Nonlocal systems of BVPs with asymptotically superlinear boundary conditions

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Abstract. In this paper we consider a coupled system of second-order boundary value problems with nonlocal, nonlinear boundary conditions, and we examine conditions under which such problems will have at least one positive solution. By imposing only an asymptotic growth condition on the nonlinear boundary functions, we are able to achieve generalizations over existing works and, in particular, we allow for the nonlocal terms to be able to be realized as Lebesgue-Stieltjes integrals possessing signed Borel measures. We conclude with a numerical example to illustrate the use of one of our two main results.

Keywords: coupled system of second-order boundary value problems, nonlocal boundary condition, nonlinear boundary condition, superlinear growth, positive solution

Classification: Primary 34B10, 34B15, 34B18; Secondary 47H07, 47H10

1. Introduction

In this paper we consider a system of nonlocal boundary value problems with nonlinear boundary conditions. In particular, we consider the nonlinear system of boundary value problems

$$\begin{aligned}
&x''(t) = -a_1(t)g_1(x(t), y(t)),\ t \in (0,1),\\
&y''(t) = -a_2(t)g_2(x(t), y(t)),\ t \in (0,1),\\
&x(0) = 0 = y(0),\\
&x(1) = H_1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right), \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^1\right)\right),\\
&y(1) = H_2\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^2\right), \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^2\right)\right),
\end{aligned} \tag{1.1}$$

where ε_0^1, $\varepsilon_0^2 > 0$ are constants, which shall be specified later, $\xi_0^1, \xi_0^2 \in (0,1)$ are fixed, ϕ_1, $\phi_2 : \mathcal{C}([0,1]) \to \mathbb{R}$ are linear functionals, which capture the nonlocal nature of the boundary conditions, and H_1, $H_2 : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, which capture the nonlinear nature of the boundary conditions. We also assume that the nonlinearities g_1, $g_2 : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are continuous functions. The nonlocal terms here are quite general since they are realized as

Lebesgue-Stieltjes integrals — that is,

$$\phi_1(x) := \int_{[0,1]} x(t)\, d\alpha_1(t) \quad\text{and}\quad \phi_2(y) := \int_{[0,1]} y(t)\, d\alpha_2(t), \tag{1.2}$$

with α_1, $\alpha_2 \in BV([0,1])$. Since it may be assumed without loss that, in fact, α_1, $\alpha_2 \in NBV([0,1])$, we get that associated to each of α_1, α_2 there exists a unique Borel measure, say μ_{α_1} and μ_{α_2}, respectively. In our context, importantly, these measures may be *signed*.

Here we study the existence of at least one positive solution to problem (1.1). To accomplish this task, we use the perturbation terms in (1.1) — namely, $\varepsilon_0^1 x(\xi_0^1)$, $\varepsilon_0^2 y(\xi_0^1)$, $\varepsilon_0^1 x(\xi_0^2)$, and $\varepsilon_0^2 y(\xi_0^2)$ — as well as a new condition on the nonlinear functions H_1 and H_2. These novelties reveal, in a way that shall be delineated momentarily, that many of the restrictions previous authors have imposed on the various terms appearing in other problems similar to (1.1) are, in fact, unnecessary in our setting. Our principal condition on these functions is to require that, for each $i = 1, 2$,

$$\lim_{z_1+z_2\to+\infty} \frac{H_i\left(z_1, z_2\right)}{z_1^{p_i^\infty} + z_2^{q_i^\infty}} = +\infty \tag{1.3}$$

holds for some p_i^∞, $q_i^\infty \in (0,1]$ with at least one of p_i^∞ and q_i^∞, for each $i = 1, 2$, able to be taken equal to unity. In particular (cf., Remark 3.2), condition (1.3) implies that each of H_1 and H_2 may enjoy asymptotically superlinear growth in at least one of the two coordinate directions (cf., Remark 3.3). We will even give an existence result associated to the somewhat more relaxed condition

$$\limsup_{z_1+z_2\to 0^+} \frac{H_i\left(z_1, z_2\right)}{z_1 + z_2} < \rho_i, \tag{1.4}$$

for each $i = 1, 2$, with ρ_i a positive constant to be selected later; importantly, the result associated to condition (1.4) will even be applicable in the unperturbed case — i.e., $\varepsilon_0^1 = \varepsilon_0^2 = 0$. It should be pointed out that, in fact, Yang [16], [17] introduced an asymptotic condition similar to (1.4), though in the context of a slightly different problem. Regardless, Yang imposes a number of other hypotheses — such as complicated conditions on the equivalent of our nonlinearities g_1 and g_2 as well as the assumption that the equivalent of μ_{α_1} and μ_{α_2} be positive — with which we completely dispense here.

In any case, to place problem (1.1) in an appropriate context, we remark that it is, in fact, most closely related to recent papers both of Kang and Wei [10] and of Infante and Pietramala [7]. Regarding [10], Kang and Wei considered a problem very similar to (1.1). However, they were forced to assume that each of the measures μ_{α_1} and μ_{α_2} was *positive*. Moreover, regarding their equivalent of the nonlinearities g_1 and g_2 appearing in (1.1), they assumed that these functions satisfied very strict growth conditions. On the other hand, regarding [7], a similar problem to (1.1) was considered, a principal difference being that the equivalent of

H_1 and H_2 were functions of a single variable only — for instance, $H_1(x)$. In any case, the authors there assumed that each of the measures μ_{α_1} and μ_{α_2} was positive. Furthermore, they assumed that the nonlinear boundary nonlinearities (i.e., the equivalent of H_1 and H_2) satisfied uniformly linear growth — that is, there were $0 \leq \alpha < \beta$ such that $\alpha z \leq H(z) \leq \beta z$, for all $z \geq 0$. This latter condition is somewhat restrictive, and we remove it completely in this work. Finally, our techniques even allow for the nonlinearities g_1 and g_2 to have completely different limiting behavior — cf., point (5) below.

In addition to [7], [10], there have been many other recent works on nonlocal, nonlinear boundary value problems — see, for example, [6], [8], [9], [16], [17]. While our work here is slightly less directly related to these, it is, nonetheless, nontrivially connected to these other papers, and we provide here techniques and insights not found in any of those other works. It is certainly important to mention that the basic cone theoretic technique used in this paper is indebted to the important paper of Infante and Webb [12]. Finally, we mention that our results here complement certain of the results which we have recently given in [4].

In summary, we provide here the following generalizations over preceding works.

(1) We allow for each of μ_{α_1} and μ_{α_2} to be *signed* measures rather than merely positive. This is an improvement over the preceding works, as intimated above.

(2) We do *not* assume a uniform linear growth condition on either H_1 or H_2. We instead assume either the asymptotic condition given in (1.3) together with an assumption that these functions possess superlinear growth as $z_1 + z_2 \to 0$ or condition (1.4). In particular, this shows that superlinear growth at $(+\infty, +\infty)$ is allowable. More generally, one need *not* assume a uniform linear growth condition as seems to appear in nearly all works on this sorts of problems — cf., [6], [7], [8], [9] — since in our setting there may be no $\beta > 0$ such that $H_i(z_1, z_2) \leq \beta(z_1 + z_2)$, for all z_1, $z_2 \geq 0$.

(3) Specifically regarding Yang's works [16], [17], we point out that our results here even provide some interesting generalizations of the methods contained therein. In particular, while the results of [16], [17] concern different problems than (1.1), those works do appear to be among the only ones to consider an asymptotic condition with respect to the nonlinear boundary functions, at least to the best of the author's knowledge. A close examination of the proofs in those works, however, reveals that they use in a very explicit way the positivity of the respective Stieltjes measures. Lacking this positivity, as we do here, we must search for alternative approaches. Consequently, we feel that our results here represent an interesting advancement over those presented in [16], [17].

(4) We believe that our techniques even allow H to be only eventually positive, though we do not prove such a theorem here — see [3] for an exemplar of this extension in a context somewhat different from this one.

(5) We show that the assumption of asymptotic superlinearity of the functions H_1 and H_2 allows for neither g_1 nor g_2 to have any particular type

of growth (e.g., sub- or superlinearity) as $\|(x,y)\| \to +\infty$. In particular, this means that g_1 and g_2 can have completely different limiting behavior. For example, g_1 could be sublinear as $\|(x,y)\| \to +\infty$, whilst g_2 is superlinear as $\|(x,y)\| \to +\infty$. While Yang also allowed for mixed asymptotic behavior of the nonlinearities in [16], a cursory examination of that paper indicates that a number of complicated conditions are required to deduce that result. By contrast, our conditions are quite simple and relatively easy to check computationally.

2. Preliminaries

We consider in this work the space $\mathcal{X} := \mathcal{B} \times \mathcal{B}$, where $\mathcal{B}$ represents the Banach space $\mathcal{C}([0,1])$ when equipped with the usual supremum norm, $\|\cdot\| := \|\cdot\|_\infty$. Note — see Dunninger and Wang [2] — that $\mathcal{X}$ becomes a Banach space when equipped with the norm $\|(x,y)\| := \|x\| + \|y\|$. It is then known that a fixed point in $\mathcal{X}$ of

(2.1)
$$\begin{aligned} &S(x,y)(t)\\ &:= (T_1(x,y), T_2(x,y))\\ &= \Bigg(tH_1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right), \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^1\right)\right) + \int_0^1 G(t,s)a_1(s)g_1(x(s),y(s))\,ds,\\ &\qquad tH_2\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^2\right), \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^2\right)\right) + \int_0^1 G(t,s)a_2(s)g_2(x(s),y(s))\,ds\Bigg) \end{aligned}$$

is a solution of problem (1.1), where $S : \mathcal{X} \to \mathcal{X}$ and $T_i : \mathcal{X} \to \mathcal{B}$, for each $i = 1, 2$. Here $G : [0,1] \times [0,1] \to \mathbb{R}$ appearing in (2.1) is the Green's function associated to the two-point conjugate problem — that is,

$$G(t,s) := \begin{cases} t(1-s), & 0 \le t \le s \le 1,\\ s(1-t), & 0 \le s \le t \le 1, \end{cases} \tag{2.2}$$

as is well known — see, for example, [11]. In the sequel, we shall assume that the set $[a,b]$ is a given fixed subinterval of $(0,1)$. With this declaration it is then well known that there is a constant $\gamma := \min_{t\in[a,b]}\{t, 1-t\}$ such that

$$\min_{t\in[a.b]} G(t,s) \ge \gamma \max_{t\in[0,1]} G(t,s) = \gamma G(s,s), \tag{2.3}$$

for each $s \in [0,1]$. Note that $\gamma \in (0,1)$. Finally, let us also recall as a preliminary lemma Krasnosel'skiĭ's fixed point theorem — see [1].

Lemma 2.1. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{K} \subseteq \mathcal{B}$ be a cone. Assume that Ω_1 and Ω_2 are bounded open sets contained in $\mathcal{B}$ such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Assume, further, that $T : \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{K}$ is a completely continuous operator. If either*

(1) $\|Ty\| \le \|y\|$ *for* $y \in \mathcal{K} \cap \partial\Omega_1$ *and* $\|Ty\| \ge \|y\|$ *for* $y \in \mathcal{K} \cap \partial\Omega_2$*; or*

(2) $\|Ty\| \geq \|y\|$ *for* $y \in \mathcal{K} \cap \partial\Omega_1$ *and* $\|Ty\| \leq \|y\|$ *for* $y \in \mathcal{K} \cap \partial\Omega_2$;
then T *has at least one fixed point in* $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main result and numerical example

We begin by listing the various structural conditions we impose on the constituent parts of problem (1.1). These conditions are the following.

H1: For each i, let $H_i : \mathbb{R}^2 \to [0, +\infty)$ be a real-valued, continuous function. Moreover, $H_i : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ — i.e., H_i is nonnegative when restricted to $[0, +\infty) \times [0, +\infty)$.

H2: For each i, the functional $\phi_i(y)$ appearing in (1.1) is linear and, in particular, has the realization

$$\phi_i(y) := \int_{[0,1]} y(t)\, d\alpha_i(t), \tag{3.1}$$

where $\alpha_i : [0, 1] \to \mathbb{R}$ satisfies $\alpha_i \in BV([0, 1])$.

H3: For each i, there is a constant $\varepsilon_1^i \in [0, \frac{1}{2})$ such that the functional ϕ_i in (1.1) satisfies the inequality

$$|\phi_i(y)| \leq \varepsilon_1^i \|y\| \tag{3.2}$$

for all $y \in \mathcal{C}([0, 1])$.

H4: For each i, there are $p_i^\infty \in (0, 1]$ and $q_i^\infty \in (0, 1]$, where for each i at least one of p_i^∞ and q_i^∞ is equal to unity, such that

$$\lim_{z_1+z_2\to+\infty} \frac{H_i(z_1, z_2)}{z_1^{p_i^\infty} + z_2^{q_i^\infty}} = +\infty \tag{3.3}$$

holds. Furthermore, for each i it holds that

$$\lim_{z_1+z_2\to0^+} \frac{H_i(z_1, z_2)}{z_1 + z_2} = 0. \tag{3.4}$$

H5: We find that

$$\lim_{x+y\to0^+} \frac{g_1(x, y)}{x + y} = 0 \quad \text{and} \quad \lim_{x+y\to0^+} \frac{g_2(x, y)}{x + y} = 0. \tag{3.5}$$

H6: The constants ε_0^1, ε_0^2, ε_1^1, and ε_1^2 satisfy

$$0 \leq \varepsilon_0^1 + \varepsilon_0^2 + \varepsilon_1^2 + \varepsilon_1^2 < \frac{1}{2}. \tag{3.6}$$

H7: For each i, each of

$$\int_{[0,1]} t\, d\alpha_i(t) \geq 0 \tag{3.7}$$

and

$$\int_{[0,1]} G(t,s)\, d\alpha_i(t) \geq 0 \tag{3.8}$$

holds, where the latter holds for each $s \in [0,1]$.

Let us make some brief remarks regarding certain of the preceding conditions.

Remark 3.1. Regarding conditions (H2)–(H3), we point out that a wide variety of functions satisfy these conditions. Indeed, consider the following pair of functionals.

$$\begin{aligned} \phi_1^i(y) &:= \int_F y(t)\, dt, \\ \phi_2^i(y) &:= \sum_{k=1}^{n} a_k y\left(\xi_k\right). \end{aligned} \tag{3.9}$$

Since each of $(3.9)_1$–$(3.9)_2$ is linear, each satisfies (H2). On the other hand, so long as $m(F) \leq \varepsilon_0^i$, say, where m is the Lebesgue measure, then $(3.9)_1$ satisfies (H3). Provided that $\sum_{k=1}^{n} |a_k| \leq \varepsilon_0^i$, then $(3.9)_2$ satisfies (H3). Example 3.9 contains another example.

Remark 3.2. Regarding condition (H4) and specifically (3.3) therein, this is the asymptotic superlinear condition which, in part, distinguishes our methods here from others. On the other hand, (3.4) appearing in condition (H4) implies that H is also superlinear as $(x,y) \to (0^+,0^+)$. Some functions, $H : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$, satisfying condition (H4), then, are the following. (In each case, $p_i^\infty = q_i^\infty = 1$, for each i.)

$$\begin{aligned} H\left(z_1,z_2\right) &:= z_1^{r_1} + z_2^{r_2},\ r_1, r_2 > 1, \\ H\left(z_1,z_2\right) &:= \left(z_1+z_2\right)^r \cos\left(\frac{1}{z_1+z_2+1}\right),\ r > 1, \\ H\left(z_1,z_2\right) &:= \begin{cases} \left(z_1+z_2\right)^2, & 0 \leq z_1+z_2 \leq 1, \\ e^{z_1+z_2-1}, & z_1+z_2 > 1. \end{cases} \end{aligned} \tag{3.10}$$

It is easy to check that each of $(3.10)_1$–$(3.10)_3$ satisfies each part of condition (H4).

Furthermore, we should mention that each of the functions above cannot be incorporated into the theory of either [7] or [10] due to the superlinear growth at $(+\infty,+\infty)$. In fact, such nonlinear boundary functions could not be incorporated into any of the results given in [6], [8], [9], [10] for that matter. So, condition (H4) allows for a vastly different variety of nonlinear boundary functions than other recent works on these sorts of problems. Moreover, as shall be explicated in the proof of Theorem 3.5, which is our first existence result, this asymptotic superlinear growth condition also allows for the mixed growth of the nonlinearities g_1 and g_2, as mentioned in Section 1.

Remark 3.3. Also regarding condition (H4), we point out that this condition allows for H_i to have different types of growth in the different coordinate directions (i.e., when either $z_1 = 0$ or $z_2 = 0$). For example, consider the continuous function $H : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ defined by

$$H(z_1, z_2) := \begin{cases} (z_1 + z_2)^2 \left(z_1^2 + \sqrt{z_2}\right), & 0 \leq z_1 + z_2 \leq 1, \\ z_1^2 + \sqrt{z_2}, & z_1 + z_2 \geq 1. \end{cases} \tag{3.11}$$

In the z_1-coordinate direction, we find that H grows superlinearly as $z_1 + z_2 \to +\infty$. On the other hand, in the z_2-coordinate direction, we find that H grows sublinearly as $z_1 + z_2 \to +\infty$. Finally, it holds that

$$\lim_{z_1+z_2\to 0^+} \frac{H(z_1, z_2)}{z_1 + z_2} = 0 \quad \text{and} \quad \lim_{z_1+z_2\to+\infty} \frac{H(z_1, z_2)}{z_1 + z_2^{0.3}} = +\infty. \tag{3.12}$$

Remark 3.4. As remarked in Section 1, we believe that the conditions imposed on H_i by condition (H4) may be changed in a manner similar to the argument presented in [3]. But we leave such investigations for future work.

Now, let γ_0 be the constant defined by

$$\gamma_0 := \min\{a, 1 - b\}, \tag{3.13}$$

where $\gamma_0 \in (0, 1)$. Then the cone, $\mathcal{K}$, we shall use in the sequel is then defined by (3.14)

$$\mathcal{K} := \left\{(x, y) \in \mathcal{X} : x, y \geq 0, \ \min_{t\in[a,b]}[x(t) + y(t)] \geq \gamma_0 \|(x, y)\|, \ \phi_1(x), \phi_2(y) \geq 0\right\},$$

which is a simple modification of a cone first introduced by Infante and Webb [12]. Let us point out at this juncture that $\mathcal{K}$ does not contain only the neutral element of $\mathfrak{X}$. Indeed, if we put, say, $\boldsymbol{\beta}_1(t) := (t, 0)$, $\boldsymbol{\beta}_2(t) := (0, t)$, and $\boldsymbol{\beta}_3(t) := (\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2)(t) = (t, t)$, then it is easy to see that $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3 \in \mathcal{K}$ so that $\mathcal{K}$ contains infinitely many nontrivial elements of $\mathfrak{X}$.

In any case, with these preliminary observations, we now state and prove our main result. We note, however, that in the statement of this theorem we assume that $p_1^\infty = p_2^\infty = 1$. In other words, it is the numbers q_1^∞, q_2^∞ that can be potentially less than unity. We do this only for definiteness and ease of exposition in the sequel.

Theorem 3.5. *Assume that $\xi_0^1, \xi_0^2 \in [a, b]$, where $[a, b]$ is a fixed set satisfying $[a, b] \Subset (0, 1)$ as in Section 2. Then there exists a number $\delta \in (0, 1)$ such that if both $q_1^\infty, q_2^\infty \in (1 - \delta, 1]$ and (H1)–(H7) hold, then problem (1.1) has at least one positive solution.*

PROOF: To begin, as in (2.1) above, we consider the operator $S : \mathfrak{X} \to \mathfrak{X}$ defined by

$$S(x, y)(t) := (T_1(x, y), T_2(x, y)) \tag{3.15}$$

where, for each $i = 1, 2$, we have that $T_i : \mathcal{X} \to \mathcal{B}$ is defined by

$$\begin{aligned} T_i(x,y) &:= tH_i\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^i\right), \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^i\right)\right) \\ &\quad + \int_0^1 G(t,s)a_i(s)g_i\left(x(s), y(s)\right)\,ds. \end{aligned} \tag{3.16}$$

We shall first argue that $S : \mathcal{K} \to \mathcal{K}$. To this end, it is obvious that for $(x, y) \in \mathcal{K}$, it follows that $T_i(x, y)(t) \geq 0$, for each $t \in [0, 1]$ and $i = 1, 2$. We also note from the definition of γ_0 in (3.13) that

$$\begin{aligned} \min_{t\in[a,b]} T_i(x,y) &\geq \gamma_0 H_i\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^i\right), \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^i\right)\right) \\ &\quad + \gamma \max_{t\in[0,1]} \int_0^1 G(t,s)a_i(s)g_i(x(s),y(s))\,ds \\ &\geq \gamma_0 \|T_i\left(x,y\right)\|. \end{aligned} \tag{3.17}$$

We conclude that

$$\min_{t\in[a,b]} \left[\left(T_1(x,y)\right)(t) + \left(T_2(x,y)\right)(t)\right] \geq \gamma_0 \|S\left(x,y\right)\|. \tag{3.18}$$

Finally, we observe that

$$\begin{aligned} \phi_1\left(T_1(x,y)\right) &= H_1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right), \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^1\right)\right) \int_{[0,1]} t\,d\alpha_1(t) \\ &\quad + \int_{[0,1]} \int_0^1 G(t,s)a_1(s)g_1(x(s),y(s))\,ds\,d\alpha_1(t) \\ &= H_1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right), \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^1\right)\right) \int_{[0,1]} t\,d\alpha_1(t) \\ &\quad + \int_0^1 \left[\int_{[0,1]} G(t,s)\,d\alpha_1(t)\right] a_1(s)g_1\left(x(s),y(s)\right)\,ds \\ &\geq 0, \end{aligned} \tag{3.19}$$

where the final inequality follows from assumption (H7). In a similar way, it follows that $\phi_2(T_2(x, y)) \geq 0$. Thus, $S : \mathcal{K} \to \mathcal{K}$, as claimed. Let us also point out at this juncture that, by a standard argument involving the Arzela-Ascoli theorem (recall here that H_i is assumed to be continuous, for each $i = 1, 2$), we find that the operator S is completely continuous; we omit the details of this argument, however.

Now, by condition (H5) we find that there is a number $r_1 > 0$ such that

$$g_1(x,y) \leq \eta_1(x+y) \tag{3.20}$$

whenever $\|(x,y)\| \le r_1$ and where $\eta_1 > 0$ satisfies

$$\eta_1 \max\left\{\int_0^1 G(s,s)a_1(s)\,ds, \int_0^1 G(s,s)a_2(s)\,ds\right\} \le \frac{1}{4}\,. \tag{3.21}$$

In addition, condition (H4) — i.e., equation (3.4) — implies the existence of a number $r_1^* > 0$ such that, for each $i = 1, 2$,

$$\begin{aligned} &H_i\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^i\right), \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^i\right)\right) \\ &\qquad < \eta_2\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^i\right) + \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^i\right)\right) \end{aligned} \tag{3.22}$$

whenever

$$\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^i\right) + \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^i\right) < r_1^*, \tag{3.23}$$

and where $\eta_2 > 0$ is defined by

$$\eta_2 := \frac{1}{8\max\left\{\varepsilon_0^1, \varepsilon_0^2, \varepsilon_1^2, \varepsilon_1^2\right\}}\,. \tag{3.24}$$

Notice that

$$\begin{aligned} &\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^i\right) + \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^i\right) \\ &\quad \le \varepsilon_1^1\|x\| + \varepsilon_1^2\|y\| + \varepsilon_0^1\|x\| + \varepsilon_0^2\|y\| \\ &\quad \le \left[\max\left\{\varepsilon_1^1, \varepsilon_1^2\right\} + \max\left\{\varepsilon_0^1, \varepsilon_0^2\right\}\right]\|(x,y)\| \\ &\quad \le 2\max\left\{\varepsilon_0^1, \varepsilon_0^2, \varepsilon_1^2, \varepsilon_1^2\right\}\|(x,y)\|. \end{aligned} \tag{3.25}$$

So, in particular, if $(x,y) \in \mathcal{K}$ satisfies

$$\|(x,y)\| < \frac{r_1^*}{2\max\left\{\varepsilon_0^1, \varepsilon_0^2, \varepsilon_1^2, \varepsilon_1^2\right\}}\,, \tag{3.26}$$

then it follows that (3.22) holds.

So, set

$$r_1^{**} := \min\left\{r_1, \frac{r_1^*}{2\max\left\{\varepsilon_0^1, \varepsilon_0^2, \varepsilon_1^2, \varepsilon_1^2\right\}}\right\}. \tag{3.27}$$

Put

$$\Omega_{r_1^{**}} := \left\{(x,y) \in \mathfrak{X} : \|(x,y)\| < r_1^{**}\right\}. \tag{3.28}$$

Then for each $(x,y)\in\mathcal{K}\cap\partial\Omega_{r_1^{**}}$, we have that

(3.29)
$$\begin{aligned}
&\|T_1(x,y)\|\\
&\le H_1\left(\phi_1\left(x\right)+\varepsilon_0^1x\left(\xi_0^1\right),\phi_2\left(y\right)+\varepsilon_0^2y\left(\xi_0^1\right)\right)+\int_0^1 G(s,s)a_1(s)g_1(x(s),y(s))\,ds\\
&\le \eta_2\left(\phi_1\left(x\right)+\varepsilon_0^1x\left(\xi_0^1\right)+\phi_2\left(y\right)+\varepsilon_0^2y\left(\xi_0^1\right)\right)+\eta_1\int_0^1 G(s,s)a_1(s)(x(s)+y(s))\,ds\\
&\le \eta_2\left(\phi_1\left(x\right)+\varepsilon_0^1x\left(\xi_0^1\right)+\phi_2\left(y\right)+\varepsilon_0^2y\left(\xi_0^1\right)\right)+\frac{1}{4}\|(x,y)\|\\
&\le \frac{1}{4}\|(x,y)\|+\frac{1}{4}\|(x,y)\|\\
&=\frac{1}{2}\|(x,y)\|.
\end{aligned}$$

Thus, we conclude that

$$\|T_1(x,y)\|\le\frac{1}{2}\|(x,y)\|, \tag{3.30}$$

for each $(x,y)\in\mathcal{K}\cap\partial\Omega_{r_1^{**}}$. A similar argument holds for the operator T_2. Consequently, we deduce that

$$\|S(x,y)\|\le\|(x,y)\|, \tag{3.31}$$

for each $(x,y)\in\mathcal{K}\cap\partial\Omega_{r_1^{**}}$.

On the other hand, let us assume without loss of generality that $p_i^{\infty}=1$ for each i so that $q_i^{\infty}\in(0,1]$, for each i. Then condition (H4) — i.e., equation (3.3) — implies the existence of a number $r_2^*:=r_2^*(\eta_3)>0$ such that

$$\begin{aligned}
&H_1\left(\phi_1\left(x\right)+\varepsilon_0^1x\left(\xi_0^1\right),\phi_2\left(y\right)+\varepsilon_0^2y\left(\xi_0^1\right)\right)\\
&\qquad\ge\eta_3\left(\left[\phi_1(x)+\varepsilon_0^1x\left(\xi_0^1\right)\right]+\left[\phi_2(y)+\varepsilon_0^2y\left(\xi_0^1\right)\right]^{q_1^{\infty}}\right)
\end{aligned} \tag{3.32}$$

whenever

$$\phi_1(x)+\varepsilon_0^1x\left(\xi_0^1\right)+\phi_2(y)+\varepsilon_0^2y\left(\xi_0^1\right)\ge r_2^* \tag{3.33}$$

for some number r_2^*. Note that by picking r_2^* sufficiently large, the same type of estimate likewise holds for H_2; we assume henceforth that this is so. Here, in (3.32), we choose η_3 to be the number

$$\eta_3:=\frac{1}{t_0\gamma_0\min\left\{\varepsilon_0^1,\varepsilon_0^2,\varepsilon_0^2\right\}}, \tag{3.34}$$

where $t_0\in(a,b)$ is fixed but arbitrary; since $(a,b)\Subset(0,1)$, it holds that $t_0\neq0$, and so, $\eta_3>0$. Importantly, η_3 depends *neither* on q_1^{∞} *nor* on q_2^{∞}. Now, notice that for $(x, y) \in \mathcal{K}$ since $\phi_1(x)$, $\phi_2(y) \geq 0$ and $\xi_0^1 \in E$, we may estimate

$$\begin{aligned}\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right) + \phi_2(y) + \varepsilon_0^2 y\left(\xi_0^1\right) &\geq \min\left\{\varepsilon_0^1, \varepsilon_0^2\right\}\left[x\left(\xi_0^1\right) + y\left(\xi_0^1\right)\right] \\ &\geq \min\left\{\varepsilon_0^1, \varepsilon_0^2\right\} \min_{t\in[a,b]}[x(t) + y(t)] \\ &\geq \gamma_0 \min\left\{\varepsilon_0^1, \varepsilon_0^2\right\} \|(x, y)\|.\end{aligned} \tag{3.35}$$

Consequently, if (x, y) satisfies

$$\|(x, y)\| \geq \frac{r_2^*}{\gamma_0 \min\left\{\varepsilon_0^1, \varepsilon_0^2\right\}}, \tag{3.36}$$

then (3.32) holds.

We next interrupt to prove an easy lemma. Suppose that $x, y \geq 0$ with $x, y \leq M$ for some $M \geq 1$ and finite. Let q satisfy $0 < q \leq 1$. Choose the constant c such that

$$c := \min\left\{1, M^{q-1}\right\}; \tag{3.37}$$

note that $-1 < q - 1 \leq 0$. Obviously, $c \in (0, 1]$ since $M \geq 1$ and $q - 1 \leq 0$. Then it follows that

$$x + y^q \geq c(x + y), \tag{3.38}$$

for all $(x, y) \in [0, M] \times [0, M]$. Indeed, we merely notice that, for $(x, y) \in [0, M] \times [0, M]$

$$cx \leq x \tag{3.39}$$

and

$$cy \leq y^q, \tag{3.40}$$

since $y \mapsto y^{q-1}$ is decreasing for $y > 0$, whereupon adding (3.39)–(3.40) we estimate

$$cx + cy \leq x + y^q, \tag{3.41}$$

which evidently proves inequality (3.38).

Now continuing with the proof, let us put

$$r_2^{**} := \max\left\{1, 2r_1^{**}, \frac{r_2^*}{\gamma_0 \min\left\{\varepsilon_0^1, \varepsilon_0^2\right\}}\right\}, \tag{3.42}$$

which is *independent* of each of q_1^∞ and q_2^∞. Define $\Omega_{r_2^{**}}$ by

$$\Omega_{r_2^{**}} := \left\{(x, y) \in \mathcal{X} : \|(x, y)\| < r_2^{**}\right\}. \tag{3.43}$$

Using estimate (3.38), then, and the fact that

$$\int_0^1 G(t_0,s)\,a_1(s)g_1(x(s),y(s))\,ds \geq 0, \tag{3.44}$$

we deduce that for each $(x,y) \in \mathcal{K} \cap \partial\Omega_{r_2^{**}}$

$$\begin{aligned}
(T_1(x,y))(t_0) &= t_0H_1\left(\phi_1(x)+\varepsilon_0^1 x\left(\xi_0^1\right),\phi_2(y)+\varepsilon_0^2 y\left(\xi_0^1\right)\right)\\
&\quad+\int_0^1 G(t_0,s)\,a_1(s)g_1(x(s),y(s))\,ds\\
&\geq t_0H_1\left(\phi_1(x)+\varepsilon_0^1 x\left(\xi_0^1\right),\phi_2(y)+\varepsilon_0^2 y\left(\xi_0^1\right)\right)\\
&\geq t_0\eta_3\left(\left[\phi_1(x)+\varepsilon_0^1 x\left(\xi_0^1\right)\right]+\left[\phi_2(y)+\varepsilon_0^2 y\left(\xi_0^1\right)\right]^{q_1^\infty}\right)\\
&\geq t_0\eta_3\left[\varepsilon_0^1 x\left(\xi_0^1\right)+\left(\varepsilon_0^2\right)^{q_1^\infty}\left[y\left(\xi_0^1\right)\right]^{q_1^\infty}\right]\\
&\geq t_0\eta_3\left[\varepsilon_0^1 x\left(\xi_0^1\right)+\varepsilon_0^2\left[y\left(\xi_0^1\right)\right]^{q_1^\infty}\right]\\
&\geq t_0\eta_3\min\left\{\varepsilon_0^1,\varepsilon_0^2\right\}\left[\left[x\left(\xi_0^1\right)\right]+\left[y\left(\xi_0^1\right)\right]^{q_1^\infty}\right]\\
&\geq t_0\eta_3\min\left\{\varepsilon_0^1,\varepsilon_0^2\right\}c_1\left[x\left(\xi_0^1\right)+y\left(\xi_0^1\right)\right]\\
&\geq t_0\eta_3\min\left\{\varepsilon_0^1,\varepsilon_0^2\right\}\gamma_0c_1\|(x,y)\|\\
&\geq c_1\|(x,y)\|,
\end{aligned} \tag{3.45}$$

where we have used the lemma of the previous paragraph to get the third-to-last inequality, and so, here $c_1 := \min\{1,(r_2^{**})^{q_1^\infty-1}\}$. We have also used both the fact that $\varepsilon_0^2 \in [0,\frac{1}{2})$ and that $q_1^\infty \in (0,1]$ so that $(\varepsilon_0^2)^{q_1^\infty} \geq \varepsilon_0^2$. In summary, it follows that

$$\|T_1(x,y)\| \geq c_1\|(x,y)\|. \tag{3.46}$$

Likewise, for each $(x,y) \in \mathcal{K} \cap \partial\Omega_{r_2^{**}}$ we deduce that for $c_2 := \min\{1,(r_2^{**})^{q_2^\infty-1}\}$

$$\|T_2(x,y)\| \geq c_2\|(x,y)\|. \tag{3.47}$$

We now conclude the argument by considering cases. If $q_1^\infty = q_2^\infty = 1$, then from (3.37), it is obvious that $c_1 = c_2 = 1$. In this case we deduce from (3.46)–(3.47) that

$$\|S(x,y)\| \geq 2\|(x,y)\| > \|(x,y)\|, \tag{3.48}$$

for each $(x,y) \in \mathcal{K} \cap \partial\Omega_{r_2^{**}}$. On the other hand, in case $0 < \max\{q_1^\infty,q_2^\infty\} < 1$, then

$$c_1 := \left(r_2^{**}\right)^{q_1^\infty-1} \quad\text{and}\quad c_2 := \left(r_2^{**}\right)^{q_2^\infty-1}. \tag{3.49}$$

In order that $c_1 + c_2 \geq 1$ be satisfied, at a minimum we must have that

$$\min\left\{2^{\frac{1}{1-q_1^{\infty}}}, 2^{\frac{1}{1-q_2^{\infty}}}\right\} \geq r_2^{**}. \tag{3.50}$$

Evidently, since r_2^{**} is finite and $(1-q_i^{\infty})^{-1} \to +\infty$ as $q_i^{\infty} \to 1^-$, there exists a $\delta > 0$ sufficiently small such that for each q_1^{∞}, $q_2^{\infty} \in (1-\delta, 1]$ we have that (3.50) holds. In this case, we again deduce that (3.48) holds with, say, the factor 2 replaced by 1. Importantly, we point out that r_2^{**} does *not* depend on q_i^{∞} for either i. Consequently, we may, in inequality (3.50) above, freely increase q_i^{∞}, for each i, without changing the previously selected and fixed value of r_2^{**}.

Finally, putting the preceding paragraphs together, we make two conclusions. Firstly, if $q_1^{\infty} = q_2^{\infty} = 1$, then by Lemma 2.1 and inequality (3.48) we deduce the existence of a function $(x_0, y_0) \in \mathcal{K}$ such that $S(x_0, y_0) = (x_0, y_0)$, where $x_0(t)$, $y_0(t)$ forms a positive solution of problem (1.1). Secondly, if q_1^{∞}, $q_2^{\infty} \leq 1$, then there exists a $\delta > 0$ sufficiently small such that if q_1^{∞}, $q_2^{\infty} \in (1-\delta, 1]$, then problem (1.1) still has at least one positive solution. And as these cases are exhaustive this completes the proof. □

We now prove a second result that demonstrates an alternative approach to problem (1.1). In particular, we begin by introducing the following condition.

H8: For each $i = 1, 2$, there is a constant $\rho_i > 0$ such that

$$\limsup_{z_1+z_2\to 0^+} \frac{H_i(z_1, z_2)}{z_1 + z_2} < \rho_i \tag{3.51}$$

holds, where $\rho_i \in [0, \frac{1}{2\max\{\varepsilon_1^1, \varepsilon_1^2\}})$.

On the one hand, condition (H8) is certainly more general than condition (H4). For instance, the continuous function $H : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ defined by

$$H(z_1, z_2) := \begin{cases} (z_1 + z_2)\cos\left(\frac{1}{z_1+z_2}\right), & z_1 + z_2 \neq 0 \\ 0, & z_1 = z_2 = 0 \end{cases} \tag{3.52}$$

satisfies

$$\limsup_{z_1+z_2\to 0^+} \frac{H(z_1, z_2)}{z_1 + z_2} = 1 \tag{3.53}$$

but $\lim_{z_1+z_2\to 0^+} \frac{H(z_1,z_2)}{z_1+z_2}$ does not exist. On the other hand, in order to prove the next result, we shall have to impose growth conditions on the nonlinearities g_1 and g_2 at infinity. Thus, we introduce condition (H9) below.

H9: We find that

$$\lim_{x+y\to+\infty} \frac{g_1(x, y)}{x + y} = +\infty \quad \text{and} \quad \lim_{x+y\to+\infty} \frac{g_2(x, y)}{x + y} = +\infty. \tag{3.54}$$

With condition (H8) and (H9) in hand we state and prove the following theorem. We first give two preliminary remarks.

Remark 3.6. We note that condition (H8) is more closely related to certain of the conditions given by Yang [16], [17], to which was alluded in Section 1. In particular, however, we note that *unlike* the results Yang gives, which admittedly were for a slightly different problem than (1.1), we do *not* require complicated conditions on the nonlinearities g_1 and g_2. Indeed, conditions (H5) and (H9) are quite straightforward and standard. Moreover, the measures here are signed. So, we consider these observations to be both interesting and noteworthy.

Remark 3.7. We also note, as will become clear in the statement and proof of Theorem 3.8 in the sequel, that with this particular assumption — namely (H8) — we may dispense with the perturbation terms appearing in (1.1). In particular and importantly, then, we may set $\varepsilon_0^1 = \varepsilon_0^2 = 0$.

Theorem 3.8. *Suppose that conditions* (H1)–(H3) *and* (H5)–(H9) *hold. In addition, suppose that* $\varepsilon_0^1 = \varepsilon_0^2 = 0$. *Then the unperturbed problem* (1.1) *has at least one positive solution.*

Proof: Due to the assumptions given in the statement of this theorem, it is still the case that $T : \mathcal{K} \to \mathcal{K}$ and that T is a completely continuous operator. So, we proceed directly to the cone theoretic part of the argument.

To this end, let $\rho_i < \frac{1}{2\max\{\varepsilon_1^1, \varepsilon_1^2\}}$ be given, for each $i = 1, 2$. Evidently, we may select $k \in \mathbb{N}$ sufficiently large such that

$$0 \leq \rho_i < \frac{2^k - 1}{2^{k+1}\max\{\varepsilon_1^1, \varepsilon_1^2\}} < \frac{1}{2\max\{\varepsilon_1^1, \varepsilon_1^2\}} \tag{3.55}$$

holds for each i. Moreover, for each i, select the number $\eta_i > 0$ such that

$$\eta_i \int_0^1 G(s,s)a_i(s)\,ds \leq \frac{1}{2^{k+1}} \tag{3.56}$$

holds. Condition (H5) implies the existence of a number $r_1 > 0$ such that $g_i(x,y) \leq \eta_i(x+y)$ for all $0 \leq x+y < r_1$ and for each i. On the other hand, from condition (H8), we may select a number $0 < \varepsilon < \min\{\rho_1, \rho_2\}$ sufficient small such that

$$H_i(z_1, z_2) < (\rho_i - \varepsilon)(z_1 + z_2) \tag{3.57}$$

holds whenever $0 \leq z_1 + z_2 < r_1^*$ for some number $r_1^* > 0$, for each $i = 1, 2$. In addition, since (3.55) holds, for each i, it evidently holds that

$$0 < \rho_i - \varepsilon < \frac{2^k - 1}{2^{k+1}\max\{\varepsilon_1^1, \varepsilon_1^2\}}. \tag{3.58}$$

Now, condition (H3) implies that

$$\phi_1(x) \le \varepsilon_1^1\|x\| \quad \text{and that} \quad \phi_2(y) \le \varepsilon_1^2\|y\|. \tag{3.59}$$

Consequently, for each $(x, y) \in \mathcal{K}$ satisfying

$$0 \le \|(x, y)\| < \min\{r_1, r_1^*\}, \tag{3.60}$$

it follows that

$$\begin{aligned} &\phi_1(x) \le \varepsilon_1^1\|x\| \le \varepsilon_1^1\|(x, y)\| < \frac{1}{2}r_1^* \quad \text{and that} \\ &\phi_2(y) \le \varepsilon_1^2\|y\| \le \varepsilon_1^2\|(x, y)\| < \frac{1}{2}r_1^*. \end{aligned} \tag{3.61}$$

Now, select $r_1^{**} > 0$ such that

$$r_1^{**} < \min\{r_1, r_1^*\} \tag{3.62}$$

and put $\Omega_{r_1^{**}} := \{(x, y) \in \mathcal{K} : \|(x, y)\| < r_1^{**}\}$. Then upon combining (3.59)–(3.62), we may estimate

$$H_i(\phi_1(x), \phi_2(y)) < (\rho_i - \varepsilon)(\phi_1(x) + \phi_2(y)), \tag{3.63}$$

for each $(x, y) \in \mathcal{K} \cap \partial\Omega_{r_1^{**}}$ and $i = 1, 2$. So, combining all of these estimates, we deduce that

$$\begin{aligned} \|T_1(x, y)\| &\le H_1(\phi_1(x), \phi_2(y)) + \int_0^1 G(s, s)a_1(s)g_1(x(s), y(s))\,ds \\ &\le (\rho_1 - \varepsilon)(\phi_1(x) + \phi_2(y)) + \frac{1}{2^{k+1}}\|(x, y)\| \\ &\le \frac{2^k - 1}{2^{k+1}\max\{\varepsilon_1^1, \varepsilon_1^2\}}\left(\varepsilon_1^1\|x\| + \varepsilon_1^2\|y\|\right) + \frac{1}{2^{k+1}}\|(x, y)\| \\ &\le \frac{2^k - 1}{2^{k+1}\max\{\varepsilon_1^1, \varepsilon_1^2\}}\max\{\varepsilon_1^1, \varepsilon_1^2\}(\|x\| + \|y\|) + \frac{1}{2^{k+1}}\|(x, y)\| \\ &= \frac{2^k - 1}{2^{k+1}\max\{\varepsilon_1^1, \varepsilon_1^2\}}\max\{\varepsilon_1^1, \varepsilon_1^2\}\|(x, y)\| + \frac{1}{2^{k+1}}\|(x, y)\| \\ &= \frac{1}{2}\|(x, y)\|. \end{aligned} \tag{3.64}$$

Similarly, we deduce that

$$\|T_2(x, y)\| \le \frac{1}{2}\|(x, y)\| \tag{3.65}$$

whence

$$\|S(x, y)\| \le \|(x, y)\|, \tag{3.66}$$

for each $(x,y) \in \mathcal{K} \cap \partial\Omega_{r_1^{**}}$.

On the other hand, select the number $\eta_3 > 0$ to satisfy

$$\eta_3 \max\left\{\int_{[a,b]} \gamma_0^2 G(s,s)a_1(s)\,ds, \int_{[a,b]} \gamma_0^2 G(s,s)a_2(s)\,ds\right\} \geq \frac{1}{2}\,. \tag{3.67}$$

Then by condition (H9), we have that

$$g_i(x,y) \geq \eta_3(x+y), \tag{3.68}$$

for all $x+y \geq r_2$ and for each $i=1,2$. Put

$$r_2^* := \max\left\{\frac{r_2}{\gamma_0}, 2r_1^{**}\right\}. \tag{3.69}$$

Then since $H_1(z_1,z_2) \geq 0$, for all $(z_1,z_2) \in [0,+\infty) \times [0,+\infty)$, we deduce that

$$\begin{aligned} \min_{t\in[a,b]} \left(T_1(x,y)\right)(t) &\geq \eta_3 \int_{[a,b]} \gamma_0 G(s,s)a_1(s)[x(s)+y(s)]\,ds \\ &\geq \|(x,y)\|\eta_3 \int_{[a,b]} \gamma_0^2 G(s,s)a_1(s)\,ds \\ &\geq \frac{1}{2}\|(x,y)\|, \end{aligned} \tag{3.70}$$

whence

$$\|T_1(x,y)\| \geq \frac{1}{2}\|(x,y)\|, \tag{3.71}$$

for each $(x,y) \in \mathcal{K} \cap \partial\Omega_{r_2^*}$. Similarly,

$$\|T_2(x,y)\| \geq \frac{1}{2}\|(x,y)\|, \tag{3.72}$$

so that $\|S(x,y)\| \geq \|(x,y)\|$, for $(x,y) \in \mathcal{K} \cap \partial\Omega_{r_2^*}$. Consequently, we may invoke Lemma 2.1 to deduce the existence of at least one positive solution to problem (1.1). $\square$

We conclude with an explicit numerical example together with some final remarks.

Example 3.9. Consider the boundary value problem

$$
\begin{aligned}
-x''(t) &= (2t+1)g_1(x(t),y(t)),\\
-y''(t) &= e^{-3t+1}g_2(x(t),y(t)),\\
x(0) &= H_1\left(\phi_1(x)+\frac{1}{40}x\left(\frac{1}{2}\right),\phi_2(y)+\frac{1}{300}y\left(\frac{2}{5}\right)\right),\\
y(0) &= H_2\left(\phi_1(x)+\frac{1}{40}x\left(\frac{1}{2}\right),\phi_2(y)+\frac{1}{300}y\left(\frac{2}{5}\right)\right),\\
x(1) &= 0 = y(1),
\end{aligned}
\tag{3.73}
$$

where we make the following declarations:
(3.74)

$$
\begin{aligned}
H_1(z_1,z_2) &:= (z_1+z_2)^3,\\
H_2(z_1,z_2) &:= z_1^{1.1}e^{z_1}+z_2^2e^{z_2},\\
\phi_1(x) &:= \frac{1}{8}x\left(\frac{1}{3}\right)-\frac{1}{40}x\left(\frac{1}{2}\right)-\frac{1}{12}x\left(\frac{3}{5}\right)+\frac{1}{2}\int_{\left[\frac{13}{20},\frac{3}{4}\right]}x(s)\,ds,\\
\phi_2(y) &:= -\frac{1}{300}y\left(\frac{2}{5}\right)+\frac{1}{15}y\left(\frac{9}{20}\right)-\frac{1}{100}y\left(\frac{11}{20}\right)+\frac{1}{10}\int_{\left[\frac{3}{5},\frac{7}{10}\right]}y(s)\,ds,\\
g_1(x,y) &:= \begin{cases}(x+y)^2, & x+y\le 1,\\ \sqrt{x+y}, & x+y\ge 1,\end{cases}\\
g_2(x,y) &:= (x+y)^3.
\end{aligned}
$$

Interestingly, note that g_1 is sublinear as $x+y\to+\infty$, whereas g_2 is superlinear. Furthermore, let us observe at this juncture that on account of the definitions of ϕ_1 and ϕ_2 given in (3.74), we may recast the boundary conditions at $t=0$ in (3.73) in the somewhat simpler form

$$
\begin{aligned}
x(0) &= H_1(\psi_1(x),\psi_2(y)) = \left[\psi_1(x)+\psi_2(y)\right]^3,\\
y(0) &= H_2(\psi_1(x),\psi_2(y)) = \left(\psi_1(x)\right)^{1.1}e^{\psi_1(x)}+\left(\psi_2(y)\right)^2e^{\psi_2(y)},
\end{aligned}
\tag{3.75}
$$

where we have put $\psi_1(x):=\phi_1(x)+\frac{1}{40}x(\frac{1}{2})$ and $\psi_2(y):=\phi_2(y)+\frac{1}{300}y(\frac{2}{5})$. Incidentally, though we do not show this explicitly, let us also remark that it is easy to show that the Stieltjes measures μ_{α_1} and μ_{α_2} are *signed* for this problem.

It is now easy to check that each of conditions (H1)–(H7) is satisfied. In particular, note that we may select $\varepsilon_1^1:=\frac{17}{60}$, $\varepsilon_1^2:=\frac{9}{100}$, $\varepsilon_0^1:=\frac{1}{40}$, and $\varepsilon_0^2:=\frac{1}{300}$. Moreover, we note that $\int_{[0,1]}t\,d\alpha_1(t)=\frac{17}{1200}\ge 0$ and that $\int_{[0,1]}t\,d\alpha_2(t)=\frac{89}{3000}\ge 0$. In any case, we conclude that we may invoke Theorem 3.5 to deduce that problem (3.73) has at least one positive solution. Likewise, problem (3.75) has at least one positive solution, too.

Remark 3.10. We note that problem (3.73) could not be addressed by any existing results. This is true for a variety of reasons, among which are the following: problem (3.73) involves a system of equations; it imposes no growth conditions on g_1 and g_2 for (x, y) large in norm; it allows for each of H_1 and H_2 to have superlinear growth as $x + y \to +\infty$; and it allows for each of ϕ_1 and ϕ_2 to be have associated signed Borel measures. In short, we are not aware that any results in the existing literature can be applied to problem (3.73). And this is the advantage of the asymptotic conditions (H4) and (H8), which we have introduced in this work.

Remark 3.11. Observe that Example 3.9 demonstrates that it is *not* necessary for the function $H_2(z_1, z_2)$ to be able to be realized in the form

$$H_2 (z_1, z_2) = \widetilde{H} (z_1 + z_2), \tag{3.76}$$

for some function $\widetilde{H}$. Indeed, while such a decomposition is an easy way in which to satisfy condition (H4), the function $H_2(z_1, z_2) = z_1^{1.1} e^{z_1} + z_2^2 e^{z_2}$ in (3.74) cannot be realized in this simpler form. Of course, the functions H_1, g_1, and g_2 need not be able to be realized as a function of $z_1 + z_2$ either. The point is that condition (H4) can still be satisfied in spite of this. In fact, for example, to ensure that (3.3) in condition (H4) is satisfied, it is enough, for instance, that

$$H_i (z_1, z_2) \geq (z_1 + z_2)^{\alpha} \tag{3.77}$$

holds for $z_1 + z_2$ sufficiently large and for some $\alpha > 1$. Evidently, (3.77) does *not* require that H_i satisfy (3.76) for some $\widetilde{H}$. Furthermore, note that an additional example of this sort was provided in both $(3.10)_1$ and (3.11).

Remark 3.12. We have elected not to give an example of Theorem 3.8 since its application would proceed in a very similar manner to Example 3.9. Nonetheless, we emphasize that in the case of Theorem 3.8, we may take the perturbation terms in (1.1) equal to zero and, hence, in this case we are recovering solutions to the *unperturbed* (i.e., $\varepsilon_0^1 = \varepsilon_0^2 = 0$) problem (1.1).

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