain

$$\begin{aligned}&u(c \odot N)(x, y, t) = \min\{c \odot N(f(x, B), f(y, B), t), (1 - \alpha_{c \odot N}) \odot w(B)(x, y)\} \\ &= \min\{1 - c + cN(f(x, B), f(y, B), t), 1 - c + c \cdot ((1 - \alpha_N) \odot w(B)(x, y))\} \\ &= 1 - c + c \min\{N(f(x, B), f(y, B), t), (1 - \alpha_N) \odot w(B)(x, y)\} \\ &= 1 - c + cu(N)(x, y, t) = c \odot u(N)(x, y, t).\end{aligned}$$

So u preserves the operation $\odot$.

Let us now prove that the operator u preserves minima of fuzzy ultrametrics. Let $N, P \in \mathcal{F}$ with $\operatorname{dom} N = \operatorname{dom} P = A$. Then $w(\operatorname{dom}(N \wedge P)) = w(A)$. We obtain

$$\begin{aligned}\alpha_{N \wedge P} &= \inf\{\min\{N, P\}(x, y, t) : x, y \in A,\ t > 0\} \\ &= \min\{\inf\{N(x, y, t) : x, y \in A,\ t > 0\}, \inf\{P(x, y, t) : x, y \in A,\ t > 0\}\} \\ &= \alpha_N \wedge \alpha_P.\end{aligned}$$

Also,

$$\begin{aligned}
&(1-\alpha_{N\wedge P})\odot w(A) = (1-\alpha_N\wedge\alpha_P)\odot w(A)\\
&= \alpha_N\wedge\alpha_P + (1-\alpha_N\wedge\alpha_P)w(A) = w(A) + (\alpha_N\wedge\alpha_P)(1-w(A))\\
&= \min\{w(A)+\alpha_N(1-w(A)), w(A)+\alpha_P(1-w(A))\}\\
&= ((1-\alpha_N)\odot w(A))\wedge((1-\alpha_P)\odot w(A)).
\end{aligned}$$

Therefore, for every $x,y\in X$ and $t>0$ we obtain

$$\begin{aligned}
&u(N\wedge P)(x,y,t) = \min\{(N\wedge P)(f(x,A),f(y,A),t),(1-\alpha_{N\wedge P})\odot w(A)(x,y)\}\\
&= \min\{\min\{N(f(x,A),f(y,A),t),P(f(x,A),f(y,A),t)\},\\
&\qquad (1-\alpha_N\wedge\alpha_P)\odot w(A)(x,y)\}\\
&= \min\big\{N(f(x,A),f(y,A),t),P(f(x,A),f(y,A),t),\\
&\qquad (1-\alpha_N)\odot w(A)(x,y),(1-\alpha_P)\odot w(A)(x,y)\big\}\\
&= u(N)(x,y,t)\wedge u(P)(x,y,t,).
\end{aligned}$$

Finally we show that $\alpha_{u(N)}=\alpha_N$ for every $N\in\mathcal{F}$. On the one hand since $u(N)$ is an extension of N over X we see that $\alpha_{u(N)}\le\alpha_N$. On the other hand since

$$(1-\alpha_N)\odot w(\operatorname{dom} N)(x,y) = \alpha_N + (1-\alpha_N)w(\operatorname{dom} N)(x,y)\in(\alpha_N,1]$$

for every $x,y\in X$ we see that $u(N)(x,y,t)\ge\alpha_N$ for all $x,y\in X$ and $t>0$. Therefore, $\alpha_{u(N)}\ge\alpha_N$. Combining the above we obtain the equality $\alpha_{u(N)}=\alpha_N$. □

4. Continuity of the extension operator

To be able to talk about continuity properties of the extension operator u constructed in the previous section we need to topologize the set of partial fuzzy ultrametrics in some reasonable way. We consider this problem for a particular case of continuous fuzzy ultrametrics. For every $A\in\exp(X)$ let $\mathcal{FC}(A)\subset\mathcal{F}(A)$ be the set of all continuous fuzzy ultrametrics on A (i.e. every element of $\mathcal{FC}(A)$ is a continuous map on the product $X\times X\times(0,\infty)$). Let $\mathcal{FC}=\bigcup\{\mathcal{FC}(A)\colon A\in\exp(X),\ |A|\ge 2\}$. We identify every fuzzy ultrametric P from $\mathcal{FC}$ with its graph

$$\Gamma_P=\{(x,y,t,P(x,y,t)) : x,y\in\operatorname{dom} P,\ t\in(0,\infty)\}$$

which is a closed subset of the space $X\times X\times(0,\infty)\times(0,1]$. Let ρ be the metric on $X\times X\times(0,\infty)\times(0,1]$ defined by

$$\rho[(a,b,t,s),(a',b',t',s')] = d_M(a,a')+d_M(b,b')+|t-t'|+|s-s'|$$

for $a,b,a',b'\in X$, $t,t'\in(0,\infty)$ and $s,s'\in(0,1]$. Recall that d_M is the ultrametric on X defined by the formula $d_M(x,y)=1-\inf_{t>0}M(x,y,t)$ for every

$x, y \in X$. The notion of the Hausdorff distance can be applied not only to closed and bounded subsets of a metric space but more generally to all its closed subsets (see for instance [3, §3.2]). Let H_ρ be the Hausdorff distance on $\exp(X \times X \times (0,\infty) \times (0,1])$ generated by the metric ρ. Then H_ρ takes values in $[0,\infty]$. We consider $\mathcal{FC}$ as a subspace of the metric space $(\exp(X \times X \times (0,\infty) \times (0,1]), H_\rho)$. Note that even though the graphs of partial fuzzy ultrametrics are unbounded sets, the Hausdorff distance between any two of them is always finite and is in fact less than 3. The unboundedness of graphs occurs due to the factor $(0,\infty)$ and all graphs extend indefinitely along this axis while the other factors X and $(0,1]$ are bounded. So $(\mathcal{FC}, H_\rho)$ is a bounded metric space.

Theorem 2. *The restriction $u|_{\mathcal{FC}} \colon \mathcal{FC} \to \mathcal{FC}(X)$ is continuous with respect to the topology on $\mathcal{FC}$ generated by the Hausdorff metric and the topology on $\mathcal{FC}(X)$ of uniform convergence on compact sets.*

Proof: First note that it is clear that u maps every continuous fuzzy ultrametric to a continuous fuzzy ultrametric on X since it is defined as a composition of continuous functions. We have to show that if $\{P_n\}$ is a sequence in $\mathcal{FC}$ converging to $P \in \mathcal{FC}$ then $u(P_n)$ converges to $u(P)$ uniformly on compact subsets of $X \times X \times (0,\infty)$. Let $\{P_n\}$ be a sequence in $\mathcal{FC}$ such that $H_\rho(\Gamma_{P_n}, \Gamma_P) \to 0$ for some $P \in \mathcal{FC}$. This implies $\operatorname{dom} P_n \to \operatorname{dom} P$ in $\exp(X)$ and also $\alpha_{P_n} \to \alpha_P$ as $n \to \infty$. Let $\operatorname{dom} P_n = C_n$ for every $n \in \mathbb{N}_+$ and $\operatorname{dom} P = C$. First let us prove that $w(C_n)$ converges to $w(C)$ uniformly on $X \times X$. Recall that σ is an ultrametric on $X \times \exp(X)$ given by

$$\sigma((x,A),(y,B)) = \max\{d_M(x,y), H_{d_M}(A,B)\}$$

for all $(x,A),(y,B) \in X \times \exp(X)$ where H_{d_M} is the Hausdorff ultrametric generated by d_M. Choose any $i_0 \in \mathbb{N}_+$. Since $C_n \to C$ there exists $n_0 \in \mathbb{N}_+$ such that $H_{d_M}(C, C_n) < 1/i_0$ for $n > n_0$. Let $x, y \in X$. We obtain

$$\sigma((x,C_n),(x,C)) < \frac{1}{i_0} \quad \text{and} \quad \sigma((y,C_n),(y,C)) < \frac{1}{i_0}$$

for $n > n_0$. This means that

$$(x,C_n),(x,C) \in W_{i_0} \subset W_{i_0-1} \subset \cdots \subset W_1$$

and

$$(y,C_n),(y,C) \in W'_{i_0} \subset W'_{i_0-1} \subset \cdots \subset W'_1,$$

where $W_i, W'_i \in \mathcal{W}_i$, $i \in \{1,\dots,i_0\}$ (recall that $\{\mathcal{W}_i\}_{i=1}^\infty$ is the system of covers of the space $X \times \exp(X)$ defined in the proof of Theorem 1). Therefore, $w_i(C_n)(x,y) = w_i(C)(x,y)$ for $i \in \{1,\dots,i_0\}$. Now for $i_1, i_2 > i_0$ and $n > n_0$ we obtain

$$\left|\frac{1}{i_1} \odot w_{i_1}(C)(x,y) - \frac{1}{i_2} \odot w_{i_2}(C_n)(x,y)\right|$$

$$\leq \left|\frac{1}{i_1}-\frac{1}{i_2}\right| + \left|\frac{1}{i_1}w_{i_1}(C)(x,y)-\frac{1}{i_2}w_{i_2}(C_n)(x,y)\right|$$
$$\leq \frac{2}{\min\{i_1,i_2\}} < \frac{2}{i_0}\,.$$

We get $|w(C)(x,y)-w(C_n)(x,y)| < 2/i_0$ and so $w(C_n)$ converges to $w(C)$ uniformly on $X\times X$.

Using the fact that the numerical sequence $\{\alpha_{P_n}\}$ converges to α_P we see that the sequence $\{(1-\alpha_{P_n})\odot w(C_n)\}$ converges uniformly to $(1-\alpha_P)\odot w(C)$ on $X\times X$.

Now fix arbitrary $\varepsilon>0$. Let $\{(x_n,y_n,t_n)\}$ be a sequence in $X\times X\times(0,\infty)$ converging to some point (x_0,y_0,t_0). We are going to prove that $u(P_n)$ converges continuously to $u(P)$ on $X\times X\times(0,\infty)$ which in our case is equivalent to the uniform convergence on compact subsets of $X\times X\times(0,\infty)$ (see [10, p. 109]). So we need to show that the sequence $\{u(P_n)(x_n,y_n,t_n)\}$ converges to $u(P)(x_0,y_0,t_0)$. Since P is continuous at the point $(f(x_0,C),f(y_0,C),t_0)$ one can find $\delta>0$ such that

$$|P(a,b,t)-P(f(x_0,C),f(y_0,C),t_0)| < \frac{\varepsilon}{2}$$

for every $a,b\in C$ with $d_M(a,f(x_0,C))<\delta$, $d_M(b,f(y_0,C))<\delta$ and $|t-t_0|<\delta$. Since $C_n\to C$ as $n\to\infty$ and f is (uniformly) continuous, there is $n_1\in\mathbb{N}_+$ such that

$$d_M(f(x_0,C),f(x_n,C_n))<\delta,\ d_M(f(y_0,C),f(y_n,C_n))<\delta \ \text{ and } \ |t_0-t_n|<\frac{\delta}{2}$$

for all $n>n_1$.

Since $\Gamma_{P_n}\to\Gamma_P$ as $n\to\infty$, there is $n_2\in\mathbb{N}_+$ such that for every $n>n_2$ there exist $a_n,b_n\in C$, $s_n>0$ with

$$d_M(a_n,f(x_n,C_n))<\delta,\ d_M(b_n,f(y_n,C_n))<\delta,\ |s_n-t_n|<\frac{\delta}{2}$$

and

$$|P(a_n,b_n,s_n)-P_n(f(x_n,C_n),f(y_n,C_n),t_n)| < \frac{\varepsilon}{2}\,.$$

Since d_M is an ultrametric we obtain

$$d_M(f(x_0,C),a_n)<\delta,\ d_M(f(y_0,C),b_n)<\delta \ \text{ and } \ |t_0-s_n|<\delta$$

for every $n>\max\{n_1,n_2\}$. Therefore,

$$\begin{aligned}
&|P(f(x_0,C),f(y_0,C),t_0)-P_n(f(x_n,C_n),f(y_n,C_n),t_n)|\\
&\leq |P(f(x_0,C),f(y_0,C),t_0)-P(a_n,b_n,s_n)|\\
&\quad+|P(a_n,b_n,s_n)-P_n(f(x_n,C_n),f(y_n,C_n),t_n)|\\
&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

From the fact that $(1-\alpha_{P_n})\odot w(C_n)$ converges uniformly to $(1-\alpha_P)\odot w(C)$ on $X\times X$ we get

$$|(1-\alpha_{P_n})\odot w(C_n)(x_n,y_n)-(1-\alpha_P)\odot w(C)(x_0,y_0)|\to 0,\ n\to\infty$$

which implies

$$|u(P_n)(x_n,y_n,t_n)-u(P)(x_0,y_0,t_0)|\to 0,\ n\to\infty.$$

So the restriction $u|_{\mathcal{FC}}\colon \mathcal{FC}\to\mathcal{FC}(X)$ is a continuous map. $\square$

References

[1] Banakh T., AE(0)*–spaces and regular operators extending (averaging) pseudometrics*, Bull. Polish Acad. Sci. Math. **42** (1994), no. 3, 197–206.

[2] Bessaga C., *On linear operators and functors extending pseudometrics*, Fund. Math. **142** (1993), no. 2, 101-122.

[3] Beer G., *Topologies on Closed and Closed Convex Sets*, Kluwer Academic, Dordrecht, 1993.

[4] George A., Veeramani P., *On some results of analysis for fuzzy metric spaces*, Fuzzy Sets and Systems **90** (1997), 365–368.

[5] Gregori V., Morillas S., Sapena A., *On a class of completable fuzzy metric spaces*, Fuzzy Sets and Systems **161** (2010), 2193–2205.

[6] de Groot J., *Non-archimedean metrics in topology*, Proc. Amer. Math. Soc. **7** (1956), 948–953.

[7] Hausdorff F., *Erweiterung einer Homöomorphie*, Fund. Math. **16** (1930), 353–360.

[8] Künzi H.P., Shapiro L.B., *On simultaneous extension of continuous partial functions*, Proc. Amer. Math. Soc. **125** (1997), 1853–1859.

[9] Miheţ D., *Fuzzy ψ-contractive mappings in non-Archimedean fuzzy metric spaces*, Fuzzy Sets and Systems **159** (2008), no. 6, 739–744.

[10] Narici L., Beckenstein E., *Topological vector spaces*, Pure and Applied Mathematics, 95, Marcel Dekker, New York-Basel, 1985.

[11] Pikhurko O., *Extending metrics in compact pairs*, Mat. Stud. **3** (1994), 103–106.

[12] Repovš D., Savchenko A., Zarichnyi M., *Fuzzy Prokhorov metric on the set of probability measures*, Fuzzy Sets and Systems **175** (2011), 96–104.

[13] Savchenko A., *Extension of fuzzy metrics*, Carpathian Mathematical Publications **2** (2010), no. 2, 111–115 (in Ukrainian).

[14] Savchenko A., Zarichnyi M., *Fuzzy ultrametrics on the set of probability measures*, Topology **48** (2009), 130–136.

[15] Stasyuk I., Tymchatyn E.D., *A continuous operator extending ultrametrics*, Comment. Math. Univ. Carolin. **50** (2009), no. 1, 141–151.

[16] Stasyuk I., Tymchatyn E.D., *On continuous linear operators extending metrics*, submitted to Proc. Amer. Math. Soc.

[17] Tymchatyn E.D., Zarichnyi M., *On simultaneous linear extensions of partial (pseudo) metrics*, Proc. Amer. Math. Soc. **132** (2004), 2799–2807.

[18] Tymchatyn E.D., Zarichnyi M., *A note on operators extending partial ultrametrics*, Comment. Math. Univ. Carolin. **46** (2005), no. 3, 515–524.

[19] Zarichnyi M., *Regular linear operators extending metrics: a short proof*, Bull. Polish Acad. Sci. Math. **44** (1996), no. 3, 267–269.

Department of Mechanics and Mathematics, Lviv National University, Universytetska St. 1, 79000 Lviv, Ukraine

Current address:
Department of Computer Science and Mathematics, Nipissing University, 100 College Drive, Box 5002, North Bay, ON, P1B 8L7, Canada

E-mail: i_stasyuk@yahoo.com

Department of Mathematics and Statistics, McLean Hall, University of Saskatchewan, 106 Wiggins Road, Saskatoon, SK, S7N 5E6, Canada

E-mail: tymchat@math.usask.ca

(*Received* November 30, 2010, *revised* July 13, 2011)