On noncompact perturbation of nonconvex sweeping process

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Abstract. We prove a theorem on the existence of solutions of a first order functional differential inclusion governed by a class of nonconvex sweeping process with a noncompact perturbation.

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1. Introduction

The aim of this paper is to prove the existence of solutions of the following nonconvex differential inclusions

$$\text{(1.1)} \qquad \begin{cases} \dot{x}(t) \in -N^p_{C(t)}(x(t)) + F(t, T(t)x) & \text{a.e on } [0,\tau]; \\ x(t) = \varphi(t) \quad \forall t \in [-a, 0]; \\ x(t) \in C(t) \quad \forall t \in [0,\tau], \end{cases}$$

where C is a nonconvex set-valued mapping, $N^p_{C(t)}(x(t))$ denotes a prescribed normal cone to the set $C(t)$ at $x(t)$, F is a set-valued mapping with nonconvex and noncompact values, and φ is a continuous function.

The evolution problem (1.1) is generally called the sweeping process. It has been introduced and studied by Moreau (without memory), in the setting where all sets $C(t)$ are assumed to be convex (see for example [9]). Note that, the sweeping process is related to the modelization of elasto-plastic materials (see for example [10] and [11]).

The differential inclusions (1.1), with $C(t)$ convex or the complement of the interior of a convex set, have been considered by several authors (see [4], [13] and the references therein). Recently, using important properties of uniformly ρ-prox-regular sets developed in [2] and [12], the existence of solutions of the sweeping process with convex or nonconvex perturbation is established (see for example [6] and [8]). Remark that, in all the cited papers, the compactness assumption on the perturbation is widely used.

In this paper, our main purpose is to obtain the existence of solutions of (1.1), in the case when $C(t)$ is uniformly ρ-prox-regular and the perturbation $F(\cdot,\cdot)$ is nonconvex, noncompact, integrably bounded, measurable with respect to the first argument and Lipschitz continuous with respect to the second argument.

2. Preliminaries and statement of the main result

Let H be a real separable Hilbert space with the norm $\|\cdot\|$ and the scalar product $\langle\cdot,\cdot\rangle$. For I a segment in $\mathbb{R}$, we denote by $\mathcal{C}(I,H)$ the Banach space of continuous functions from I to H equipped with the norm $\|x(\cdot)\|_\infty := \sup\{\|x(t)\|; t\in I\}$. For a a positive number, we put $\mathcal{C}_a := \mathcal{C}([-a,0],H)$ and for any $t\in[0,T]$, $T>0$, we define the operator $T(t)$ from $\mathcal{C}([-a,T],H)$ to $\mathcal{C}_a$ with $(T(t)(x(\cdot)))(s) := (T(t)x)(s) := x(t+s)$, $s\in[-a,0]$. For $x\in H$ and $r>0$ let $B(x,r) := \{y\in H; \|y-x\|<r\}$ be the open ball centered at x with radius r and $\overline{B}(x,r)$ be its closure and put $B = B(0,1)$. For $\varphi\in\mathcal{C}_a$ and $r>0$ let $B_a(\varphi,r) := \{\psi\in\mathcal{C}_a; \|\psi-\varphi\|_\infty<r\}$ be the open ball centered at φ with radius r and $\overline{B}_a(\varphi,r)$ be its closure. For $x\in H$ and for nonempty subsets A,B of H we denote $d_A(x)$ or $d(x,A)$ the real $\inf\{\|y-x\|; y\in A\}$, $e(A,B) := \sup\{d_B(x); x\in A\}$ and $H(A,B) = \max\{e(A,B),e(B,A)\}$. For measurability purpose, H (resp. $\Omega\subset H$) is endowed with the σ-algebra $B(H)$ (resp. $B(\Omega)$) of Borel subsets for the strong topology and the segment I is endowed with Lebesgue measure and the σ-algebra of Lebesgue measurable subsets. A multifunction is said to be measurable if its graph is measurable. For more details on measurability theory, we refer the reader to book of Castaing-Valadier [3].

We need first to recall some notations and definitions that will be used in all the paper.

Let $V : H\to\mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous function and x be any point where V is finite. The proximal subdifferential $\partial^p V(x)$ of V at x is the set of all $y\in H$, for which there exist $\delta,\sigma>0$ such that for all $x'\in x+\delta\overline{B}$

$$\langle y, x'-x\rangle \le V(x')-V(x)+\sigma\|x'-x\|^2.$$

Let S be a nonempty closed subset of H and x be a point in S. We recall (see [5]) that the proximal normal cone of S at x is defined by $N_S^p(x) := \partial^p\psi_S(x)$, where $\psi_S(\cdot)$ denotes the indicator function of S, i.e., $\psi_S(x)=0$ if $x\in S$ and $+\infty$ otherwise.

Recall now that for a given $\rho\in]0,+\infty]$, a subset S is uniformly ρ-prox-regular (see [12]), or equivalently ρ-proximally smooth (see [5]), if and only if every nonzero proximal normal to S can be realized by a ρ-ball, this means that for all $\bar{x}\in S$ and all $\xi\in N_S^p(\bar{x})\setminus\{0\}$ one has

$$\left\langle \frac{\xi}{\|\xi\|}, x-\bar{x}\right\rangle \le \frac{1}{2\rho}\|x-\bar{x}\|^2$$

for all $x\in S$. We make the convention $\frac{1}{\rho}=0$ for $\rho=+\infty$. Recall that for $\rho=+\infty$ the uniform ρ-prox-regularity of S is equivalent to the convexity of S.

The following propositions summarize some important consequences of uniform prox-regularity needed in the sequel.

Proposition 2.1 ([12]). *Let S be a nonempty closed subset in H and $x \in S$. The following assertions hold.*

(i) $\partial^p d(x,S) = N_S^p(x) \cap \overline{B}$.

(ii) *Let $\rho \in]0,+\infty]$. If S is uniformly ρ-prox-regular, then for all $x \in H$ with $d(x,S) < \rho$ one has $\mathrm{Proj}_S(x) \neq \emptyset$ and $\partial^P d(x,S) = \partial^C d(x,S)$, where $\partial^C d(x,S)$ is the Clarke subdifferential of $d(\cdot,S)$ at x. So, in such a case, the subdifferential $\partial d(x,S) := \partial^P d(x,S) = \partial^C d(x,S)$ is a closed convex set in H.*

(iii) *If S is uniformly ρ-prox-regular, then for all $x_i \in S$ and all $v_i \in N_S^p(x_i)$ with $\|v_i\| \leq \rho$ $(i = 1,2)$ one has*

$$\langle v_1 - v_2, x_1 - x_2\rangle \geq -\|x_1 - x_2\|^2.$$

As a consequence of (iii) we get that for uniformly ρ-prox-regular sets, the proximal normal cone to S coincides with all the normal cones contained in the Clarke normal cone at all points $x \in S$, i.e., $N_S^P(x) = N_S^C(x)$. In such a case, we put $N_S(x) := N_S^P(x) = N_S^C(x)$.

Proposition 2.2 ([2]). *Let $\rho \in]0,+\infty]$ and Ω be an open subset in H and let $C : \Omega \to 2^H$ be a Hausdorff-continuous set-valued mapping. Assume that C has uniformly ρ-prox-regular values. Then, the set-valued mapping given by $(z,x) \to \partial d_{C(z)}(x)$ from $\Omega \times H$ (endowed with the strong topology) to H (endowed with the weak topology) is upper semicontinuous, which is equivalent to the upper semicontinuity of the function $(z,x) \to \sigma(\partial d_{C(z)}(x),p)$ for any $p \in H$. Here $\sigma(S,p)$ denotes the support function associated with S, i.e., $\sigma(S,p) = \sup_{s\in S}\langle s,p\rangle$.*

Let us recall the following lemmas that will be used in the sequel.

Lemma 2.3 ([15]). *Let Ω be a nonempty set in H. Assume that $F : [a,b] \times \Omega \to 2^H$ is a multifunction with nonempty closed values satisfying:*

- *for every $x \in \Omega$, $F(\cdot,x)$ is measurable on $[a,b]$;*
- *for every $t \in [a,b]$, $F(t,\cdot)$ is (Hausdorff) continuous on Ω.*

Then for any measurable function $x(\cdot) : [a,b] \to \Omega$, the multifunction $F(\cdot,x(\cdot))$ is measurable on $[a,b]$.

Lemma 2.4 ([15]). *Let $G : [a,b] \to 2^H$ be a measurable multifunction and $y(\cdot) : [a,b] \to H$ a measurable function. Then for any positive measurable function $r(\cdot) : [a,b] \to \mathbb{R}^+$, there exists a measurable selection $g(\cdot)$ of G such that for almost all $t \in [a,b]$*

$$\|g(t) - y(t)\| \leq d\big(y(t), G(t)\big) + r(t).$$

Assume that the following hypotheses hold:

(H1) $C : [0,b] \to 2^H$ is a set-valued map with nonempty compact values satisfying

(a) for each $t \in [0,b]$, $C(t)$ is ρ-prox-regular for some fixed $\rho \in]0,+\infty]$;
(b) there exists an absolutely continuous function $v : [0,b] \to \mathbb{R}$ such that

$$\Big| d(x, C(t)) - d(x, C(s)) \Big| \leq |v(t) - v(s)|$$

for all $x \in H$ and $s, t \in [0,b]$;

(H2) $F : [0,b] \times \mathcal{C}_a \to 2^H$ is a set-valued map with nonempty closed values satisfying
(i) for each $\psi \in \mathcal{C}_a$, $t \mapsto F(t,\psi)$ is measurable;
(ii) there is a function $m(\cdot) \in L^1([0,b],\mathbb{R}^+)$ such that for all $t \in [0,b]$ and for all $\psi_1, \psi_2 \in \mathcal{C}_a$

$$H\Big(F(t,\psi_1), F(t,\psi_2)\Big) \leq m(t)\|\psi_1 - \psi_2\|_\infty;$$

(iii) for all bounded subset S of $\mathcal{C}_a$, there exist two functions $g_S(\cdot), p_S(\cdot) \in L^1([0,b],\mathbb{R}^+)$ such that for all $t \in [0,b]$ and for all $\psi \in S$

$$\|F(t,\psi)\| := \sup_{y \in F(t,\psi)} \|y\| \leq g_S(t) + p_S(t)\|\psi\|_\infty.$$

We established the following result:

Theorem 2.5. *If assumptions* (H1) *and* (H2) *are satisfied, then for all* $\varphi \in \mathcal{C}_a$ *such that* $\varphi(0) \in C(0)$, *there exist* $T > 0$, $r > 0$, *and a continuous function* $x(\cdot) : [-a,T] \to H$, *that is absolutely continuous on* $[-a,T]$ *such that* $x(\cdot)$ *is solution of*

$$\begin{cases} \dot{x}(t) \in -N_{C(t)}(x(t)) + F(t, T(t)x) & \text{a.e on [0,T];} \\ x(t) = \varphi(t) \quad \forall t \in [-a,0]; \\ x(t) \in C(t) \quad \forall t \in [0,T], \end{cases}$$

and satisfies

$$\|\dot{x}(t)\| \leq |\dot{v}(t)| + g(t) + p(t)(\|\varphi\|_\infty + r), \qquad \text{for almost all } t \in [0,T].$$

3. Proof of the main result

Fix $\varphi \in \mathcal{C}_a$ such that $\varphi(0) \in C(0)$. Let $r > 0$ and $g(\cdot), p(\cdot) \in L^1([0,b],\mathbb{R}^+)$ be such that

$$\|F(t,\psi)\| \leq g(t) + p(t)\|\psi\|_\infty \quad \forall (t,\psi) \in [0,b] \times \overline{B}_a(\varphi, r). \tag{3.1}$$

Let $T_1 > 0$ be such that

$$\int_0^{T_1} \Big(2g(t) + 2p(t)(\|\varphi\|_\infty + r) + |\dot{v}(t)|\Big)\, dt < \inf\Big\{\frac{r}{2}, \frac{\rho}{2}\Big\}. \tag{3.2}$$

The idea of such T_1 has been used in [7]. For $\varepsilon > 0$ set
(3.3)
$$\eta(\varepsilon) = \sup\Big\{\gamma \in]0,\varepsilon] : \Big|\int_{t_1}^{t_2}\Big(|\dot v(s)| + 2g(s) + 2p(s)(\|\varphi\|_\infty + r)\Big)\,ds\Big| < \varepsilon,$$
$$\text{and } \|\varphi(t_1) - \varphi(t_2)\| < \varepsilon \ \text{ if } \ |t_1 - t_2| < \gamma\Big\}.$$

Put

$$T = \min\Big\{T_1, \frac{1}{2}\eta(\frac{r}{2}), b\Big\}. \tag{3.4}$$

We will used the following lemma to prove the main result.

Lemma 3.1. *If assumptions* (H1) *and* (H2) *are satisfied, then for all $n \in \mathbb{N}^*$ and for all $y(\cdot) \in L^1([0,T],H)$, there exist a continuous mapping $x_n(\cdot) : [-a,T] \to H$, a step functions $\theta_n(\cdot), \bar\theta_n(\cdot) : [0,T] \to [0,T]$ and $f_n(\cdot) \in L^1([0,T],H)$ such that*

- $f_n(t) \in F(t, T(\theta_n(t))x_n)$, $x_n(\bar\theta_n(t)) \in C(\bar\theta_n(t))$, *for all $t \in [0,T]$;*
- $\|f_n(t) - y(t)\| \le d(y(t), F(t, T(\theta_n(t))x_n)) + \frac{1}{n}$ *for all $t \in [0,T]$;*
- $\Big(\dot x_n(t) - f_n(t)\Big) \in -N\Big(C(\bar\theta_n(t)), x_n(\bar\theta_n(t))\Big)$ *for almost all $t \in [0,T]$;*
- $\|\dot x_n(t) - f_n(t)\| \le |\dot v(t)| + g(t) + p(t)(r + \|\varphi\|_\infty)$ *for almost every $t \in [0,T]$.*

Proof: Fix $n \in \mathbb{N}^*$ and let $y(\cdot) : [0,T] \to H$ be a measurable function. Consider a sequence $(P_n)_n$ of subdivisions of $[0,T]$:

$$P_n = \Big\{0 = t_0^n < t_1^n < \cdots < t_i^n < \cdots < t_{2^n}^n = T\Big\}$$

where $t_i^n = i\frac{T}{2^n}$, $0 < i < 2^n$. Let us define a sequence $(x_n)_n$ of approximate solutions as follows. Set $x_n(s) = \varphi(s)$ for all $s \in [-a,0]$. Put $x_0^n = \varphi(0) \in C(t_0^n)$. In view of Lemma 2.4, there exists a function $f_0^n \in L^1([0,t_1^n],H)$ such that $f_0^n(t) \in F(t, T(0)x_n)$ and

$$\|f_0^n(t) - y(t)\| \le d\big(y(t), F(t, T(0)x_n)\big) + \frac{1}{n}$$

for all $t \in [0, t_1^n]$. By (H1), (3.1) and (3.2), we have

$$\begin{aligned} d_{C(t_1^n)}\left(x_0^n + \int_{t_0^n}^{t_1^n} f_0^n(s)\,ds\right) &\le d_{C(t_0^n)}\left(x_0^n + \int_{t_0^n}^{t_1^n} f_0^n(s)\,ds\right) + |v(t_1^n) - v(t_0^n)| \\ &\le \int_{t_0^n}^{t_1^n} \|f_0^n(s)\|\,ds + \int_{t_0^n}^{t_1^n} |\dot v(s)|\,ds \end{aligned}$$

$$\le \int_{t_0^n}^{t_1^n} \Big(g(s) + p(s)\|\varphi\|_\infty + |\dot v(s)|\Big)\, ds$$
$$\le \frac{\rho}{2}.$$

As C has uniformly ρ-prox-regular values, by Proposition 2.1, we have

$$\mathrm{Proj}_{C(t_1^n)}\left(x_0^n + \int_{t_0^n}^{t_1^n} f_0^n(s)\, ds\right) \neq \emptyset.$$

Then, one can choose a point x_1^n in

$$\mathrm{Proj}_{C(t_1^n)}\left(x_0^n + \int_{t_0^n}^{t_1^n} f_0^n(s)\, ds\right).$$

Note that $x_1^n \in C(t_1^n)$ and

$$\begin{aligned} \left\| x_1^n - \left(x_0^n + \int_{t_0^n}^{t_1^n} f_0^n(s)\, ds\right)\right\| &= d_{C(t_1^n)}\left(x_0^n + \int_{t_0^n}^{t_1^n} f_0^n(s)\, ds\right) \\ &\le \int_{t_0^n}^{t_1^n} \Big(g(s) + p(s)\|\varphi\|_\infty + |\dot v(s)|\Big)\, ds. \end{aligned}$$

Remark that

$$\begin{aligned} \|x_1^n - \varphi(0)\| &\le \left\| x_1^n - \left(x_0^n + \int_{t_0^n}^{t_1^n} f_0^n(s)\, ds\right)\right\| + \int_{t_0^n}^{t_1^n} \|f_0^n(s)\|\, ds \\ &\le \int_{t_0^n}^{t_1^n} \Big(2g(s) + 2p(s)\|\varphi\|_\infty + |\dot v(s)|\Big)\, ds \\ &\le \frac{r}{2}. \end{aligned}$$

Then $x_1^n \in \overline{B}(\varphi(0), r)$. Now, set

$$x_n(t) = x_0^n + \frac{\alpha(t) - \alpha(t_0^n)}{\alpha(t_1^n) - \alpha(t_0^n)}\left(x_1^n - x_0^n - \int_{t_0^n}^{t_1^n} f_0^n(s)\, ds\right) + \int_{t_0^n}^{t} f_0^n(s)\, ds$$

for all $t \in [t_0^n, t_1^n]$, where

$$\alpha(t) = \int_0^t \Big(|\dot v(s)| + g(s) + p(s)(r + \|\varphi\|_\infty)\Big)\, ds, \ \forall t \in [0, T].$$

So for all $t \in [t_0^n, t_1^n]$

$$\begin{aligned}
\|x_n(t) - \varphi(0)\| &\leq \frac{\alpha(t) - \alpha(t_0^n)}{\alpha(t_1^n) - \alpha(t_0^n)} \left\| x_1^n - x_0^n - \int_{t_0^n}^{t_1^n} f_0^n(s)\, ds \right\| + \int_{t_0^n}^{t} \|f_0^n(s)\|\, ds \\
&\leq \alpha(t) - \alpha(t_0^n) + \int_{t_0^n}^{t} g(s) + p(t)\|\varphi\|_\infty\, ds \\
&\leq \int_{t_0^n}^{t} \Big(|\dot{v}(s)| + 2g(s) + 2p(s)(r + \|\varphi\|_\infty) \Big)\, ds \\
&\leq \frac{r}{2}
\end{aligned}$$

which is equivalent to $x_n(t) \in \overline{B}(\varphi(0), \frac{r}{2})$ for all $t \in [t_0^n, t_1^n]$. Now, we have to estimate $\|(T(t_1^n)x_n)(s) - \varphi(s)\|$ for each $s \in [-a, 0]$. If $-t_1^n \leq s \leq 0$, then $t_1^n + s \in [0, t_1^n]$. Thus, by the fact that $|s| \leq t_1^n \leq T < \eta(\frac{r}{2})$, we have

$$\begin{aligned}
\|(T(t_1^n)x_n)(s) - \varphi(s)\| &= \|x_n(t_1^n + s) - \varphi(s)\| \\
&\leq \|x_n(t_1^n + s) - \varphi(0)\| + \|\varphi(s) - \varphi(0)\| \\
&\leq r.
\end{aligned}$$

If $-a \leq s \leq -t_1^n$, then $t_1^n + s \in [-a, 0]$ and

$$\begin{aligned}
\|(T(t_1^n)x_n)(s) - \varphi(s)\| &= \|\varphi(t_1^n + s) - \varphi(s)\| \\
&\leq r.
\end{aligned}$$

Therefore, $T(t_1^n)x_n \in \overline{B}_a(\varphi(\cdot), r)$.

We reiterate this process for constructing sequences $(f_i^n(\cdot))_i, (x_i^n)_i$ satisfying, for all $0 \leq i \leq 2^n - 1$ and for all $t \in [t_i^n, t_{i+1}^n]$, the following assertions:

$$f_i^n(t) \in F(t, T(t_i^n)x_n),\ x_0^n \in C(t_0^n),\ x_{i+1}^n \in C(t_{i+1}^n) \cap \overline{B}(\varphi(0), r), \tag{3.5}$$

$$x_n(t) \in \overline{B}(\varphi(0), r),\ T(t_i^n)x_n \in \overline{B}_a(\varphi(\cdot), r), \tag{3.6}$$

$$\|f_i^n(t) - y(t)\| \leq d\big(y(t), F(t, T(t_i^n)x_n)\big) + \frac{1}{n}\,, \tag{3.7}$$

$$x_{i+1}^n \in \mathrm{Proj}_{C(t_{i+1}^n)} \left(x_i^n + \int_{t_i^n}^{t_{i+1}^n} f_i^n(s)\, ds \right), \tag{3.8}$$

$$\left\| x_{i+1}^n - \left(x_i^n + \int_{t_i^n}^{t_{i+1}^n} f_i^n(s)\, ds \right) \right\| \leq \int_{t_i^n}^{t_{i+1}^n} \Big(|\dot{v}(s)| + g(s) + p(s)(r + \|\varphi\|_\infty) \Big)\, ds, \tag{3.9}$$

$$(3.10)\quad x_n(t) = x_i^n + \frac{\alpha(t)-\alpha(t_i^n)}{\alpha(t_{i+1}^n)-\alpha(t_i^n)}\left(x_{i+1}^n - x_i^n - \int_{t_i^n}^{t_{i+1}^n} f_i^n(s)\,ds\right) + \int_{t_i^n}^{t} f_i^n(s)\,ds.$$

Now, we define the functions $\theta_n(\cdot), \bar{\theta}_n(\cdot) : [0,T] \to [0,T]$ and $f_n(\cdot) \in L^1([0,T],H)$ by setting for all $t \in [t_i^n, t_{i+1}^n[$

$$\bar{\theta}_n(t) = t_{i+1}^n,\ \bar{\theta}_n(T) = T,\ \ f_n(t) = f_i^n(t),$$

and for all $t \in]t_i^n, t_{i+1}^n]$

$$\theta_n(t) = t_i^n,\ \theta_n(0) = 0.$$

We claim that $x_n(\cdot)$ is absolutely continuous. Indeed, for all $0 \le i \le 2^n - 1$ and for all t and s in $[t_i^n, t_{i+1}^n]$, $s < t$, one has

$$x_n(t) - x_n(s) = \frac{\alpha(t)-\alpha(s)}{\alpha(t_{i+1}^n)-\alpha(t_i^n)}\left(x_{i+1}^n - x_i^n - \int_{t_i^n}^{t_{i+1}^n} f_i^n(s)\,ds\right) + \int_{s}^{t} f_i^n(s)\,ds.$$

Then, by (3.1) and (3.9) we get

$$\begin{aligned}\|x_n(t)-x_n(s)\| &= \frac{\alpha(t)-\alpha(s)}{\alpha(t_{i+1}^n)-\alpha(t_i^n)}\left\|x_{i+1}^n - x_i^n - \int_{t_i^n}^{t_{i+1}^n} f_i^n(s)\,ds\right\| \\ &\quad + \int_s^t \Big(g(\tau)+p(\tau)(\|\varphi\|_\infty + r)\Big)\,d\tau \\ &\le \alpha(t)-\alpha(s) + \int_s^t \Big(g(\tau)+p(\tau)(\|\varphi\|_\infty + r)\Big)\,d\tau.\end{aligned}$$

Hence

$$(3.11)\qquad \|x_n(t)-x_n(s)\| \le \int_s^t |\dot{v}(\tau)| + 2g(\tau) + 2p(\tau)(r+\|\varphi\|_\infty)\,d\tau.$$

By addition this last inequality holds for all $s,t \in [0,T]$ with $s < t$. Hence $x_n(\cdot)$ is absolutely continuous. Remark that for all $0 \le i \le 2^n - 1$ and for almost every t in $[t_i^n, t_{i+1}^n]$,

$$(3.12)\qquad \dot{x}_n(t) = \frac{\dot{\alpha}(t)}{\alpha(t_{i+1}^n)-\alpha(t_i^n)}\left(x_{i+1}^n - x_i^n - \int_{t_i^n}^{t_{i+1}^n} f_i^n(s)\,ds\right) + f_n(t).$$

Then, by (3.9) we obtain for almost every $t \in [0,T]$

$$\|\dot{x}_n(t) - f_n(t)\| \le |\dot{v}(t)| + g(t) + p(t)(\|\varphi\|_\infty + r).$$

Observe that by construction, we have $f_n(t) \in F(t, T(\theta_n(t))x_n)$ and

$$\|f_n(t) - y(t)\| \le d\big(y(t), F(t, T(\theta_n(t))x_n)\big) + \frac{1}{n}$$

for all $t \in [0,T]$. Also, by construction and the relation (3.8), we have for almost every $t \in [0,T]$

$$\Big(\dot{x}_n(t) - f_n(t)\Big) \in -N\Big(C(\bar{\theta}_n(t)), x_n(\bar{\theta}_n(t))\Big).$$

Then the proof is complete. □

Proof of the Theorem. In view of Lemma 3.1, we can define inductively sequences $(f_n(\cdot))_{n\geq 1} \subset L^1([0,T],H)$, $(x_n(\cdot))_{n\geq 1} \subset \mathcal{C}([-a,T],H)$ and $(\theta_n(\cdot))_{n\geq 1}, (\bar{\theta}_n(\cdot))_{n\geq 1} \subset S([0,T],[0,T])$; where $S([0,T],[0,T])$ denotes the space of step functions from $[0,T]$ into $[0,T]$; such that

(1) $f_n(t) \in F(t, T(\theta_n(t))x_n)$, $x_n(\bar{\theta}_n(t)) \in C(\bar{\theta}_n(t))$, for all $t \in [0,T]$;

(2) $\|f_{n+1}(t) - f_n(t)\| \leq d(f_n(t), F(t, T(\theta_{n+1}(t))x_{n+1})) + \frac{1}{n+1}$ for all $t \in [0,T]$;

(3) $(\dot{x}_n(t) - f_n(t)) \in -N(C(\bar{\theta}_n(t)), x_n(\bar{\theta}_n(t)))$ for almost all $t \in [0,T]$;

(4) $\|\dot{x}_n(t) - f_n(t)\| \leq |\dot{v}(t)| + g(t) + p(t)(\|\varphi\|_\infty + r)$ for almost every $t \in [0,T]$.

For all $t \in [0,T]$, there exists $0 \leq i \leq 2^n - 1$ such that $t \in [t_i^n, t_{i+1}^n]$. By (H1) and (3.11), we have

$$\begin{aligned} d(x_n(t), C(t)) &\leq \|x_n(t) - x_n(t_i^n)\| + d(x_n(t_i^n), C(t)) \\ &\leq \int_{t_i^n}^t \Big(|\dot{v}(s)| + 2g(s) + 2p(s)(\|\varphi\|_\infty + r)\Big)\, ds + |v(t) - v(t_i^n)|. \end{aligned}$$

The right term of the above inequality converges to 0 if $n \to +\infty$. This and the compactness of $C(t)$ ensure that the set $\{x_n(t), n \geq 1\}$ is relatively compact in H. Moreover, from (4) we deduce

$$\|\dot{x}_n(t)\| \leq |\dot{v}(t)| + 2g(t) + 2p(t)(\|\varphi\|_\infty + r)$$

for almost every $t \in [0,T]$. Then, by Arzela-Ascoli's theorem (see [1]), we can select a subsequence, again denoted by $(x_n(\cdot))_n$ which converges uniformly to an absolutely continuous function $x(\cdot)$ on $[0,T]$, moreover $\dot{x}_n(\cdot)$ converges weakly to $\dot{x}(\cdot)$ in $L^1([0,T],H)$. Also, since all functions $x_n(\cdot)$ agree with $\varphi(\cdot)$ on $[-a,0]$, we can obviously say that $x_n(\cdot)$ converges uniformly to $x(\cdot)$ on $[-a,T]$, if we extend $x(\cdot)$ in such a way that $x(\cdot) \equiv \varphi(\cdot)$ on $[-a,0]$. Additionally, observe that $x_n(\bar{\theta}_n(t))$ converges uniformly to $x(t)$ on $[0,T]$. Indeed, by (3.11) for all $t \in [0,T]$, we have

$$\begin{aligned} \|x_n(\bar{\theta}_n(t)) - x(t)\| &\leq \|x_n(\bar{\theta}_n(t)) - x_n(t)\| + \|x_n(t) - x(t)\| \\ &\leq \int_t^{\bar{\theta}_n(t)} \Big(|\dot{v}(s)| + 2g(s) + 2p(s)(r + \|\varphi\|_\infty)\Big)\, ds \\ &\quad + \|x_n(t) - x(t)\|. \end{aligned}$$

The right term of the above inequality converges to 0, it follows $x_n(\bar{\theta}_n(\cdot))$ converges uniformly to $x(\cdot)$ on $[0,T]$. Therefore, as $d(x_n(t), C(t))$ converges to 0 on

$[0,T]$, we conclude that $x(t) \in C(t)$ for all $t \in [0,T]$. On the other hand, for $t \in [0,T]$ and $y \in C(t)$, we have by (H1)

$$d_{C(\bar{\theta}_n(t))}(y) \leq |v(\bar{\theta}_n(t)) - v(t)|;$$

thus

$$\sup_{y \in C(t)} d_{C(\bar{\theta}_n(t))}(y) \leq |v(\bar{\theta}_n(t)) - v(t)|.$$

By the same way we can prove that

$$\sup_{y \in C(\bar{\theta}_n(t))} d_{C(t)}(y) \leq |v(\bar{\theta}_n(t)) - v(t)|.$$

Hence

$$H\Big(C(\bar{\theta}_n(t)), C(t)\Big) \leq |v(\bar{\theta}_n(t)) - v(t)|,$$

consequently, $C(\bar{\theta}_n(t))$ converges to $C(t)$.

Claim 3.2. *$T(\theta_n(t))x_n$ converges to $T(t)x$ in $\mathcal{C}_a$.*

Proof: Let us denote the modulus continuity of a function $\psi(\cdot)$ defined on an interval I of $\mathbb{R}$ by

$$\omega(\psi(\cdot), I, \eta) := \sup\Bigg\{\|\psi(t) - \psi(s)\|; s, t \in I, |s-t| < \eta\Bigg\}.$$

Let $\varepsilon > 0$ and let $t, t' \in [0,T]$, assume that $0 \leq t' - t < \eta(\frac{\varepsilon}{2})$. By (3.3) and (3.11), we have

$$\begin{aligned} \|x_n(t) - x_n(t')\| &\leq \int_t^{t'} \Big(|\dot{v}(s)| + 2g(s) + 2p(s)(\|\varphi\|_\infty + r)\Big)\, ds \\ &\leq \frac{\varepsilon}{2}. \end{aligned}$$

Hence

$$\omega\Big(x_n(\cdot), [0,T], \eta(\frac{\varepsilon}{2})\Big) \leq \frac{\varepsilon}{2}.$$

Also for $t, t' \in [-a, 0]$ such that $|t' - t| < \eta(\frac{\varepsilon}{2})$, we have by (3.3)

$$\|\varphi(t) - \varphi(t')\| < \frac{\varepsilon}{2}.$$

Then

$$\omega\Big(\varphi(\cdot), [-a,0], \eta(\frac{\varepsilon}{2})\Big) \leq \frac{\varepsilon}{2}.$$

Now, let $t \in [0,T]$. Since $\theta_n(t)$ converges to t, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $|\theta_n(t) - t| < \eta(\frac{\varepsilon}{2})$. Then, for all $n \geq n_0$

$$\begin{aligned} \|T(\theta_n(t))x_n - T(t)x_n\|_\infty &= \sup_{-a\leq s\leq 0} \|x_n(\theta_n(t)+s) - x_n(t+s)\| \\ &\leq \omega\Big(x_n(\cdot), [-a,T], \eta(\frac{\varepsilon}{2})\Big) \\ &\leq \omega\Big(\varphi(\cdot), [-a,0], \eta(\frac{\varepsilon}{2})\Big) + \omega\Big(x_n(\cdot), [0,T], \eta(\frac{\varepsilon}{2})\Big) \\ &\leq \varepsilon, \end{aligned}$$

hence $\|T(\theta_n(t))x_n - T(t)x_n\|_\infty$ converges to 0 as $n \to +\infty$. Therefore, since the uniform convergence of $x_n(\cdot)$ to $x(\cdot)$ on $[-a,T]$ implies that $T(t)x_n$ converges to $T(t)x$ uniformly on $[-a,0]$, we deduce that

$$T(\theta_n(t))x_n \text{ converges to } T(t)x \text{ in } \mathcal{C}_a. \tag{3.13}$$

On the other hand, from (1) and (2) we deduce

$$\begin{aligned} \|f_{n+1}(t) - f_n(t)\| &\leq H\Big(F(t, T(\theta_n(t))x_n), F(t, T(\theta_{n+1}(t))x_{n+1})\Big) + \frac{1}{n+1} \\ &\leq m(t)\|T(\theta_n(t))x_n - T(\theta_{n+1}(t))x_{n+1}\|_\infty + \frac{1}{n+1}\,. \end{aligned} \tag{3.14}$$

By (3.13), $\|T(\theta_n(t))x_n - T(\theta_{n+1}(t))x_{n+1}\|_\infty$ converges to 0, thus the right term of the relation (3.14) converges to 0. Hence $(f_n(t))_{n\geq 1}$ is a Cauchy sequence and $(f_n(\cdot))_{n\geq 1}$ converges pointwise to $f(\cdot)$. Moreover, observe that by (1),

$$\begin{aligned} d\big(f(t), F(t, T(t)x)\big) &\leq \|f(t) - f_n(t)\| + H\Big(F(t, T(\theta_n(t))x_n), F(t, T(t)x)\Big) \\ &\leq \|f(t) - f_n(t)\| + m(t)\|T(\theta_n(t))x_n - T(t)x\|_\infty. \end{aligned}$$

Since $f_n(t)$ converges to $f(t)$ and by (3.13) the last term converges to 0. So that $f(t) \in F(t, T(t)x)$ for all $t \in [0,T]$.

Now, we can apply Castaing techniques (see for example [14]). The weak convergence of $\dot{x}_n(\cdot)$ to $\dot{x}(\cdot)$ in $L^1([0,T], H)$ and the Mazur's Lemma entail

$$\dot{x}(t) - f(t) \in \bigcap_n \overline{\text{co}}\Big\{\dot{x}_m(t) - f_m(t) : m \geq n\Big\}, \text{ for a.e. on } [0,T].$$

For any $t \in [0,T]$ and $y \in H$

$$\langle y, \dot{x}(t) - f(t)\rangle \leq \inf_n \sup_{k\geq n} \langle y, \dot{x}_k(t) - f_k(t)\rangle.$$

By (3) and (4), one has

$$\Big(\dot{x}_n(t) - f_n(t)\Big) \in -N\Big(C(\bar{\theta}_n(t)), x_n(\bar{\theta}_n(t))\Big) \cap \Big(|\dot{v}(t)| + g(t) + p(t)(\|\varphi\| + r)\Big)B$$

for almost all $t \in [0,T]$. Hence, by Proposition 2.1 we get

$$\Big(\dot{x}_n(t) - f_n(t)\Big) \in -\Big(|\dot{v}(t)| + g(t) + p(t)(\|\varphi\| + r)\Big)\partial d_{C(\bar{\theta}_n(t))}(x_n(\bar{\theta}_n(t))).$$

In view of Proposition 2.2, we deduce

$$\begin{aligned}
&\langle y, \dot{x}(t) - f(t)\rangle \\
&\le \Big(|\dot{v}(t)| + g(t) + p(t)(\|\varphi\| + r)\Big) \limsup_{n\to\infty} \sigma\Big(y, -\partial d_{C(\bar{\theta}_n(t))}(x_n(\bar{\theta}_n(t)))\Big) \\
&\le \Big(|\dot{v}(t)| + g(t) + p(t)(\|\varphi\| + r)\Big)\sigma\Big(y, -\partial d_{C(t)}(x(t))\Big).
\end{aligned}$$

So, the convexity and the closedness of the set $\partial d_{C(t)}(x(t))$ ensure

$$\Big(\dot{x}(t) - f(t)\Big) \in -\Big(|\dot{v}(t)| + g(t) + p(t)(\|\varphi\| + r)\Big)\partial d_{C(t)}(x(t)) \subset -N_{C(t)}(x(t)).$$

Finally, we have for almost all $t \in [0,T]$, $\dot{x}(t) \in -N_{C(t)}(x(t)) + F(t, T(t)x)$ and for all $t \in [0,T]$, $x(t) \in C(t)$. The proof is complete. $\square$

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