Comment.Math.Univ.Carolin. 53,1 (2012) 1–17 

# Berezin transform for non-scalar holomorphic discrete series

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*Abstract.* Let $M = G/K$ be a Hermitian symmetric space of the non-compact type and let $\pi$ be a discrete series representation of $G$ which is holomorphically induced from a unitary irreducible representation $\rho$ of $K$. In the paper [B. Cahen, *Berezin quantization for holomorphic discrete series representations: the non-scalar case*, Beiträge Algebra Geom., DOI 10.1007/s13366-011-0066-2], we have introduced a notion of complex-valued Berezin symbol for an operator acting on the space of $\pi$. Here we study the corresponding Berezin transform and we show that it can be extended to a large class of symbols. As an application, we construct a Stratonovich-Weyl correspondence associated with $\pi$.

*Keywords:* Berezin quantization, Berezin symbol, Stratonovich-Weyl correspondence, discrete series representation, Hermitian symmetric space of the non-compact type, semi-simple non-compact Lie group, coherent states, reproducing kernel, adjoint orbit

*Classification:* 22E46, 32M10, 32M15, 81S10

## 1. Introduction

Let $G$ be a connected semi-simple non-compact Lie group with finite center. Let $K$ be a maximal compact subgroup of $G$. We assume that the center of $K$ has positive dimension. Then the Hermitian symmetric space of the non-compact type $G/K$ is diffeomorphic to a bounded symmetric domain $\mathcal{D}$. We consider a discrete series representation $\pi$ of $G$ which is holomorphically induced from a unitary irreducible representation $\rho$ of $K$. The space of $\rho$ is then a finite-dimensional complex vector space $V$ and $\pi$ can be realized in a Hilbert space $\mathcal{H}$ of holomorphic functions on $\mathcal{D}$ with values in $V$.

When $\rho$ is a unitary character of $K$, we can directly define the Berezin symbol $S(A)$ of an operator $A$ on $\mathcal{H}$ as a complex-valued function on $\mathcal{D}$ and the map $S : A \to S(A)$ is a bounded operator from $L^2(\mathcal{D}, \mu)$, where $\mu$ is an invariant measure on $\mathcal{D}$, to the space $\mathcal{L}_2(\mathcal{H})$ of the Hilbert-Schmidt operators on $\mathcal{H}$, see for instance [29]. The Berezin transform is then the map $B := SS^*$, which plays an important role in quantization on symmetric domains [5], [6]. In that case, Berezin transforms have been intensively studied (see in particular [29], [27], [16], [33] and [34]).

In the general case, we have constructed in [13] a Berezin map $S : A \to S(A)$ from a class of operators acting on $\mathcal{H}$ to a space of complex-valued functions on

$G/K \times o$, where $o$ denotes the coadjoint orbit of $K$ associated with $\rho$. The map $S$ has some nice properties (symmetry, covariance ...) and then can be considered as the natural generalization of the Berezin calculus to the non-scalar case.

In the present paper, we introduce and study the Berezin transform $B$ corresponding to the map $S$. In particular, we show that $B$ extends to a bounded operator acting on a space of square-integrable functions on $G/K \times o$ (Proposition 5.2). This generalizes some well-known results on the usual Berezin transform, see for instance [29, 1.19]. Moreover, we study the functions $S(d\pi(X_1X_2\cdots X_q))$ for $X_1, X_2, \ldots, X_q$ in the Lie algebra of $G$ and we prove that $B$ can be also extended to these functions, generalizing the results of [12]. As an application, we construct a Stratonovich-Weyl correspondence associated with $\pi$ (see Section 7 for a precise definition).

This paper is organized as follows. In Section 2, we introduce some notation on Hermitian symmetric spaces and holomorphic discrete series. In Section 3, we recall the results of [13] about the construction of the map $S$ and its properties. In Section 4, we introduce the Berezin transform $B$ and we show that $B$ is an integral operator. The Sections 5 and 6 are devoted to the study and the extension of $B$. Our main results are then Proposition 5.2 ($L^2$-extension of $B$) and Proposition 6.5 (extension of $B$ to the symbols of some differential operators). Finally, in Section 7, we construct a Stratonovich-Weyl correspondence associated with $\pi$.

## 2. Preliminaries

In this section, we introduce the notation and we collect some facts on Hermitian symmetric spaces of the non-compact type and holomorphic discrete series representations. Our main references are [21, Chapter VIII], [26, Chapter XII], [23, Chapter 6], [17] and [31].

Let $G$ be a connected semi-simple non-compact real Lie group with finite center and let $K$ be a maximal compact subgroup of $G$. We assume that the center of the Lie algebra of $K$ is non-trivial. Then the homogeneous space $G/K$ is a Hermitian symmetric space of the non-compact type.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of $G$ and $K$, respectively. Let $\mathfrak{g}^c$ and $\mathfrak{k}^c$ be the complexifications of $\mathfrak{g}$ and $\mathfrak{k}$ and $G^c$, $K^c$ the corresponding complex Lie groups containing $G$ and $K$, respectively. We denote by $\beta$ the Killing form of $\mathfrak{g}^c$, that is, $\beta(X,Y) = \mathrm{Tr}(\mathrm{ad}\, X\, \mathrm{ad}\, Y)$ for $X$, $Y \in \mathfrak{g}^c$. Let $\mathfrak{p}$ be the ortho-complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $\beta$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$.

We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$. Then $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}$. We denote by $\mathfrak{h}^c$ the complexification of $\mathfrak{h}$. Let $\Delta$ be the root system of $\mathfrak{g}^c$ relative to $\mathfrak{h}^c$ and let $\mathfrak{g}^c = \mathfrak{h}^c \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}^c$. Then we have the direct decompositions $\mathfrak{k}^c = \mathfrak{h}^c \oplus \sum_{\alpha\in\Delta_c} \mathfrak{g}_\alpha$ and $\mathfrak{p}^c = \sum_{\alpha\in\Delta_n} \mathfrak{g}_\alpha$ where $\mathfrak{p}^c$ denotes the complexification of $\mathfrak{p}$ and $\Delta_c$ (resp. $\Delta_n$) denotes the set of compact (resp. non-compact) roots. We choose an ordering on $\Delta$ as in [21, p. 384] and we denote by $\Delta^+$, $\Delta_c^+$ and $\Delta_n^+$ the corresponding sets of positive roots, positive compact roots and positive non-compact roots, respectively. We set $\mathfrak{p}^+ = \sum_{\alpha\in\Delta_n^+} \mathfrak{g}_\alpha$ and $\mathfrak{p}^- = \sum_{\alpha\in\Delta_n^+} \mathfrak{g}_{-\alpha}$. Then we have $[\mathfrak{k}^c, \mathfrak{p}^\pm] \subset \mathfrak{p}^\pm$ and

$\mathfrak{p}^+$ and $\mathfrak{p}^-$ are abelian subalgebras [21, Proposition 7.2]. Since $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}$, we also have $[\mathfrak{p}^+,\mathfrak{p}^-]\subset\mathfrak{k}^c$. We denote by $P^+$ and $P^-$ the analytic subgroups of $G^c$ with Lie algebras $\mathfrak{p}^+$ and $\mathfrak{p}^-$, respectively.

For each $\mu\in(\mathfrak{h}^c)^*$, we denote by $H_\mu$ the element of $\mathfrak{h}^c$ satisfying $\beta(H,H_\mu)=\mu(H)$ for all $H\in\mathfrak{h}^c$. Note that if $\mu$ is real-valued on $i\mathfrak{h}$ then $iH_\mu\in\mathfrak{g}$. For $\mu$, $\nu\in(\mathfrak{h}^c)^*$, we set $(\mu,\nu):=\beta(H_\mu,H_\nu)$.

Let $\theta$ denote the conjugation induced by the real form $\mathfrak{g}$ of $\mathfrak{g}^c$. For $X\in\mathfrak{g}^c$, we set $X^*=-\theta(X)$. We denote by $g\to g^*$ the involutive anti-automorphism of $G^c$ which is obtained by exponentiating $X\to X^*$ to $G^c$. Recall that the multiplication map $(z,k,y)\to zky$ is a diffeomorphism from $P^+\times K^c\times P^-$ onto an open submanifold of $G^c$ containing $G$ [21, Lemma 7.9]. Following [26, p. 497], we introduce the projections $\zeta:P^+K^cP^-\to P^+$, $\kappa:P^+K^cP^-\to K^c$ and $\eta:P^+K^cP^-\to P^-$. Then the map $gK\to\log\zeta(g)$ from $G/K$ to $\mathfrak{p}^+$ induces a diffeomorphism from $G/K$ onto a bounded domain $\mathcal{D}\subset\mathfrak{p}^+$ [21, p. 392]. The natural action of $G$ on $G/K$ corresponds to the action of $G$ on $\mathcal{D}$ given by $g\cdot Z=\log\zeta(g\exp Z)$. The $G$-invariant measure on $\mathcal{D}$ is $d\mu(Z)=\chi_0(\kappa(\exp Z^*\exp Z))\,d\mu_L(Z)$ where $\chi_0$ is the character on $K^c$ defined by $\chi_0(k)=\mathrm{Det}_{\mathfrak{p}^+}(\mathrm{Ad}\,k)$ and $d\mu_L(Z)$ is a Lebesgue measure on $\mathcal{D}$ [26, p. 538].

Note that by fixing an Iwasawa decomposition $G=NAK$, we get a smooth section $G/K\to NA\subset G$. Then we obtain a smooth section $\mathcal{D}\to G$, $Z\to g_Z$, that is, we have $g_Z\cdot 0=Z$ for $Z\in\mathcal{D}$.

Now, let $(\rho,V)$ be a unitary irreducible representation of $K$ with highest weight $\lambda$ (relative to $\Delta_c^+$). We also denote by $\rho$ the extension of $\rho$ to $K^c$. Let $\mathcal{H}$ be the Hilbert space of all holomorphic functions on $\mathcal{D}$ with values in $V$ such that

$$\|f\|^2:=\int_{\mathcal{D}}\langle K(Z,Z)^{-1}f(Z),f(Z)\rangle_V\,d\mu(Z)<+\infty$$

where $K(Z,W):=\rho(\kappa(\exp W^*\exp Z))^{-1}$ for $Z$, $W\in\mathcal{D}$.

For $g\in G$ and $Z\in\mathcal{D}$, we set $J(g,Z):=\rho(\kappa(g\exp Z))$.

**Proposition 2.1** ([26, p. 542], [17]). *The space $\mathcal{H}$ is non-zero if and only if $(\lambda+\delta,\alpha)<0$ for each non-compact positive root $\alpha$, where $\delta$ stands for half of the sum of the positive roots. In that case, $\mathcal{H}$ contains all $V$-valued polynomials. Moreover, the action of $G$ on $\mathcal{H}$ defined by*

$$\pi(g)f(Z)=J(g^{-1},Z)^{-1}\,f(g^{-1}\cdot Z)$$

*is a unitary irreducible representation of $G$ which belongs to the holomorphic discrete series of $G$.*

In the rest of the paper, we assume that the condition of the preceding proposition is fulfilled.

The evaluation maps $K_Z:\mathcal{H}\to V$, $f\to f(Z)$ are continuous [26, p. 539]. The generalized coherent states of $\mathcal{H}$ are the maps $E_Z=K_Z^*:V\to\mathcal{H}$ defined by $\langle f(Z),v\rangle_V=\langle f,E_Zv\rangle$ for $f\in\mathcal{H}$ and $v\in V$.

We have the following result, see [26, p. 540] and [17].

**Proposition 2.2.** (1) *There exists a constant $c_\rho > 0$ such that $E_Z^* E_W = c_\rho K(Z,W)$ for each $Z$, $W \in \mathcal{D}$.*

(2) *For $g \in G$ and $Z \in \mathcal{D}$, we have $E_{g\cdot Z} = \pi(g) E_Z J(g,Z)^*$.*

## 3. Berezin symbols

In this section, we first introduce the Berezin calculus associated with $\rho$, see [4], [32] and [9].

Let $\lambda \in \mathfrak{h}^*$ be the highest weight of $\rho$ relative to $\Delta_c^+$. Let $\varphi_0 \in \mathfrak{h}$ be such that $\lambda(H) = i\beta(\varphi_0, H)$ for each $H \in \mathfrak{h}$, that is, $\varphi_0 = -iH_\lambda$. In the rest of the paper, we assume that $\varphi_0$ is regular in the sense that $\alpha(\varphi_0) \neq 0$ for each $\alpha \in \Delta$. Then the orbit $o(\varphi_0)$ of $\varphi_0$ under the adjoint action of $K$ is said to be associated with $\rho$ [8], [32].

Note that a complex structure on $o(\varphi_0)$ is then defined by the diffeomorphism $o(\varphi_0) \simeq K/H \simeq K^c/H^cN^-$ where $N^-$ is the analytic subgroup of $K^c$ with Lie algebra $\sum_{\alpha\in\Delta_c^+} \mathfrak{g}_\alpha$.

Without loss of generality, we can assume that $V$ is a space of holomorphic functions on $o(\varphi_0)$ as in [9]. Since $V$ is finite-dimensional, for each $\varphi \in o(\varphi_0)$ there exists a unique function $e_\varphi \in V$ (called a coherent state) such that $a(\varphi) = \langle a, e_\varphi\rangle_V$ for each $a \in V$. The Berezin calculus on $o(\varphi_0)$ associates with each operator $B$ on $V$ the complex-valued function $s(B)$ on $o(\varphi_0)$ defined by

$$s(B)(\varphi) = \frac{\langle Be_\varphi, e_\varphi\rangle_V}{\langle e_\varphi, e_\varphi\rangle_V}$$

which is called the symbol of $B$.

The following properties of the Berezin calculus can be found in [14], [4] and [9].

**Proposition 3.1.** (1) *The map $B \to s(B)$ is injective.*

(2) *For each operator $B$ on $V$, we have $s(B^*) = \overline{s(B)}$.*

(3) *For $\varphi \in o(\varphi_0)$, $k \in K$ and $B \in \mathrm{End}(V)$, we have*

$$s(B)(\mathrm{Ad}(k)\varphi) = s(\rho(k)^{-1}B\rho(k))(\varphi).$$

(4) *For $U \in \mathfrak{k}$ and $\varphi \in o(\varphi_0)$, we have $s(d\rho(U))(\varphi) = i\beta(\varphi, U)$.*

Now, in order to define the Berezin symbol $S(A)$ of an operator $A$ on $\mathcal{H}$, we first define the pre-symbol $S_0(A)$ of $A$ as a $\mathrm{End}(V)$-valued function on $\mathcal{D}$.

Let $\mathcal{H}^0$ be the subspace of $\mathcal{H}$ generated by the functions $E_Z v$ for $Z \in \mathcal{D}$ and $v \in V$. Clearly, $\mathcal{H}^0$ is a dense subspace of $\mathcal{H}$. Let $\mathcal{C}$ be the space consisting of all operators $A$ on $\mathcal{H}$ such that the domain of $A$ contains $\mathcal{H}^0$ and the domain of $A^*$ also contains $\mathcal{H}^0$. For $Z \in \mathcal{D}$, we denote $h_Z := \kappa(g_Z) \in K^c$. We define the pre-symbol $S_0(A)$ of $A \in \mathcal{C}$ by

$$S_0(A)(Z) = c_\rho^{-1}\, \rho(h_Z^{-1}) E_Z^* A E_Z \rho(h_Z^{-1})^*$$

and the Berezin symbol $S(A)$ of $A$ is then defined as the complex-valued function on $\mathcal{D}\times o(\varphi_0)$ given by

$$S(A)(Z,\varphi)=s(S_0(A)(Z))(\varphi).$$

In [13], we proved the following properties of $S$.

**Proposition 3.2.** (1) *The map $A\to S(A)$ is injective on $\mathcal{C}$.*

(2) *For each $A\in\mathcal{C}$, we have $S(A^*)=\overline{S(A)}$.*

(3) *We have $S(I)=1$.*

(4) *For each $A\in\mathcal{C}$, $g\in G$, $Z\in\mathcal{D}$ and $\varphi\in o(\varphi_0)$, we have*

$$S(A)(g\cdot Z,\varphi)=S(\pi(g)^{-1}A\pi(g))(Z,\mathrm{Ad}(k(g,Z))\varphi)$$

*where $k(g,Z):=h_Z^{-1}\kappa(g\exp Z)^{-1}h_{g\cdot Z}$ is an element of $K$.*

(5) *For each $X\in\mathfrak{g}^c$, $Z\in\mathcal{D}$ and $\varphi\in o(\varphi_0)$, we have*

$$S(d\pi(X))(Z,\varphi)=i\beta(\mathrm{Ad}(g_Z)\varphi,X).$$

Let $\mathcal{O}(\varphi_0)$ be the orbit of $\varphi_0$ under the adjoint action of $G$ on $\mathfrak{g}$. In [13], we have also proved that the map $\Psi:\mathcal{D}\times o(\varphi_0)\to\mathcal{O}(\varphi_0)$ defined by $\Psi(Z,\varphi)=\mathrm{Ad}(g_Z)\varphi$ is a diffeomorphism such that

$$\mathrm{Ad}(g)\,\Psi(Z,\varphi)=\Psi(g\cdot Z,\mathrm{Ad}(k(g,Z))^{-1}\varphi)\tag{3.1}$$

for $g\in G$, $Z\in\mathcal{D}$ and $\varphi\in o(\varphi_0)$.

We fix a $K$-invariant measure $\nu$ on $o(\varphi_0)$ normalized as in [9, Section 2]. Then the measure $\tilde{\mu}:=\mu\otimes\nu$ on $\mathcal{D}\times o(\varphi_0)$ is invariant under the action of $G$ on $\mathcal{D}\times o(\varphi_0)$ given by $g\cdot(Z,\varphi):=(g\cdot Z,\mathrm{Ad}(k(g,Z))^{-1}\varphi)$. Moreover, the measure $\mu_{\mathcal{O}(\varphi_0)}:=(\Psi^{-1})^*(\tilde{\mu})$ is a $G$-invariant measure on $\mathcal{O}(\varphi_0)$.

## 4. The Berezin transform

We denote by $\mathcal{L}_2(\mathcal{H})$ (respectively $\mathcal{L}_2(V)$) the space of Hilbert-Schmidt operators on $\mathcal{H}$ (respectively $V$) endowed with the Hilbert-Schmidt norm $\|\cdot\|_2$ defined by $\|A\|_2^2=\mathrm{Tr}(A^*A)$. Since $V$ is finite-dimensional, we have $\mathcal{L}_2(V)=\mathrm{End}(V)$. We denote by $L^2(\mathcal{D}\times o(\varphi_0))$ (respectively $L^2(\mathcal{D}),L^2(o(\varphi_0))$) the space of functions on $\mathcal{D}\times o(\varphi_0)$ (resp. $\mathcal{D}$, $o(\varphi_0)$) which are square-integrable with respect to the measure $\tilde{\mu}$ (resp. $\mu$, $\nu$). We define similarly the spaces $L^1(\mathcal{D}\times o(\varphi_0))$, $L^\infty(\mathcal{D}\times o(\varphi_0))$, etc.

In [11], we proved the following proposition.

**Proposition 4.1.** *For each $\varphi\in o(\varphi_0)$, let $p_\varphi$ denote the orthogonal projection of $V$ on the line generated by $e_\varphi$. Then the adjoint $s^*$ of the operator $s:\mathcal{L}_2(V)\to L^2(o(\varphi_0))$ is given by*

$$s^*(a)=\int_{o(\varphi_0)}a(\varphi)p_\varphi\,d\nu(\varphi)$$

*for each $a\in L^2(o(\varphi_0))$.*

Our aim is to obtain a similar result for $S$. To this goal, we introduce the operator $T$ defined by

$$T(f)=\int_{\mathcal{D}\times o(\varphi_0)} P_{Z,\varphi}f(Z,\varphi)\,d\mu(Z)d\nu(\varphi)$$

where $P_{Z,\varphi}:=c_\rho^{-1}E_Z\rho(h_Z^{-1})^*p_\varphi\rho(h_Z^{-1})E_Z^*$.

**Proposition 4.2.** (1) *$P_{Z,\varphi}$ is the orthogonal projection of $\mathcal{H}$ on the line generated by $E_Z\rho(h_Z^{-1})^*e_\varphi$.*

(2) *For each $A\in\mathcal{L}_2(\mathcal{H})$, we have $S(A)\in L^\infty(\mathcal{D}\times o(\varphi_0))$.*

(3) *For each $f\in L^1(\mathcal{D}\times o(\varphi_0))$, we have $T(f)\in\mathcal{L}_2(\mathcal{H})$.*

(4) *For each $A\in\mathcal{L}_2(\mathcal{H})$, we have $\mathrm{Tr}(AP_{Z,\varphi})=S(A)(Z,\varphi)$.*

(5) *The operators $S:\mathcal{L}_2(\mathcal{H})\to L^\infty(\mathcal{D}\times o(\varphi_0))$ and $T:L^1(\mathcal{D}\times o(\varphi_0))\to\mathcal{L}_2(\mathcal{H})$ are adjoint in the sense that*

$$\int_{\mathcal{D}\times o(\varphi_0)} S(A)(Z,\varphi)\overline{f(Z,\varphi)}\,d\mu(Z)d\nu(\varphi)=\langle A,T(f)\rangle_2$$

*for each $A\in\mathcal{L}_2(\mathcal{H})$ and $f\in L^1(\mathcal{D}\times o(\varphi_0))$.*

Proof: (1) Let $Z\in\mathcal{D}$. We can decompose $g_Z$ as $g_Z=\exp Zh_Zy$ where $y\in P^-$. Then we have $e=g_Z^*g_Z=y^*h_Z^*\exp Z^*\exp Zh_Zy$ where $e$ is the unit element of $G^c$. This implies that $\kappa(\exp Z^*\exp Z)^{-1}=h_Zh_Z^*$. Therefore, by applying (1) of Proposition 2.2, we obtain

$$E_Z^*E_Z=c_\rho\rho(\kappa(\exp Z^*\exp Z))^{-1}=c_\rho\rho(h_Zh_Z^*). \tag{4.1}$$

By using this equality, we immediately verify that $P_{Z,\varphi}^2=P_{Z,\varphi}$. Moreover, it is clear that $P_{Z,\varphi}^*=P_{Z,\varphi}$. Then $P_{Z,\varphi}$ is an orthogonal projection of $\mathcal{H}$. Using Equality (4.1) again, we get $P_{Z,\varphi}E_Z\rho(h_Z^{-1})^*e_\varphi=E_Z\rho(h_Z^{-1})^*e_\varphi$. Finally, since $p_\varphi$ is a rank one operator, we see that $P_{Z,\varphi}$ is also a rank one operator, hence the orthogonal projection on the line generated by $E_Z\rho(h_Z^{-1})^*e_\varphi$.

(2) Let $A\in\mathcal{L}_2(\mathcal{H})$. We have

$$\|S_0(A)(Z)\|_2\le c_\rho^{-1}\|\rho(h_Z^{-1})E_Z^*\|_{\mathrm{op}}\|A\|_2\|E_Z\rho(h_Z^{-1})^*\|_{\mathrm{op}}.$$

Since

$$\|\rho(h_Z^{-1})E_Z^*\|_{\mathrm{op}}\|E_Z\rho(h_Z^{-1})^*\|_{\mathrm{op}}=\|\rho(h_Z^{-1})E_Z^*E_Z\rho(h_Z^{-1})^*\|_{\mathrm{op}}=\|c_\rho\mathrm{id}_V\|_{\mathrm{op}}=c_\rho,$$

we get $\|S_0(A)(Z)\|_2\le\|A\|_2$. Then we have

$$|S(A)(Z,\varphi)|\le\|S_0(A)(Z)\|_{\mathrm{op}}\le\|S_0(A)(Z)\|_2\le\|A\|_2.$$

Hence $S(A)\in L^\infty(\mathcal{D}\times o(\varphi_0))$.

(3) Let $f\in L^1(\mathcal{D}\times o(\varphi_0))$. Since $\|P_{Z,\varphi}\|_2=1$, we see that $T(f)$ is well-defined as a Bochner integral and that $\|T(f)\|_2\le\|f\|_1$.

(4) Let $A \in \mathcal{L}_2(\mathcal{H})$. Recall that $P_{Z,\varphi}$ is the orthogonal projection on the line generated by $E_Z\rho(h_Z^{-1})^*e_\varphi$. Then, by considering an orthonormal basis $(h_k)_{k\geq 1}$ of $\mathcal{H}$ such that $h_1 = \|E_Z\rho(h_Z^{-1})^*e_\varphi\|_2^{-1}E_Z\rho(h_Z^{-1})^*e_\varphi$, we get

$$\mathrm{Tr}(AP_{Z,\varphi}) = \frac{\langle AE_Z\rho(h_Z^{-1})^*e_\varphi, E_Z\rho(h_Z^{-1})^*e_\varphi\rangle}{\langle E_Z\rho(h_Z^{-1})^*e_\varphi, E_Z\rho(h_Z^{-1})^*e_\varphi\rangle}.$$

Thus, since we have

$$\langle E_Z\rho(h_Z^{-1})^*e_\varphi, E_Z\rho(h_Z^{-1})^*e_\varphi\rangle = \langle \rho(h_Z^{-1})E_Z^*E_Z\rho(h_Z^{-1})^*e_\varphi, e_\varphi\rangle_V = c_\rho\langle e_\varphi, e_\varphi\rangle_V,$$

we find

$$\begin{aligned}\mathrm{Tr}(AP_{Z,\varphi}) &= c_\rho^{-1}\frac{\langle \rho(h_Z^{-1})E_Z^*AE_Z\rho(h_Z^{-1})^*e_\varphi, e_\varphi\rangle_V}{\langle e_\varphi, e_\varphi\rangle_V}\\ &= s(S_0(A)(Z))(\varphi) = S(A)(Z,\varphi).\end{aligned}$$

(5) This is an immediate consequence of (4). □

Now, we can consider the Berezin transform $B := ST$ as an operator from $L^1(\mathcal{D}\times o(\varphi_0))$ to $L^\infty(\mathcal{D}\times o(\varphi_0))$. The following proposition shows that $B$ can be expressed as an integral operator.

**Proposition 4.3.** *For each $f \in L^1(\mathcal{D}\times o(\varphi_0))$, we have*

$$B(f)(Z,\psi) = \int_{\mathcal{D}\times o(\varphi_0)} k(Z,W,\psi,\varphi)\, f(W,\varphi)\, d\mu(W)d\nu(\varphi)$$

*where*

$$k(Z,W,\psi,\varphi) := \frac{|\langle \rho(\kappa(g_Z^{-1}g_W))^{-1}e_\psi, e_\varphi\rangle_V|^2}{\langle e_\varphi, e_\varphi\rangle_V\langle e_\psi, e_\psi\rangle_V}.$$

Proof: We begin with the following remark. Let $Z, W \in \mathcal{D}$. We can write $g_Z = \exp Zh_Zy$ and $g_W = \exp Wh_Wy'$ where $y, y' \in P^-$. Then we have

$$\exp W^*\exp Z = h_W^{*-1}y'^{*-1}g_W^*g_Zy^{-1}h_Z^{-1}.$$

Hence we get $\kappa(\exp W^*\exp Z) = h_W^{*-1}\kappa(g_W^*g_Z)h_Z^{-1}$. Using this equality, we see that

$$\begin{aligned}\rho(h_Z^{-1})E_Z^*E_W\rho(h_W^{-1})^* &= c_\rho\rho(h_Z^{-1})\rho(\kappa(\exp W^*\exp Z))^{-1}\rho(h_W^{-1})^*\\ &= c_\rho\rho(\kappa(g_W^*g_Z))^{-1}.\end{aligned}$$

Now, let $f \in L^1(\mathcal{D}\times o(\varphi_0))$. We have

$$\begin{aligned}&S_0(T(f))(Z) = c_\rho^{-1}\rho(h_Z^{-1})E_Z^*T(f)E_Z\rho(h_Z^{-1})^*\\ &= c_\rho^{-2}\int_{\mathcal{D}\times o(\varphi_0)}\rho(h_Z^{-1})E_Z^*E_W\rho(h_W^{-1})^*p_\varphi\rho(h_W^{-1})E_W^*E_Z\rho(h_Z^{-1})^*\, f(W,\varphi)d\mu(W)d\nu(\varphi).\end{aligned}$$

By the preceding remark, we get

$$S_0(T(f))(Z) = \int_{\mathcal{D}\times o(\varphi_0)} \rho(\kappa(g_W^* g_Z))^{-1} p_\varphi \rho(\kappa(g_W^* g_Z)^{-1})^* \, f(W,\varphi)\, d\mu(W) d\nu(\varphi).$$

Now we aim to compute $S(T(f))(Z,\psi) = s(S_0(T(f))(Z))(\psi)$. We note that, putting $h := \kappa(g_Z^* g_W)$, we have

$$\begin{aligned} s(\rho(h^{-1})^* p_\varphi \rho(h^{-1}))(\psi) &= \frac{\langle \rho(h^{-1})^* p_\varphi \rho(h^{-1}) e_\psi, e_\psi\rangle_V}{\langle e_\psi, e_\psi\rangle_V} \\ &= \frac{\langle p_\varphi \rho(h^{-1}) e_\psi, \rho(h^{-1}) e_\psi\rangle_V}{\langle e_\psi, e_\psi\rangle_V} \\ &= \frac{|\langle \rho(h^{-1}) e_\psi, e_\varphi\rangle_V|^2}{\langle e_\psi, e_\psi\rangle_V \langle e_\varphi, e_\varphi\rangle_V} \end{aligned}$$

since

$$p_\varphi \rho(h^{-1}) e_\psi = \frac{\langle \rho(h^{-1}) e_\psi, e_\varphi\rangle_V}{\langle e_\varphi, e_\varphi\rangle_V}\, e_\varphi.$$

Finally, we obtain

$$s(S_0(T(f))(Z))(\psi) = \int_{\mathcal{D}\times o(\varphi_0)} \frac{|\langle \rho(h^{-1}) e_\psi, e_\varphi\rangle_V|^2}{\langle e_\psi, e_\psi\rangle_V \langle e_\varphi, e_\varphi\rangle_V}\, f(W,\varphi)\, d\mu(W) d\nu(\varphi)$$

as desired. □

## 5. Extension of the Berezin transform to $L^2$-spaces

In this section, we show that the Berezin transform $B := ST$ can be extended to the space $L^2(\mathcal{D}\times o(\varphi_0))$. We retain the notation from Section 4. The first step is to show that the integral

$$I(Z,\psi) := \int_{\mathcal{D}\times o(\varphi_0)} k(Z,W,\psi,\varphi)\, d\mu(W) d\nu(\varphi)$$

is finite for each $(Z,\psi) \in \mathcal{D}\times o(\varphi_0)$. More precisely, we have the following result.

**Lemma 5.1.** *For each $(Z,\psi) \in \mathcal{D}\times o(\varphi_0)$, we have $I(Z,\psi) = c_\rho^{-1}$.*

Proof: First recall that for $a \in V$ we have

$$\int_{o(\varphi_0)} |\langle a, e_\varphi\rangle_V|^2 \|e_\varphi\|_V^2 \, d\nu(\varphi) = \|a\|_V^2$$

(see [9]). Then

$$I(Z,\psi) = \frac{1}{\|e_\psi\|_V^2} \int_{\mathcal{D}} \|\rho(\kappa(g_Z^{-1} g_W))^{-1} e_\psi\|_V^2 \, d\mu(W).$$

Now we perform the change of variables $W \to g_Z \cdot W$ in this integral. Remark that, since $(g_Z g_W)^{-1} g_{g_Z \cdot W} \cdot 0 = 0$, we have $(g_Z g_W)^{-1} g_{g_Z \cdot W} \in K^c P^- \cap G = K$. Denoting this element by $k$, we get $\kappa(g_Z^{-1} g_{g_Z \cdot W}) = \kappa(g_W k) = h_W k$. Then

$$\|\rho(\kappa(g_Z^{-1} g_{g_Z \cdot W}))^{-1} e_\psi\|_V = \|\rho(k^{-1} h_W^{-1}) e_\psi\|_V = \|\rho(h_W)^{-1} e_\psi\|_V.$$

Hence we obtain

$$\begin{aligned} I(Z,\psi) &= \frac{1}{\|e_\psi\|_V^2} \int_{\mathcal{D}} \|\rho(h_W)^{-1} e_\psi\|_V^2 \, d\mu(W) \\ &= \frac{1}{\|e_\psi\|_V^2} \int_{\mathcal{D}} \langle K(W,W)^{-1} e_\psi, e_\psi \rangle_V \, d\mu(W) \end{aligned}$$

since we have $\rho(h_W h_W^*) = K(W,W)$ by Equality (4.1).

On the other hand, recall the reproducing property

$$\langle f(Z), v\rangle_V = \langle f, E_Z v\rangle = \int_{\mathcal{D}} \langle K(W,W)^{-1} f(W), (E_Z v)(W)\rangle_V \, d\mu(W).$$

Applying this equality to the constant function $f(W) = v$ and evaluating at $Z = 0$, we get

$$\|v\|_V^2 = \int_{\mathcal{D}} \langle K(W,W)^{-1} v, (E_0 v)(W)\rangle_V \, d\mu(W).$$

Since we have $(E_0 v)(W) = E_W^* E_0 v = c_\rho v$, we obtain

$$\|v\|_V^2 = c_\rho \int_{\mathcal{D}} \langle K(W,W)^{-1} v, v\rangle_V \, d\mu(W).$$

Finally, applying this equality to $v = e_\psi$, we obtain $I(Z,\psi) = c_\rho^{-1}$. $\square$

**Proposition 5.2.** (1) *The map $B := ST$ can be extended to a bounded operator of $L^2(\mathcal{D} \times o(\varphi_0))$ and we have $\|B\|_{\mathrm{op}} \leq c_\rho^{-1}$.*

(2) *$T$ extends to a bounded operator from $L^2(\mathcal{D} \times o(\varphi_0))$ to $\mathcal{L}_2(\mathcal{H})$, $S$ extends to a bounded operator from $\mathcal{L}_2(\mathcal{H})$ to $L^2(\mathcal{D} \times o(\varphi_0))$ and these operators are adjoint to each other.*

Proof: (1) Let $f \in L^1(\mathcal{D} \times o(\varphi_0)) \cap L^2(\mathcal{D} \times o(\varphi_0))$. Then, using Lemma 5.1 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &|B(f)(Z,\psi)|^2 \\ &\leq \int_{\mathcal{D}\times o(\varphi_0)} k(Z,W,\psi,\varphi) \, d\mu(W) d\nu(\varphi) \\ &\quad \times \int_{\mathcal{D}\times o(\varphi_0)} k(Z,W,\psi,\varphi) |f(W,\varphi)|^2 \, d\mu(W) d\nu(\varphi) \\ &\leq c_\rho^{-1} \int_{\mathcal{D}\times o(\varphi_0)} k(Z,W,\psi,\varphi) |f(W,\varphi)|^2 \, d\mu(W) d\nu(\varphi). \end{aligned}$$

Integrating this inequality and using Lemma 5.1 again, we then obtain

$$\begin{aligned}\int_{\mathcal{D}\times o(\varphi_0)} &|B(f)(Z,\psi)|^2\, d\mu(Z)d\nu(\psi)\\ &\le c_\rho^{-1}\int_{\mathcal{D}\times o(\varphi_0)} k(Z,W,\psi,\varphi)|f(W,\varphi)|^2\, d\mu(Z)d\mu(W)d\nu(\varphi)d\nu(\psi)\\ &\le c_\rho^{-2}\int_{\mathcal{D}\times o(\varphi_0)} |f(W,\varphi)|^2\, d\mu(W)d\nu(\varphi).\end{aligned}$$

Therefore, the result follows.

(2) Let $f\in L^1(\mathcal{D}\times o(\varphi_0))\cap L^2(\mathcal{D}\times o(\varphi_0))$. By applying (5) of Proposition 4.2 to $A=T(f)$ and using (1), we get

$$\|T(f)\|_2^2\le\langle ST(f),f\rangle\le\|f\|_2\|ST(f)\|_2\le c_\rho^{-1}\|f\|_2^2.$$

This implies that $T$ extends to an operator (also denoted by $T$) from $L^2(\mathcal{D}\times o(\varphi_0))$ to $\mathcal{L}_2(\mathcal{H})$. Let $T^*:\mathcal{L}_2(\mathcal{H})\to L^2(\mathcal{D}\times o(\varphi_0))$ be the adjoint of $T$. Recall that we have

$$\langle S(A),f\rangle=\langle A,T(f)\rangle_2=\langle T^*(A),f\rangle$$

for each $A\in\mathcal{L}_2(\mathcal{H})$ and each $f\in L^1(\mathcal{D}\times o(\varphi_0))\cap L^2(\mathcal{D}\times o(\varphi_0))$. This shows that $S$ extends to the operator $T^*:\mathcal{L}_2(\mathcal{H})\to L^2(\mathcal{D}\times o(\varphi_0))$. □

Now we establish that $B$ is $G$-covariant. We denote by $\tau$ the left-regular representation of $G$ on $L^2(\mathcal{D}\times o(\varphi_0))$ defined by $(\tau(g)(f))(Z,\varphi)=f(g^{-1}\cdot(Z,\varphi))$. Then $\tau$ is unitary. We have the following proposition.

**Proposition 5.3.** *For each $f\in L^2(\mathcal{D}\times o(\varphi_0))$ and each $g\in G$, we have $B(\tau(g)f)=\tau(g)(B(f))$.*

Proof: By (4) of Proposition 3.2, we have $\tau(g)S(A)=S(\pi(g)A\pi(g)^{-1})$ for each $A\in\mathcal{L}_2(\mathcal{H})$ and $g\in G$. Since $\tau$ is unitary, the corresponding property for $T=S^*$ is $S^*(\tau(g)f)=\pi(g)S^*(f)\pi(g)^{-1}$ for each $f\in L^2(\mathcal{D}\times o(\varphi_0))$ and $g\in G$. This gives

$$SS^*(\tau(g)f)=S(\pi(g)S^*(f)\pi(g)^{-1})=\tau(g)SS^*(f)$$

for each $f\in L^2(\mathcal{D}\times o(\varphi_0))$, hence the result. □

## 6. Extension of the Berezin transform to symbols of differential operators

Let us introduce some additional notation as in [12, Section 4]. Let $(E_\alpha)_{\alpha\in\Delta_n^+}$ be a basis for $\mathfrak{p}^+$ as in [21, Chapter VIII, Corollary 7.6]. In particular, we have $\mathfrak{g}_\alpha=\mathbb{C}E_\alpha$ and $[E_\alpha,E_{-\alpha}]=\frac{2}{\alpha(H_\alpha)}H_\alpha$ for each $\alpha\in\Delta_n^+$. Let $\alpha_1,\alpha_2,\ldots,\alpha_n$ be an enumeration of $\Delta_n^+$. Let $Z=\sum_{k=1}^n z_kE_{\alpha_k}$ be the decomposition of $Z\in\mathfrak{p}^+$ in the basis $(E_{\alpha_k})$. If $f$ is a holomorphic function on $\mathcal{D}$, then we denote by $\partial_kf$ the partial derivative of $f$ with respect to $z_k$. We say that a function $f(Z)$ on $\mathcal{D}$

is a polynomial of degree $q$ in the variable $Z$ if $f(\sum_{k=1}^{n} z_k E_{\alpha_k})$ is a polynomial of degree $q$ in the variables $z_1, z_2, \ldots, z_n$. For $Z, W \in \mathcal{D}$, we set $l_Z(W) := \log \eta(\exp Z^* \exp W) \in \mathfrak{p}^-$.

Moreover, if $L$ is a Lie group and $X$ is an element of the Lie algebra of $L$ then we denote by $X^+$ the right invariant vector field on $L$ generated by $X$, that is, $X^+(h) = \frac{d}{dt}(\exp tX)h|_{t=0}$ for $h \in L$.

We first recall some useful results, in particular an explicit expression for the derived representation $d\pi$. Let $p_{\mathfrak{p}^+}$, $p_{\mathfrak{k}^c}$ and $p_{\mathfrak{p}^-}$ be the projections of $\mathfrak{g}^c$ onto $\mathfrak{p}^+$, $\mathfrak{k}^c$ and $\mathfrak{p}^-$ associated with the direct decomposition $\mathfrak{g}^c = \mathfrak{p}^+ \oplus \mathfrak{k}^c \oplus \mathfrak{p}^-$. By differentiating the multiplication map from $P^+ \times K^c \times P^-$ onto $P^+K^cP^-$, we can easily prove the following result.

**Lemma 6.1** ([10]). *Let $X \in \mathfrak{g}^c$ and $g = z\,k\,y$ where $z \in P^+$, $k \in K^c$ and $y \in P^-$. We have*

(1) $d\zeta_g(X^+(g)) = (\mathrm{Ad}(z)\, p_{\mathfrak{p}^+}(\mathrm{Ad}(z^{-1})\, X))^+(z)$.
(2) $d\kappa_g(X^+(g)) = (p_{\mathfrak{k}^c}(\mathrm{Ad}(z^{-1})\, X))^+(k)$.
(3) $d\eta_g(X^+(g)) = (\mathrm{Ad}(k^{-1})\, p_{\mathfrak{p}^-}(\mathrm{Ad}(z^{-1})\, X))^+(y)$.

From this result, we immediately deduce the following proposition (see [26, Proposition XII.2.1] and also [10]).

**Proposition 6.2.** *For $X \in \mathfrak{g}^c$ and $f \in \mathcal{H}$, we have*

$$d\pi(X)f(Z) = d\rho(p_{\mathfrak{k}^c}(\mathrm{Ad}((\exp Z)^{-1})\, X))\, f(Z) - (df)_Z\big(p_{\mathfrak{p}^+}(e^{-\operatorname{ad} Z}\, X)\big).$$

*In particular, we have*

(1) *if $X \in \mathfrak{p}^+$ then $d\pi(X)f(Z) = -(df)_Z(X)$;*
(2) *if $X \in \mathfrak{k}^c$ then $d\pi(X)f(Z) = d\rho(X)f(Z) + (df)_Z([Z,X])$;*
(3) *if $X \in \mathfrak{p}^-$ then $d\pi(X)f(Z) = -d\rho([Z,X])f(Z) - \frac{1}{2}(df)_Z([Z,[Z,X]])$.*

Now, we study the form of the Berezin symbols of the operators $d\pi(X_1X_2 \cdots X_q)$ for $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$. The following lemma is the generalization of [12], Lemma 4.1 and Lemma 4.2.

**Lemma 6.3.** (1) *For each $Z, W \in \mathcal{D}$, $W' \in \mathfrak{p}^+$ and $v \in V$, we have*

$$\frac{d}{dt}\,(E_Z v)(W + tW')\big|_{t=0} = -c_\rho d\rho([l_Z(W), W'])\rho(\kappa(\exp Z^* \exp W))^{-1} v.$$

(2) *For $Z, W \in \mathcal{D}$ and $W' \in \mathfrak{p}^+$, we have*

$$\frac{d}{dt}\, l_Z(W + tW')\big|_{t=0} = \frac{1}{2}[l_Z(W), [l_Z(W), W']].$$

(3) *The function $(\partial_{k_1}\partial_{k_2} \cdots \partial_{k_q} E_Z v)(W)$ is of the form $Q(l_Z(W))(E_Z v)(W)$ where $Q$ is a polynomial of degree $\leq q$ with values in* $\mathrm{End}(V)$.
(4) *For each $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$, the operator $d\pi(X_1X_2 \cdots X_q)$ is a sum of terms of the form $P(Z)\partial_{k_1}\partial_{k_2} \cdots \partial_{k_r}$ where $r \leq q$ and $P$ is a polynomial of degree $\leq 2q$ with values in* $\mathrm{End}(V)$.

(5) *For each* $X_1, X_2, \dots, X_q \in \mathfrak{g}^c$, *the pre-symbol* $S_0(d\pi(X_1 X_2 \cdots X_q))$ *is a sum of terms of the form* $\rho(h_Z)^{-1} P(Z) Q(l_Z(Z)) \rho(h_Z)$ *where* $P$ *is a polynomial of degree* $\leq 2q$ *with values in* $\mathrm{End}(V)$ *and* $Q$ *is a polynomial of degree* $\leq q$ *with values in* $\mathrm{End}(V)$.

PROOF: The proof, based on Lemma 6.1 and Proposition 6.2, is similar to that of [12, Lemma 4.1]. Note that (5) is an immediate consequence of (4). □

In the following lemma, we give some expressions for $\|\rho(h_Z)\|_{\mathrm{op}}$ and $\|\rho(h_Z)^{-1}\|_{\mathrm{op}}$ which will be needed in the proof of Proposition 6.5. Recall that we have denoted by $\lambda$ the highest weight of $\rho$ relative to $\Delta_c^+$. We also denote the lowest weight of $\rho$ by $\lambda_{lw}$ (see [30, p. 326]). Moreover, let $\gamma_1, \gamma_2, \dots, \gamma_r$ be a subset of $\Delta_n^+$ consisting of strongly orthogonal roots (see for instance [21, p. 385]). We also set $H_s = [E_{\gamma_s}, E_{-\gamma_s}]$ for $s = 1, 2, \dots, r$.

**Lemma 6.4.** *Let* $Z = \mathrm{Ad}(k)(\sum_{s=1}^r t_s E_{\gamma_s})$ *where* $k \in K$ *and* $1 \geq t_1 \geq t_2 \geq \dots \geq t_r \geq 0$. *Then we have* $\|\rho(h_Z)\|_{\mathrm{op}}^2 = \prod_{s=1}^r (1 - t_s^2)^{\lambda_{lw}(H_s)}$ *and* $\|\rho(h_Z)^{-1}\|_{\mathrm{op}}^2 = \prod_{s=1}^r (1 - t_s^2)^{-\lambda(H_s)}$.

PROOF: If $Z = \mathrm{Ad}(k)(\sum_{s=1}^r t_s E_{\gamma_s})$ where $1 \geq t_1 \geq t_2 \geq \dots \geq t_r \geq 0$ then we have $\kappa(\exp Z^* \exp Z) = k \exp(-\sum_{s=1}^r \log(1 - t_s^2) H_s) k^{-1}$, see for instance [31, p. 3] or [17, p. 231]. Hence the eigenvalues of

$$\rho(h_Z^*)^{-1} \rho(h_Z)^{-1} = \rho(\kappa(\exp Z^* \exp Z)) = \exp\bigl(-\sum_{s=1}^r \log(1 - t_s^2) d\rho(H_s)\bigr)$$

are the $\exp(-\sum_{s=1}^r \log(1 - t_s^2) \mu(H_s))$ for $\mu$ weight of $\rho$. Now, since

$$\log \frac{1}{1 - t_1^2} \geq \log \frac{1}{1 - t_2^2} \geq \cdots \log \frac{1}{1 - t_r^2}$$

we have

$$\mu\left(\sum_{s=1}^r \log \frac{1}{1 - t_s^2} H_s\right) \leq \lambda\left(\sum_{s=1}^r \log \frac{1}{1 - t_s^2} H_s\right)$$

for each weight $\mu$ of $\rho$, [22, p. 16]. This implies that

$$\|\rho(h_Z)^{-1}\|_{\mathrm{op}}^2 = \exp\left(\sum_{s=1}^r \log \frac{1}{1 - t_s^2} \lambda(H_s)\right).$$

The second equality is proved similarly. □

Now we are in position to establish the main result of this section.

**Proposition 6.5.** *Let* $\lambda_0 := d\chi_0|_{\mathfrak{h}^c}$ *and let* $q_\rho := \mathrm{Min}_{1 \leq s \leq r}(-\frac{3}{2}\lambda - \lambda_0 + \frac{1}{2}\lambda_{lw})(H_s)$. *If* $q \leq q_\rho$ *then for each* $X_1, X_2, \dots, X_q \in \mathfrak{g}^c$, *the Berezin transform of* $S(d\pi(X_1 X_2 \cdots X_q))$ *is well-defined.*

Proof: Let $(Z,\psi)\in\mathcal{D}\times o(\varphi_0)$. Fix $g\in G$ such that $g\cdot(0,\varphi_0)=(Z,\psi)$. Then, by using Proposition 5.3, we see that

$$B(f)(Z,\psi)=\int_{\mathcal{D}\times o(\varphi_0)} k(0,W,\varphi_0,\varphi)\,f(g\cdot(W,\varphi))\chi_0(\kappa(\exp W^*\exp W))\,d\mu_L(W)d\nu(\varphi).$$

In particular, if $f=S(d\pi(X_1X_2\cdots X_q))$ then by (4) of Proposition 3.2 we have

$$f(g\cdot(W,\varphi))=S(\pi(g)^{-1}d\pi(X_1X_2\cdots X_q)\pi(g))(W,\varphi)=S(d\pi(Y_1Y_2\cdots Y_q))(W,\varphi)$$

where $Y_k=\mathrm{Ad}(g^{-1})X_k$ for $k=1,2,\ldots,q$.

Now assume that $q\leq q_\rho$. In order to show that $B(f)(Z,\psi)$ is well-defined, we will prove that the integrand

$$J(W,\varphi):=k(0,W,\varphi_0,\varphi)S(d\pi(Y_1Y_2\cdots Y_q))(W,\varphi)\chi_0(\kappa(\exp W^*\exp W))$$

is bounded hence integrable for the measure $d\mu_L(W)d\nu(\varphi)$ on $\mathcal{D}\times o(\varphi_0)$. We begin by the following observations.

(1) We have

$$k(0,W,\varphi_0,\varphi)=\frac{|\langle\rho(h_W^{-1})e_{\varphi_0},e_\varphi\rangle_V|^2}{\|e_{\varphi_0}\|_V^2\|e_\varphi\|_V^2}\leq\|\rho(h_W^{-1})\|_V^2$$

for each $W\in\mathcal{D}$ and $\varphi\in o(\varphi_0)$.

(2) Recall that

$$S_0(d\pi(Y_1Y_2\cdots Y_q))(W)=\sum_{i=1}^{l}\rho(h_W)^{-1}P_i(W)Q_i(l_W(W))\rho(h_W)$$

where the $P_i$ are polynomials of degree $\leq 2q$ with values in $\mathrm{End}(V)$ and the $Q_i$ are polynomials of degree $\leq q$ with values in $\mathrm{End}(V)$ for $i=1,2,\ldots,l$. Then we have

$$\begin{aligned}|S(d\pi(Y_1Y_2\cdots Y_q))(W,\varphi)|&\leq\|S_0(d\pi(Y_1Y_2\cdots Y_q))(W)\|_{\mathrm{op}}\\&\leq C\,\|\rho(h_W)^{-1}\|_{\mathrm{op}}\|\rho(h_W)\|_{\mathrm{op}}\sum_{i=1}^{l}\|Q_i(l_W(W))\|_{\mathrm{op}}\end{aligned}$$

where $C$ is a constant (independent of $W$).

(3) For each $W\in\mathcal{D}$, we can write $W=\mathrm{Ad}(k)(\sum_{s=1}^r t_sE_{\gamma_s})$ where $k\in K$ and $1\geq t_1\geq t_2\geq\ldots\geq t_r\geq 0$ as in Lemma 6.4, see [22, p. 16] and [25, Theorem 3]. Then we have

$$\chi_0\big(\kappa(\exp W^*\exp W)\big)=\exp\left(\sum_{s=1}^r\log\frac{1}{1-t_s^2}\lambda_0(H_s)\right)$$

and

$$\log \eta(\exp W^* \exp W) = \mathrm{Ad}(k)\left(-\sum_{s=1}^{r} \frac{t_s}{1-t_s^2} E_{-\gamma_s}\right),$$

see for instance [17, p. 231].

Using these observations and Lemma 6.4, we obtain

$$\begin{aligned}|J(W,\varphi)| &\leq C\, \|\rho(h_W)^{-1}\|^3_{\mathrm{op}} \|\rho(h_W)\|_{\mathrm{op}} \chi_0(\kappa(\exp W^* \exp W)) \sum_{i=1}^{l} \|Q_i(l_W(W))\|_{\mathrm{op}} \\ &\leq C' \prod_{i=1}^{l} (1-t_s^2)^{q_\rho - q}\end{aligned}$$

where $C'$ is a constant. Hence the result follows. □

**Example.** In order to illustrate the previous proposition, we consider the case $G = SU(2,1)$ and $K = S(U(2) \times U(1)) \simeq U(2)$. Then we have $\mathfrak{g}^c = sl(3,\mathbb{C})$ and $\mathfrak{h}^c$ is the abelian subalgebra of $\mathfrak{g}^c$ consisting of the matrices $\mathrm{Diag}(a_1, a_2, a_3)$ where $a_i \in \mathbb{C}$ for $i = 1, 2, 3$ and $a_1 + a_2 + a_3 = 0$. The set of roots of $\mathfrak{h}^c$ on $\mathfrak{g}^c$ is $\{\lambda_i - \lambda_j \,:\, 1 \leq i \neq j \leq 3\}$ where $\lambda_i(X) = a_i$ for $X \in \mathfrak{h}^c$ as above. We take the set of positive roots to be $\lambda_1 - \lambda_2$ (compact root), $\lambda_1 - \lambda_3$ and $\lambda_2 - \lambda_3$ (non-compact roots). Hence the system of strongly orthogonal roots reduces to $\gamma = \lambda_1 - \lambda_3$.

Let $H_1 = \mathrm{Diag}(1, 1, -2)$ and $H_2 = \mathrm{Diag}(1, -1, 0)$ in $\mathfrak{h}^c$. Let $\rho_m$ be the unitary irreducible representation of $SU(2)$ of dimension $m+1$. Here we consider $SU(2)$ as a subgroup of $K \simeq U(2)$. The highest weight $\tilde{\lambda}_m$ of $\rho_m$ is defined by $\tilde{\lambda}_m(H_2) = m$. Let $\mathbb{S}^1$ be the group of diagonal matrices of the form $\mathrm{Diag}(e^{i\theta}, e^{i\theta}, e^{-2i\theta})$ where $\theta \in \mathbb{R}$. Let $n \in \mathbb{Z}$ be such that $m+n$ is even. Then $\rho_{m,n}(ug) := u^n \rho_m(g)$ is a unitary irreducible representation of $K$ and all the unitary irreducible representations of $K$ are of this form [7, p. 87].

The highest weight $\lambda$ of $\rho_{m,n}$ is defined by $\lambda(H_1) = n$ and $\lambda(H_2) = m$. Moreover, we have $\lambda_{lw}(H_1) = n$ and $\lambda_{lw}(H_2) = -m$. Also, note that $\lambda_0(H_1) = 2$ and $\lambda_0(H_2) = 0$ (see [26, p. 541]).

Then the condition of Proposition 2.1 is $n + 2 < m < -n - 4$. Furthermore, since we have $[E_\gamma, E_{-\gamma}] = \frac{2}{\gamma(H_\gamma)} H_\gamma$ where $H_\gamma = \mathrm{Diag}(\frac{1}{6}, 0, -\frac{1}{6})$, we easily obtain that $q_\rho = -\frac{1}{2}n - m - 3$.

## 7. Stratonovich-Weyl correspondence

In this section, we construct a Stratonovich-Weyl correspondence associated with $\pi$ by using the method of [19], [11] and [12]. Recall that the notion of Stratonovich-Weyl correspondence was introduced in [28] as a natural generalization of the classical Weyl correspondence [1], [18]. Stratonovich-Weyl correspondences were systematically studied, especially by J.M. Gracia-Bondìa, J.C. Vàrilly and their co-workers, see in particular [19], [15] and [20] (see also the work of J. Arazy and H. Upmeier on invariant symbolic calculi [2], [3]).

**Definition 7.1.** Let $G_0$ be a Lie group and $\pi_0$ a unitary representation of $G_0$ on a Hilbert space $\mathcal{H}_0$. Let $M$ be a homogeneous $G_0$-space and let $\mu_0$ be a (suitably normalized) $G_0$-invariant measure on $M$. Then a Stratonovich-Weyl correspondence for the triple $(G_0, \pi_0, M)$ is an isomorphism $W$ from a vector space of operators on $\mathcal{H}_0$ to a space of functions on $M$ satisfying the following properties:

(1) the function $W(A^*)$ is the complex-conjugate of $W(A)$;
(2) Covariance: we have $W(\pi_0(g)\, A\, \pi_0(g)^{-1})(x) = W(A)(g^{-1}\cdot x)$;
(3) Traciality: we have

$$\int_M W(A)(x)W(B)(x)\, d\mu(x) = \mathrm{Tr}(AB).$$

The previous definition is adapted from [15, p. 906]. Note that here we have dropped the requirement that $W$ maps the identity operator $I$ of $\mathcal{H}_0$ to the constant function 1 since it is not adapted to the present situation where $I$ is not Hilbert-Schmidt. However, in general, this requirement should hold in some generalized sense, up to a suitable normalization of $\mu$, see [15].

The basic example is the case when $G_0$ is the $(2n+1)$-dimensional Heisenberg group $H_n$ which acts on $\mathbb{R}^{2n}$ by translations and $\pi_0$ is the Schrödinger representation of $H_n$ on $L^2(\mathbb{R}^n)$. In that case, the classical Weyl correspondence gives a Stratonovich-Weyl correspondence for the triple $(H_n, \pi_0, \mathbb{R}^{2n})$ [18], [20].

When $G_0$ is a compact semi-simple Lie group, $\pi_0$ a unitary irreducible representation of $G_0$ and $M$ the coadjoint orbit of $G_0$ which is associated with $\pi_0$ by the Kostant-Kirillov method of orbits [24], a Stratonovich-Weyl correspondence for $(G_0, \pi_0, M)$ was constructed in [19] and [11] by a taking the isometric part in the polar decomposition of the Berezin calculus on $M$. The same method also works for the holomorphic discrete series representations of scalar type of a semi-simple Lie group, see [12]. Now, we will apply this method to construct a Stratonovich-Weyl correspondence associated with $\pi$ as an application of the results of Section 5.

We introduce the polar decomposition of $S : \mathcal{L}_2(\mathcal{H}) \to L^2(\mathcal{D} \times o(\varphi_0))$. We have $S = (SS^*)^{1/2}W = B^{1/2}W$ where $W = B^{-1/2}S$ is a unitary operator from $\mathcal{L}_2(\mathcal{H})$ onto $L^2(\mathcal{D} \times o(\varphi_0))$. Then we have the following proposition.

**Proposition 7.2.** (1) *The map $W : \mathcal{L}_2(\mathcal{H}) \to L^2(\mathcal{D} \times o(\varphi_0))$ is a Stratonovich-Weyl correspondence for the triple $(G, \pi, \mathcal{D} \times o(\varphi_0))$.*
(2) *The map $\mathcal{W}$ from $\mathcal{L}_2(\mathcal{H})$ to $L^2(\mathcal{O}(\varphi_0), \mu_{\mathcal{O}(\varphi_0)})$ defined by $\mathcal{W}(f) = W(f \circ \Psi)$ is a Stratonovich-Weyl correspondence for the triple $(G, \pi, \mathcal{O}(\varphi_0))$.*

Proof: (1) Since $W$ is unitary, we have just to verify that the properties (1) and (2) of Definition 7.1 are satisfied. Since we have the properties $S(A^*) = \overline{S(A)}$ and $S^*(\overline{f}) = (S^*f)^*$, we see that $B$ hence $B^{-1/2}$ commute with complex conjugation. This gives Property (1). Finally, Property (2) is a consequence of the covariance properties of $S$, $S^*$ and $B$, see (4) of Proposition 3.2 and Proposition 5.3.

(2) This is an immediate consequence of (1). □

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(*Received* November 27, 2011)