On special partitions of Dedekind- and Russell-sets

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This paper is about one of the most bizarre areas of set theory
From the referee's report on our original manuscript

Abstract. A *Russell set* is a set which can be written as the union of a countable pairwise disjoint set of pairs no infinite subset of which has a choice function and a *Russell cardinal* is the cardinal number of a Russell set. We show that if a Russell cardinal a has a ternary partition (see Section 1, Definition 2) then the Russell cardinal $a+2$ fails to have such a partition. In fact, we prove that if a ZF-model contains a Russell set, then it contains Russell sets with ternary partitions as well as Russell sets without ternary partitions. We then consider generalizations of this result.

Keywords: Axiom of Choice, Dedekind sets, Russell sets, generalizations of Russell sets, odd sized partitions, permutation models

Classification: 03E10, 03E25, 03E35, 05A18

1. Introduction, terminology and known results

The Axiom of Choice AC, i.e., the statement that for every family $\mathcal{A}$ consisting of non-empty sets there is a function (called a choice function) $f : \mathcal{A} \to \bigcup \mathcal{A}$ such that for every $x \in \mathcal{A}$, $f(x) \in x$, was formulated by Zermelo in 1904 as part of his development of axiomatic set theory (Zermelo-Fraenkel set theory). In spite of the controversy which first surrounded the axiom due to its non-constructive nature (it asserts the existence of f but suggests no way to construct it) it is accepted and used by most mathematicians today. This fact is basically due to the work of Kurt Gödel who constructed a model for Zermelo-Fraenkel set theory in 1938, *the model of constructible sets*, in which AC was true and thus consistent with the rest of the axioms of set theory. As a result, mathematicians could be released from any fears of introducing inconsistencies by using AC.

The consequences of AC include such fundamental results as "Every vector space has a basis", "The Tychonoff product of compact topological spaces is compact" (in fact, the latter two propositions are equivalent to AC in Zermelo-Fraenkel set theory minus AC; see, e.g., [9] or [3]), "Every infinite set has a countably infinite subset" and the countable union theorem (The union of a countable collection of countable sets is countable.) There are, however, some non-intuitive (and perhaps even undesirable) consequences of the axiom. For example the existence of a non-Lebesgue measurable set and the Banach-Tarski paradox which asserts that it is possible to partition a ball into a finite number of pieces and reassemble the pieces to form two balls of the same size as the original. If one is considering replacing the axiom of choice by some weaker statement or possibly eliminating choice altogether then it seems desirable to investigate the degree to which mathematics without choice may differ from mathematics with it.

One possible difference was described by Bertrand Russell when he described how the union of a countable set of pairs might fail to be countable in his metaphor about choosing from an infinite collection of pairs of shoes versus pairs of socks. But it was Fraenkel who first proved that without the axiom of choice it was possible for many of the results, which were regarded as fundamental, to fail if the axiom of choice was not assumed. He did this by constructing models in which all of the set theoretic axioms other than the axiom of choice held and then noting that other standard theorems also failed in the models. For example, in Fraenkel's first model there was an infinite set without a countably infinite subset and in his second model a countable set of pairs whose union had no countable subset.

Actually, Fraenkel's models (which are now called *Fraenkel-Mostowski* models or *permutation models*) were models of a version of Zermelo's set theory weakened to permit the existence of atoms, elements which were not themselves sets. It was not until 1963 that Cohen discovered his method of *forcing* by which he constructed models of the full Zermelo-Fraenkel set theory (without the axiom of choice) in which there were infinite sets without countable subsets and in particular, in the second Cohen model, a countable set of pairs whose union had no countable subset. We refer the reader to [10] for details.

Sets like the one existing in the second Cohen model which are the union of a countable set of pairs but have no countable subset and which live in some universe of set theory are called *Russell sets*. They were introduced in [7] after Bertrand Russell's metaphor about choosing from an infinite collection of pairs of shoes versus pairs of socks.

In several recent papers Russell sets and their properties have been investigated (see [7], [4], [8], [6], [5]). We shall continue the research here proving a number of results about Russell sets and generalizations of Russell sets. Before giving an overview of the aims and the orientation of this paper, let us first supply some terminology.

Definition 1. Let X and Y be sets.

1. $|X| \leq |Y|$ if there exists an injection $f : X \to Y$.
2. $|X| = |Y|$ if there exists a bijection $f : X \to Y$.
3. $|X| < |Y|$ if $|X| \leq |Y|$ and $|X| \neq |Y|$.

Definition 2.

1. A *Russell sequence* is a (countable) sequence $(X_i)_{i\in\omega}$ of disjoint pairs no infinite subset of which has a choice function. A *Russell set* X is the union $X = \bigcup_{i\in\omega} X_i$ of a Russell sequence. A *Russell cardinal* is the cardinal number of a Russell set.
2. Let n be an integer such that $n \geq 2$. An *n-Russell set* is a set X which can be written $X = \bigcup_{k\in\omega} X_k$ where

(a) for each $k \in \omega$, $|X_k| = n$,
(b) for i and j in ω, if $i \neq j$ then $X_i \cap X_j = \emptyset$,
(c) no infinite subset of $\{X_k \mid k \in \omega\}$ has a choice function.

The sequence $(X_k)_{k\in\omega}$ is called an *n-Russell sequence*. An *n-Russell cardinal* is the cardinal number of an n-Russell set. According to this terminology a Russell cardinal (from part (2)) is a 2-Russell cardinal.

3. For $n \in \mathbb{N}$ (= the set of positive integers), an *n-ary partition* of a set X is a partition of X into sets each with exactly n elements. n-ary partitions with odd n are called *odd sized partitions*.
4. Let $n \in \mathbb{N}$. *A cardinal a is divisible by n* if and only if there exists a cardinal c with $a = nc$. (Equivalently, if X is any set, $|X|$ is divisible by n if and only if there exists a set Y such that $|X| = |n \times Y|$).
5. A set X is called a *Dedekind set* if it is infinite and *Dedekind finite*, i.e., $\aleph_0 \not\leq |X|$ (if and only if $|A| < |X|$ for every proper subset A of X if and only if $|X| < |X| + 1$, where $|X| + 1$ is the cardinality of $X \cup \{x\}$, $x \notin X$). A *Dedekind cardinal* is the cardinal number of a Dedekind set.

Definition 3. ZF will denote Zermelo-Fraenkel set theory minus the Axiom of Choice and ZFA will denote ZF set theory with the axiom of extensionality weakened to allow the existence of atoms.

In [8] Herrlich and Tachtsis have studied the possible partitions of a Russell set in set theory without the Axiom of Choice. Among other results, the authors in [8] established the following propositions which concern odd sized partitions of Russell sets and which shall be useful to us in the sequel.

Proposition 1 ([8, Proposition 2.1, Proposition 2.2, Theorem 2.6])**.**

(1) *For any positive integer n and any Russell set X, the set $X \times n$ is a Russell set that has an n-ary partition.*
(2) *Any odd sized partition of a Russell set is a Dedekind set.*
(3) *No Russell set has a countable odd sized partition.*
(4) *For odd n, a Russell set X has an n-ary partition if and only if its cardinal number $|X|$ is divisible by n.*

Lemma 1 ([8, Lemma 2.4])**.** *Let n be odd and let $\mathcal{V} = \{V_i : i \in I\}$ be an n-ary partition of the Russell set $X = \bigcup_{m\in\mathbb{N}} X_m$. Define the trace map* $\mathrm{tr} : I \longrightarrow \mathcal{P}_{\mathrm{fin}}(\mathbb{N})$, *where $\mathbb{N}$ is the set of positive integers and $\mathcal{P}_{\mathrm{fin}}(\mathbb{N})$ is the set of all finite subsets of $\mathbb{N}$, by* $\mathrm{tr}(i) = \{m \in \mathbb{N} : X_m \cap V_i \neq \emptyset\}$, *and let J consist of those $i \in I$ for which there exists some $i' \in I$, $i' \neq i$, with $\mathrm{tr}(i) = \mathrm{tr}(i')$. Then $I - J$ is finite.*

A question left open in [8] is the following:

Is there a model of ZF *in which Russell sets exist and all Russell sets can be partitioned into 3-element sets?*

In Section 2 we answer this question **in the negative**. Herrlich and Tachtsis also asked the same question with 3 replaced by an arbitrary odd natural number n. We shall also answer this more general question **in the negative**. In particular, in Theorem 2 we prove that if the Russell set $X = \bigcup_{i\in\omega} X_i$ (where $(X_i)_{i\in\omega}$ is a Russell sequence) has a p-ary partition, where p is an odd natural number greater than 1, then the Russell set $Y = \bigcup_{i\in\omega, i>0} X_i$ does not have such a partition. In Corollary 1 we provide the negative answer to the above question by concluding that if a ZF-model contains a Russell set, then it contains Russell sets with p-ary partitions, p an odd integer greater than 1, as well as Russell sets without p-ary partitions.

With regard to n-Russell sets, $n \geq 3$ (see Definition 2), the natural question[1] which arises is the following generalization of the Herrlich-Tachtsis question, namely

Problem 1. *Given $n, k \in \omega$, is it consistent with set theory without* AC *that there is an n-Russell set and all n-Russell sets have a k-partition?*

The primary purpose of this paper is to give a partial answer to this problem. First we note that for every natural number $n \geq 2$ it is consistent with set theory without choice that n-Russell sets do exist. Indeed, for $n = 2$, the second Fraenkel model (see the discussion above for this model) contains Russell sets (actually infinitely many; see Proposition 1(1)). For every $n \geq 3$, a permutation model is constructed in the proof of Theorem 11 which contains an n-Russell set (but no Russell sets). Now if k is a multiple of n, say $k = sn$, X is an n-Russell set and $X = \bigcup_{i\in\omega} X_i$ where $(X_i)_{i\in\omega}$ is a disjoint sequence of n-element sets, then grouping them by taking the union of the first s of the X_is, the union of the second s of the X_is and so on yields a k-partition of X. So, in this case (of n, $k = sn$) the answer to Problem 1 is in the affirmative.

The situation becomes obscure when n and k are relatively prime. As we have stated in the paragraph preceding Problem 1, the answer to the problem is in the negative when $n = 2$ and $k = 3$. We also generalize this fact in Corollary 2 where we consider partitions P of Russell sets such that for every $z \in P$, $|z|$ is an odd multiple of a given odd natural number $p > 1$.

For the case $n = 3$ and k a natural number relatively prime to 3, we also provide a negative answer to Problem 1. This is the result of Corollary 3. The keys for its proof are the results of Theorem 4 (for any 3-Russell set X and any integer p relatively prime to 3, the condition "X has a partition consisting of sets such that the cardinality of each is a multiple of p which is relatively prime to 3" is equivalent to the condition "X has a p-ary partition"), of Theorem 6 (If $p > 1$ is a natural number which is relatively prime to 3, then a 3-Russell set X has a p-ary partition if and only if $|X|$ is divisible by p) and of Theorem 1 (If a is Dedekind cardinal, p and n are natural numbers, p positive, then $p \mid (pa + n)$ if and only if $p \mid n$, where $\mid$ means "divides").

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[1]We are grateful to the referee for suggesting this question which provided a focus for our results.

For $n \geq 5$, things change dramatically. First we note that here, our results are incomplete. We show among other things that the method used for the cases $n = 2$ and $n = 3$ will not work if $n \geq 5$. That is, we cannot show:

(1) *If the n-Russell set $X = \bigcup_{i\in\omega} X_i$ (where $(X_i)_{i\in\omega}$ is a Russell sequence) has a p-ary partition then the n-Russell set $Y = \bigcup_{i\in\omega, i>0} X_i$ does not have such a partition.*

We do this by constructing a model $\mathcal{M}$ of set theory in which there is an n-Russell set for which (1) is false. Specifically, we show that for all natural numbers n and p both greater than or equal to 5, there is (in $\mathcal{M}$) an n-Russell sequence $(X_i)_{i\in\omega}$ with the property that both $\bigcup_{i\in\omega} X_i$ and $\bigcup_{i\in\omega, i>0} X_i$ have p-ary partitions which are in the model; see Theorem 8. Moreover, the permutation model of Theorem 8 sheds light on Problem 1 for the case $n = 5$ and $k > 6$. In particular, we show in Theorem 9 that in the model of Theorem 8, every 5-Russell set has a k-ary partition for every $k > 6$ and consequently the answer to Problem 1 is in the affirmative for the case $n = 5$ and $k > 6$.

For $n = 4$ and p relatively prime to 4, the question of whether (1) holds is open.

In Theorem 10 we establish that the results of Proposition 1(4) and Theorem 6 cease to be true if $n \geq 5$. Therefore, one should no longer expect to rely on similar such results (for $n \geq 5$) in order to attack Problem 1.

Finally, in Section 4 we study partitions of generalized Russell sets (see Definition 4 for this notion) and prove results via the method of Fraenkel-Mostowski permutation models (Theorems 12 and 13) which clarify that the situation, as far as 3-ary partitions, divisibility by 3 and their interrelation are concerned, is strikingly different from the corresponding one for Russell sets.

2. Odd sized and other partitions of Russell sets

In this section we answer the question of Herrlich and Tachtsis from [8] mentioned in the introduction, namely "Is there a model of ZF in which Russell sets exist and all Russell sets can be partitioned into 3-element sets (or in general into n-element sets, n an odd natural number)."

Theorem 1. *If a is Dedekind cardinal, p and n are natural numbers, p positive, then $p \mid (pa + n)$ if and only if $p \mid n$, where $\mid$ means "divides".*

Proof: $(\rightarrow)$ Assume that $pa+n = pb$ for some cardinal number b. Then $pa \leq pb$, hence by a result of Tarski [11] (in ZF, given a natural number $m \neq 0$ and two arbitrary cardinals p and q, if $mp \leq mq$, then $p \leq q$) we obtain that $a \leq b$. So there exists a cardinal x such that $a + x = b$. Therefore,

$$pa + px = p(a + x) = pb = pa + n.$$

Since pa is a Dedekind cardinal, this implies that $px = n$, thus $p \mid n$ as required.

($\leftarrow$) This is straightforward. $\square$

Theorem 2. *If X is a Russell set, p is an odd natural number greater than 1, and X has a p-ary partition, then the set Y obtained from X by the removal (or addition) of a finite number n of pairs has a p-ary partition if and only if $2n$ is divisible by p. Consequently, if the Russell set $X = \bigcup_{i\in\omega} X_i$ (where $(X_i)_{i\in\omega}$ is a Russell sequence) has a p-ary partition then the Russell set $Y = \bigcup_{i\in\omega, i>0} X_i$ does not have such a partition.*

Proof: Immediate from the fact that a Russell set is a Dedekind set and from Theorem 1 and Proposition 1(4). $\square$

Corollary 1. *Let p be an odd natural number greater than 1. Then the following holds: If a ZF-model contains a Russell set, then it contains Russell sets with p-ary partitions as well as Russell sets without p-ary partitions.*

Proof: This follows from Theorem 2 and Proposition 1(1). $\square$

Theorem 3. *Assume that p is an odd natural number greater than 1. Then the Russell set $X = \bigcup_{i\in\omega} X_i$ (where $(X_i)_{i\in\omega}$ is a Russell sequence) has a partition $\mathcal{P}$ such that $\forall z \in \mathcal{P}$, $|z|$ is an odd multiple of p if and only if X has a p-ary partition.*

Proof: ($\leftarrow$) This is straightforward.

($\rightarrow$) We first need the following slight modification of Lemma 1.

Lemma 2. *Let $\mathcal{R}$ be a partition of the Russell set $X = \bigcup_{i\in\omega} X_i$ such that for all $z \in \mathcal{R}$, $|z|$ is an odd natural number greater than 1. Then the set $\mathcal{R}_0 = \{z \in \mathcal{R} : g \restriction z \text{ is injective}\}$, where for $r \in X$ we let $g(r) =$ the unique $i \in \omega$ such that $r \in X_i$, has a finite complement in $\mathcal{R}$.*

Proof: We use the idea from the proof of Lemma 2.4 in [8] (Lemma 1 in this paper). Let $\mathcal{R}_1 = \{z \in \mathcal{R} : \exists i \in \omega, X_i \subset z\}$ then $\mathcal{R}_1 = \mathcal{R} - \mathcal{R}_0$ so the proof will be completed by showing that $\mathcal{R}_1$ is finite. Assume that $z \in \mathcal{R}_1$. Then there is some $i \in \omega$ such that $X_i \subseteq z$. Therefore for any other $w \in \mathcal{R}_1$, $X_i \cap w = \emptyset$ and hence $i \in g[z]$ but $i \notin g[w]$. It follows that $g[z] \neq g[w]$. This shows that the function h on $\mathcal{R}_1$ defined by $h(z) = g[z]$ for all $z \in \mathcal{R}_1$, is injective. But the range of h is a subset of the collection of all finite subsets of ω which is countable. It follows that $\mathcal{R}_1$ is either finite or countably infinite. Suppose that $\mathcal{R}_1$ is countably infinite. Since every $z \in \mathcal{R}_1$ is odd-sized and $|X_i| = 2$ for all $i \in \omega$, it follows that for every $z \in \mathcal{R}_1$ there is an $n_z \in \omega$ such that $|z \cap X_{n_z}| = 1$. On this basis and via induction we may define a subsequence of $(X_i)_{i\in\omega}$ with a choice function. This contradicts the fact that X is a Russell set. Therefore, $\mathcal{R}_1$ is a finite set as required. $\square$

By Lemma 2 the set $\mathcal{P}_0 = \{z \in \mathcal{P} : g \restriction z \text{ is injective}\}$ has a finite complement in $\mathcal{P}$. For every $z \in \mathcal{P}$, let k_z be the odd natural number such that $|z| = k_z p$. Since for every $z \in \mathcal{P}_0$, $g \restriction z$ is injective and $g[z]$ is well ordered (being a subset of ω), we may effectively define (i.e., using no choice principles) a well ordering of z and consequently we may define a partition $\{\mathcal{U}_{z,j} : j \in k_z\}$ of z such that $|\mathcal{U}_{z,j}| = p$ for all $j \in k_z$. On the other hand, since $\mathcal{P}_1 = \mathcal{P} - \mathcal{P}_0$ is a finite family of finite sets, it follows that $\bigcup \mathcal{P}_1$ is well ordered, hence for every $z \in \mathcal{P}_1$ we may similarly define a p-ary partition $\{\mathcal{U}_{z,j} : j \in k_z\}$ of z. Then $\mathcal{U} = \{\mathcal{U}_{z,j} : z \in \mathcal{P}, j \in k_z\}$ is a p-ary partition of X. This completes the proof of the theorem. □

Corollary 2. *Assume that p is an odd natural number greater than 1. If the Russell set $X = \bigcup_{i\in\omega} X_i$ (where $(X_i)_{i\in\omega}$ is a Russell sequence) has a partition $\mathcal{P}$ such that $\forall z \in \mathcal{P}$, $|z|$ is an odd multiple of p then the Russell set $Y = \bigcup_{i\in\omega, i>0} X_i$ does not have such a partition.*

PROOF: The result follows from Theorem 3 and from Theorem 2. □

3. Partitions of n-Russell sets, $n \geq 3$

Proposition 2. *n-Russell sets are Dedekind sets.*

PROOF: This follows immediately from Definition 2. □

Question 1. Assume that $X = \bigcup_{i\in\omega} X_i$ is an n-Russell set where $(X_i)_{i\in\omega}$ is an n-Russell sequence and that X has a partition $\mathcal{P}$ such that $\forall z \in \mathcal{P}$, there is an integer k such that k is relatively prime to n and $|z| = kp$. Is it possible for $\bigcup_{i\in\omega, i>0} X_i$ to have such a partition?

We are able to answer this question for $n = 3$. For $n = 4$ the question remains open. For $n > 4$ we have an answer for all $p > 4$. The remaining cases are open.

Next we answer Question 1 **in the negative** for the case $n = 3$.

Theorem 4. *Assume*

(1) *$X = \bigcup_{i\in\omega} X_i$ is a 3-Russell set where $(X_i)_{i\in\omega}$ is a 3-Russell sequence;*

(2) *p is a natural number which is larger than 1 and relatively prime to 3;*

(3) *X has a partition $\mathcal{P}$ such that $\forall z \in \mathcal{P}$, there is an integer k such that k is relatively prime to 3 and $|z| = kp$.*

Then X has a p-ary partition.

PROOF: We first prove the following replacement for Lemma 2.

Lemma 3. *Let $\mathcal{R}$ be a partition of the 3-Russell set $X = \bigcup_{i\in\omega} X_i$ such that $\forall z \in \mathcal{R}$, there is an integer k such that k is relatively prime to 3 and $|z| = kp$. Then the set $\mathcal{R}_0 = \{z \in \mathcal{R} : g \restriction z$ is injective$\}$, where g is defined as in the statement of Lemma 2, has a finite complement in $\mathcal{R}$.*

PROOF: The proof is by contradiction. Assume the hypotheses of the lemma and assume that $\mathcal{R}_1 = \{z \in \mathcal{R} : g \restriction z$ is not injective$\}$ is infinite. If $z \in \mathcal{R}_1$ then $\exists i \in \omega$ such that $|z \cap X_i| > 1$. Therefore we can write $\mathcal{R}_1 = \mathcal{R}_2 \cup \mathcal{R}_3$ where $\mathcal{R}_2 = \{z \in \mathcal{R} : \exists i \in \omega$ such that $|z \cap X_i| = 2\}$ and $\mathcal{R}_3 = \{z \in \mathcal{R} : \exists i \in \omega$ such that $|z \cap X_i| = 3\}$. If $\mathcal{R}_2$ is infinite we get a choice function for the infinite set $\{X_i : \exists z \in \mathcal{R}$ such that $|z \cap X_i| = 2\}$ by defining $F(X_i) = \bigcup(X_i \setminus z)$

where z is the unique element of $\mathcal{R}$ such that $|z \cap X_i| = 2$. This is not possible since $(X_i)_{i\in\omega}$ is a 3-Russell sequence. It follows that $\mathcal{R}_3$ is infinite. The function $H : \mathcal{R}_3 \to \mathcal{P}_{\text{fin}}(\omega)$ defined by $H(z) = \{i \in \omega : |z \cap X_i| = 3\}$ is injective and therefore $\mathcal{R}_3$ is well orderable, say by $\preccurlyeq$. Further, for each $z \in \mathcal{R}_3, \exists i \in \omega$ such that $|z \cap X_i| = 2$ or $|z \cap X_i| = 1$ (since $|z|$ is not a multiple of 3). It follows that the set $\mathcal{Z} = \{X_i : i \in \omega$ and $\exists z \in \mathcal{R}_3$ such that $|z \cap X_i| = 1$ or $|z \cap X_i| = 2\}$ is infinite. For $X_i \in \mathcal{Z}$ and z such that $|z \cap X_i| = 1$ or $|z \cap X_i| = 2$, let

$$f_z(X_i) = \begin{cases} z \cap X_i, & \text{if } |z \cap X_i| = 1 \\ X_i \setminus z, & \text{if } |z \cap X_i| = 2 \end{cases}.$$

We arrive at a contradiction by defining a choice function for $\mathcal{Z}$ by $K(X_i) = \bigcup f_z(X_i)$ where z is the $\preccurlyeq$ least element of $\mathcal{R}_3$ such that $|z \cap X_i| = 1$ or $|z \cap X_i| = 2$. $\square$

By Lemma 3 the set $\mathcal{P}_0 = \{z \in \mathcal{P} : g \restriction z$ is injective$\}$ has a finite complement in $\mathcal{P}$. For every $z \in \mathcal{P}$, let k_z be the natural number which is relatively prime to 3 and is such that $|z| = k_z p$. We may finish off the proof now as in the proof of $(\to)$ of Theorem 3. $\square$

Theorem 5. *Let $p > 1$ be a natural number which is relatively prime to 3 and let $\mathcal{V} = \{V_i : i \in I\}$ be a p-ary partition of the 3-Russell set $X = \bigcup_{m\in\omega} X_m$. Let $\text{tr} : I \longrightarrow \mathcal{P}_{\text{fin}}(\omega)$, $\text{tr}(i) = \{m \in \omega : X_m \cap V_i \neq \emptyset\}$, $i \in I$, be the trace map and let J consist of those $i \in I$ for which there exist $i', i'' \in I$, i, i', i'' pairwise distinct, with $\text{tr}(i) = \text{tr}(i') = \text{tr}(i'')$. Then $I - J$ is finite.*

Proof: We follow the ideas of the proof of [8, Lemma 2.4]. First, by virtue of Lemma 3, we may assume without loss of generality that for every $i \in I$ if $m \in \omega$ is such that $V_i \cap X_m \neq \emptyset$, then $|V_i \cap X_m| = 1$. Let $R = \text{tr}[I]$ be the countable range of the function tr (since $R \subseteq \mathcal{P}_{\text{fin}}(\omega)$ and it is known that, in ZF, $\mathcal{P}_{\text{fin}}(\omega)$ is countable). Clearly, $R = R_1 \cup R_2 \cup R_3$, where

$$R_i = \{r \in R : \text{tr}^{-1}(r) \text{ has precisely } i \text{ elements}\}, \quad i = 1, 2, 3.$$

Let $J_i = \text{tr}^{-1}(R_i)$, $i = 1, 2, 3$. Then $I = J_1 \cup J_2 \cup J_3$. The function tr is injective on J_1 and since R is countable and $(X_m)_{m\in\omega}$ has no subsequence with a choice function, we may easily conclude that R_1, hence J_1, is a finite set.

Now assume that the set R_2 is infinite. Put $J_2' = \{\text{tr}^{-1}(r) : r \in R_2\}$. Then the function $h : J_2' \to R_2$ defined by $h(j) = \bigcup\{\text{tr}(u) : u \in j\}$ for all $j \in J_2'$ is injective. Since R_2 is countable, J_2' is countable and let $J_2' = \{j_k : k \in \omega\}$ be an enumeration of J_2'. Put $W = \{\bigcup j_k : k \in \omega\}$. Clearly, W is a countable set. Since for each $m \in \omega$, X_m is a 3-element set and for each $m \in \omega$ and each $k \in \omega$ such that $X_m \cap (\bigcup j_k) \neq \emptyset$ we have that $|X_m \backslash (\bigcup j_k)| = 1$ we may easily define via induction a subsequence of $(X_m)_{m\in\omega}$ with a choice function. But this contradicts the fact that X is a 3-Russell set. Therefore, we may conclude that R_2 is a finite set and consequently J_2', hence $J_2 = \bigcup J_2'$ is also a finite set.

From the above we deduce that J_3 is a cofinite subset of I. Furthermore, note that for every $i \in J_3$, $\mathrm{tr}(i)$ has exactly p elements, say $m_1(i) < \ldots < m_p(i)$ and there exists a unique pair (i', i'') of pairwise distinct elements of $J_3 \backslash \{i\}$ with $\mathrm{tr}(i) = \mathrm{tr}(i') = \mathrm{tr}(i'')$. Clearly, for every $i \in J_3$, $V_i \cup V_{i'} \cup V_{i''} = X_{m_1(i)} \cup \cdots \cup X_{m_p(i)}$. Letting $J = J_3$, the proof of the theorem is complete. □

Theorem 6. *If $p > 1$ is a natural number which is relatively prime to 3, then a 3-Russell set X has a p-ary partition if and only if $|X|$ is divisible by p.*

Proof: This can be established using the result of Lemma 3 or the result of Theorem 5 and following the proof of [8, Theorem 2.6 (1), p. 187], so we simply refer the reader to the latter result in [8]. □

Theorem 7. *Assume*

(1) *$X = \bigcup_{i\in\omega} X_i$ is a 3-Russell set;*

(2) *p is a natural number which is larger than 1 and relatively prime to 3;*

(3) *X has a partition $\mathcal{P}$ such that $\forall z \in \mathcal{P}$, there is an integer k such that k is relatively prime to 3 and $|z| = kp$.*

Then the 3-Russell set $Y = \bigcup_{i\in\omega, i>0} X_i$ does not have such a partition.

Proof: By Theorem 4 we may assume without loss of generality that $\mathcal{P}$ is a p-ary partition of X, hence by Theorem 6 $|X|$ is divisible by p. If Y has also a partition as in the statement of the Theorem, then again by Theorem 4 Y has a p-ary partition, hence by Theorem 6 $|Y| = pa$ for some infinite cardinal a. Then $|X| = |Y| + 3 = pa + 3$ and since Y is a Dedekind set (being a 3-Russell set), a is a Dedekind cardinal. Thus, by Theorem 1, 3 is divisible by p. Since $p > 1$ this contradicts our assumption that 3 and p are relatively prime. Therefore, Y has no such partitions and the proof of the theorem is complete. □

Corollary 3. *Let $p > 1$ be a natural number which is relatively prime to 3. Then the following holds: If a ZF-model contains a 3-Russell set, then it contains 3-Russell sets with p-ary partitions as well as 3-Russell sets without p-ary partitions.*

We show next that the answer to Question 1 is **positive** if n and p are both greater than or equal to 5.

Theorem 8. *There is a model $\mathcal{M}$ of ZFA such that for all natural numbers n and p both greater than or equal to 5, there is (in $\mathcal{M}$) an n-Russell sequence $(X_i)_{i\in\omega}$ with the property that both $\bigcup_{i\in\omega} X_i$ and $\bigcup_{i\in\omega, i>0} X_i$ have p-ary partitions which are in the model.*

Proof: We start with a ground model of AC with a countable set $A = \bigcup\{A_i \cup B_i : i \in \omega\}$ of atoms such that:

1. for every $i \in \omega$, A_i is the three element set $A_i = \{a_{i1}, a_{i2}, a_{i3}\}$ and B_i is the two element set $B_i = \{b_{i1}, b_{i2}\}$;
2. for all $i \in \omega$, $A_i \cap B_i = \emptyset$ and for all $i, j \in \omega$, if $i \neq j$, then $(A_i \cup B_i) \cap (A_j \cup B_j) = \emptyset$.

G is the group of permutations of A generated by the cycles (a_{i1}, a_{i2}, a_{i3}) and (b_{i1}, b_{i2}), $i \in \omega$. The normal ideal $\mathcal{I}$ of supports is the set of all finite subsets of A. Let $\mathcal{M}$ be the permutation model determined by G and $\mathcal{I}$.

Lemma 4. *If u and v are any natural numbers and $m \geq 5$ is a natural number then, in $\mathcal{M}$,*

$$\left(\bigcup_{i\geq u} A_i\right) \cup \left(\bigcup_{j\geq v} B_j\right)$$

is an m-Russell set.

PROOF: The proof depends on whether or not $m = 6$.

Case 1. Assume that $m \neq 6$. Then m can be written in the form $m = 3r + 2s$ where r and s are positive natural numbers. Choose such an r and s. Let $X_0 = (\bigcup_{u\leq i<u+r} A_i) \cup (\bigcup_{v\leq j<v+s} B_j)$ and in general for $k \in \omega$, $X_k = (\bigcup_{u+kr\leq i<u+(k+1)r} A_i) \cup (\bigcup_{v+ks\leq j<v+(k+1)s} B_j)$.

Since any union of the A_is and the B_is is in the model with empty support each X_k has empty support. Therefore the sequence $(X_k)_{k\in\omega}$ is in $\mathcal{M}$ with empty support. It is also clear from the definition that for $k \in \omega$, $|X_k| = m$ and that for $k_1, k_2 \in \omega$, if $k_1 \neq k_2$ then $X_{k_1} \cap X_{k_2} = \emptyset$. Further, $\{X_k : k \in \omega\}$ can have no infinite subset with a choice function since for any finite support E only finitely many of the sets X_k meet E and therefore for all but finitely many of the sets X_k there is a permutation in G which fixes E pointwise (and fixes X_k) but moves every element of X_k. (If $\mathcal{A}$ is an infinite subfamily of $\{X_k : k \in \omega\}$ with a choice function, say f with support E, then let $k \in \omega$ such that $X_k \in \mathcal{A}$ and $X_k \cap E = \emptyset$. Let $f(X_k) = x$. Let $\psi = (\prod_{u+kr\leq i<u+(k+1)r}(a_{i1}, a_{i2}, a_{i3})) \cdot (\prod_{v+ks\leq j<v+(k+1)s}(b_{j1}, b_{j2}))$, i.e., ψ moves all the elements of X_k but fixes pointwise all the other atoms. Since $X_k \cap E = \emptyset$, we have that ψ fixes E pointwise hence $\psi(f) = f$. Furthermore since $\psi(X_k) = X_k$ we deduce that $(X_k, \psi(x)) \in f$. Since $x \in X_k$ and ψ moves every element of X_k we have that $\psi(x) \neq x$ meaning that f is not a function, a contradiction.) Therefore $(X_k)_{k\in\omega}$ is an m-Russell sequence in the model $\mathcal{M}$. We leave to the reader the proof that $\bigcup_{k\in\omega} X_k = (\bigcup_{i\geq u} A_i) \cup (\bigcup_{j\geq v} B_j)$.

Case 2. Assume that $m = 6$. The proof proceeds as in Case 1 except that $X_0 = A_u \cup A_{u+1}$, $X_1 = B_v \cup B_{v+1} \cup B_{v+2}$, and in general $X_{2m} = A_{u+2m} \cup A_{u+2m+1}$ and $X_{2m+1} = B_{v+3m} \cup B_{v+3m+1} \cup B_{v+3m+2}$. □

Now assume that $n \geq 5$. By the lemma with $m = n$, $u = 0$ and $v = 0$ we see that A is an n-Russell set, $A = \bigcup_{k\in\omega} X_k$ where $(X_k)_{k\in\omega}$ is an n-Russell sequence. Further by the proof of the lemma X_0 has the form $X_0 = (\bigcup_{i<u_0} A_i) \cup (\bigcup_{j<v_0} B_j)$ where u_0 and v_0 are in ω. Assume that $p \geq 5$. Using the lemma with $m = p$, $u = 0$ and $v = 0$ we get a p-Russell sequence $(Y_a)_{a\in\omega}$ whose union is A and therefore we obtain a p-ary partition of $A = \bigcup_{k\in\omega} X_k$. Using the lemma again

in a similar way with $m = p$, $u = u_0$ and $v = v_0$ we get a p-ary partition of $(\bigcup_{i\geq u_0} A_i) \cup (\bigcup_{j\geq v_0} B_j) = \bigcup_{k\in\omega,k>0} X_k$. $\square$

Remark 1. From the proof of Theorem 8 we infer that it is relatively consistent with ZFA that there exist Dedekind sets X such that X as well as sets obtained by adding $5k$, $k \in \mathbb{N}$, elements to X both have m-ary partitions for every natural number $m \geq 5$.

Theorem 9. *In the model of Theorem 8, for every $k > 6$, every 5-Russell set has a k-ary partition.*

Proof: We begin by noting that every integer greater than 6 can be written in the form $2r+3s$ where r and s are positive integers (as in the proof of Theorem 8, Lemma 4, Case 1). Secondly, we note the following easy lemma.

Lemma 5. *If k is a natural number which can be written in the form $k = 2r+3s$ where r and s are positive integers and X is a set which can be written as a countable disjoint union $X = \bigcup_{i\in\omega} Y_i$ where $\forall i \in \omega$, $|Y_i| = 2$ or $|Y_i| = 3$ and both of the sets $\{i \in \omega : |Y_i| = 2\}$ and $\{i \in \omega : |Y_i| = 3\}$ are infinite then X has a k-ary partition.*

The theorem will now follow as soon as we prove

Lemma 6. *In the model $\mathcal{M}$ of Theorem 8 every 5-Russell set can be written as a countable disjoint union $X = \bigcup_{i\in\omega} Y_i$ where $\forall i \in \omega$, $|Y_i| = 2$ or $|Y_i| = 3$ and both of the sets $\{i \in \omega : |Y_i| = 2\}$ and $\{i \in \omega : |Y_i| = 3\}$ are infinite.*

Proof: We shall use the notation given in the proof of Theorem 8 for the atoms of $\mathcal{M}$ and, as in Theorem 8, G will denote the group of permutations used to construct $\mathcal{M}$. In addition, for any finite set E of atoms $\mathrm{fix}_G(E)$ or simply $\mathrm{fix}(E)$ denotes the subgroup $\{\phi \in G : \forall a \in E, \phi(a) = a\}$. Finally, for any subgroup H of G and any element t of $\mathcal{M}$ we let $\mathrm{Orb}_H(t)$ denote the H orbit of t, that is, $\mathrm{Orb}_H(t) = \{\phi(t) : \phi \in H\}$. For the proof of the lemma we first make the following claim

Claim 1. For any t in $\mathcal{M}$ and any finite subset E of the atoms A, $|\operatorname{Orb}_{\mathrm{fix}(E)}(t)| = 2^i 3^j$ where i and j are natural numbers.

Proof: Assume that $t \in \mathcal{M}$ and that $E \subseteq A$ is finite. Choose a finite subset F of A so that F is a support of t and

$$\forall i \in \omega,\ (A_i \subseteq F \ \text{ or } \ A_i \cap F = \emptyset) \ \text{ and } \ (B_i \subseteq F \ \text{ or } \ B_i \cap F = \emptyset). \tag{2}$$

For $\phi \in \mathrm{fix}(E)$ let ϕ_F be the function that agrees with ϕ on F and is equal to the identity function outside of F. By (2) $\phi_F \in \mathrm{fix}(E)$ and it is also the case that $\phi_F(t) = \phi(t)$ since ϕ_F and ϕ agree on a support of t. Therefore if we let $H = \{\phi_F : \phi \in \mathrm{fix}(E)\}$, $\mathrm{Orb}_H(t) = \mathrm{Orb}_{\mathrm{fix}(E)}(t)$. Let $K = \{\psi \in H : \psi(t) = t\}$ then the following facts are easy to verify.

1. The set of pairs $\{(\phi K, \phi(t)) : \phi \in H\}$ is a one to one function from the quotient group H/K onto $\mathrm{Orb}_H(t)$.

2. $|H| = 2^{c_1}3^{c_2}$ where c_1 and c_2 are in ω. (c_1 is the number of sets B_i contained in F which do not meet E and similarly c_2 is the number of sets A_i contained in F which do not meet E.)
3. It follows from item 3 that the cardinality of the quotient group H/K is $2^{d_1}3^{d_2}$ where d_1 and d_2 are in ω.

Using items 3 and 3 we conclude that $|\operatorname{Orb}_{\operatorname{fix}(E)}(t)| = |\operatorname{Orb}_H(t)| = |H/K| = 2^{d_1}3^{d_2}$. □

Now let X be a 5-Russell set in $\mathcal{M}$ and say that X is the disjoint union of a countable set of 5 element sets, $X = \bigcup_{i\in\omega} X_j$ where the sequence $(X_j)_{j\in\omega}$ is in $\mathcal{M}$ and has support E. Then for each $j \in \omega$, every ϕ in $\operatorname{fix}(E)$ fixes X_j and therefore $\forall t \in X_j$, $\phi(t) \in X_j$. We therefore have that for all $t \in X_j$, $\operatorname{Orb}_{\operatorname{fix}(E)}(t) \subset X_j$. From this we conclude that the fix_E orbits of elements of X_j form a partition of X_j each element of which has support E. By the claim and the fact that $|X_j| = 5$ we conclude that each of the fix_E orbits of an element of X_j has size 1, 2 or 3. By taking unions of orbits if necessary this gives us a partition of X_j into two sets, P_j of size 2 and Q_j of size 3 both with support E.

We can write X as a countable disjoint union $X = \bigcup_{i\in\omega} Y_i$ as required by Lemma 6 by letting $Y_{2i} = P_i$ and $Y_{2i+1} = Q_i$ for all $i \in \omega$. □

The proof of the theorem is now complete. □

According to Proposition 1, *if n is an odd natural number, then a Russell set has an n-ary partition if and only if $|X|$ is divisible by n* (and we note that for every set X, if $|X|$ is divisible by n, then X has an n-ary partition) and according to Theorem 6, *if $p > 1$ is a natural number which is relatively prime to 3, then a 3-Russell set X has a p-ary partition if and only if $|X|$ is divisible by p.* However, the situation with n-Russell sets, $n \geq 5$, is **strikingly different** as shown by the subsequent theorem.

Theorem 10. *There is a model $\mathcal{M}$ of* ZFA *and a set A in $\mathcal{M}$ such that for every natural number $n \geq 5$, A is an n-Russell set, hence has an n-ary partition, but for every natural number $p \geq 2$, $|A|$ is not divisible by p.*

Proof: Let $\mathcal{M}$ be the permutation model defined in the proof of Theorem 8 and let A be its set of atoms. From Lemma 4 of the proof of Theorem 8 we obtain that A is an n-Russell set for every natural number $n \geq 5$. So in order to complete the proof we need to show that $|A|$ is not divisible by p for every natural number $p \geq 2$. To this end, fix an integer $p \geq 2$ and, toward a proof by contradiction, assume that $|A|$ is divisible by p and let $\{U_1, U_2, \ldots, U_p\}$ be a partition of A into p pairwise equipollent infinite sets. Let f be a bijection in $\mathcal{M}$ from U_1 to U_2 and let E be a support of f. Since A is a 5-Russell set, it is not hard to verify that every infinite subset Y of A must satisfy that $Q_Y = \{n \in \omega : 0 < |Y \cap A_n| < 3\}$ is finite and $R_Y = \{n \in \omega : 0 < |Y \cap B_n| < 2\}$ is finite and for all $n \in \omega - (Q_Y \cup R_Y)$, either $A_n \subseteq Y$ or $B_n \subseteq Y$ or both, i.e., $X_n \subseteq Y$ where for $n \in \omega$, $X_n = A_n \cup B_n$. (For example, if Q_Y is infinite, let $Y^* = \cup\{Y \cap A_n : n \in Q_Y\}$ and let E be a support for Y^*. Since Q_Y is infinite, let $n \in Q_Y$ be such that $E \cap A_n = \emptyset$. Let $x \in A_n - Y^*$, $y \in Y \cap A_n$ and let z be the third element of A_n. Then the permutation $\psi = (y, x, z)$ (ψ moves only the atoms x, y, z) fixes E pointwise hence it fixes Y^*. However, $x = \psi(y) \notin Y^*$, a contradiction. Similarly, one shows that R_Y is a finite set.)

Now let $n_0 = \max\{n \in \omega : E \cap X_n \neq \emptyset\}$. In view of the above observations, there exists a natural number $n > n_0$ such that $A_n \subseteq U_1$ or $B_n \subseteq U_1$ or $X_n \subseteq U_1$. Without loss of generality assume that $A_n = \{a_{n1}, a_{n2}, a_{n3}\} \subseteq U_1$. Suppose that $f(a_{n1}) = u$ for some $u \in U_2$. Since $U_1 \cap U_2 = \emptyset$, we have that $u \notin U_1$, hence $u \notin A_n$. We may consider now the permutation ψ to be the 3-cycle (a_{n1}, a_{n2}, a_{n3}), i.e., ψ moves only $a_{nj}, j = 1, 2, 3$, and fixes all the other atoms. Then ψ fixes E pointwise, hence it fixes the function f (not necessarily pointwise). Since $\psi(a_{n1}) = a_{n2}$, $\psi(u) = u$, and $\psi(f) = f$, we may conclude that $f(a_{n2}) = u$ meaning that f is not injective. This contradicts our assumption on f. Therefore, $|A|$ is not divisible by p as required.

This completes the proof of the theorem. □

From Proposition 1 we see that it is provable in ZF that no Russell set can be a p-Russell set, where p is an odd natural number, and vice versa. That is, for every odd natural number p, a p-Russell set cannot be a Russell set.

On the other hand, every Russell set is easily seen to be a $2n$-Russell set for every natural number $n \geq 1$. However, the reverse implication may fail to be true. In fact, in [5, Theorem 3] we have shown that *for every* natural number $n \geq 3$, it is relatively consistent with ZFA that there exists an n-Russell set which is not a Russell set. (However, there were Russell sets in *each* of these models; see [5, Remark 4]). Yet, even more may be true. In particular, for every natural number $n \geq 3$ it is relatively consistent with ZFA that there is an n-Russell set and there are no Russell sets at all. We prove this in the next theorem.

Theorem 11. *Let n be a natural number such that $n \geq 3$. Then there is a model of* ZFA *which has an n-Russell set but has no Russell sets.*

Proof: We consider two cases.

Case 1. $n = 3$ or $n \geq 5$. We start with a ground model of AC with a countable set $A = \cup\{A_i : i \in \omega\}$ of atoms such that:

1. for every $i \in \omega$, $A_i = \{a_{i1}, a_{i2}, \ldots, a_{in}\}$ (hence $\forall i \in \omega$, $|A_i| = n$);
2. for all $i, j \in \omega$, if $i \neq j$, then $A_i \cap A_j = \emptyset$.

The group G of permutations of A is the set of all permutations π such that for every $i \in \omega$, $\pi \restriction A_i$ is an even permutation of A_i. The normal ideal $\mathcal{I}$ of supports is the set of all finite subsets of A. Let $\mathcal{N}$ be the permutation model determined by G and $\mathcal{I}$.

First we note that the family $\mathcal{A} = \{A_i : i \in \omega\}$ does not have a partial choice function in $\mathcal{N}$. Assume the contrary and let $\mathcal{B}$ be an infinite subfamily of $\mathcal{A}$ having a choice function $f \in \mathcal{N}$ with support E. Since E is finite, we may fix an

$i \in \omega$ such that $A_i \in \mathcal{B}$ and $A_i \cap E = \emptyset$. Without loss of generality assume that f chooses a_{i1} from the set A_i. Consider the permutation π which is the identity on A_j, for all $j \in \omega - \{i\}$, and $\pi \restriction A_i = (a_{i1}, a_{i2})(a_{i3}, a_{i4})$. Then π fixes E pointwise, hence $\pi(f) = f$. It follows that $(i, a_{i2}) \in f$, meaning that f is not a function, a contradiction. Therefore, $\mathcal{A}$ cannot have a partial choice function in $\mathcal{N}$ and the set A of atoms is an n-Russell set in $\mathcal{N}$.

We show now that the model $\mathcal{N}$ does not admit any Russell sets. Assume the contrary and let $X = \bigcup_{i\in\omega} X_i \in \mathcal{N}$ be a Russell set. Let $E = A_0 \cup A_1 \cup \cdots \cup A_k$, for some $k \in \omega$, be a support of X_i for each $i \in \omega$. We will prove that for every $i \in \omega$ and for every element $x \in X_i$, E is a support of x. This will give us that X is a well orderable set in $\mathcal{N}$, hence we shall obtain a contradiction to the fact that X is a Russell set.

To this end, assume that there exists an $i \in \omega$, an element $x \in X_i$ and a permutation ψ such that ψ fixes E pointwise but $\psi(x) \neq x$. Let E_x be a support of x. Since E does not support x, we may assume without loss of generality that $E_x = E \cup A_{k+1}$ and that ψ fixes $A - A_{k+1}$ pointwise. Let $\mathbf{G}$ be the subgroup of G consisting of all permutations in G which fix $A - A_{k+1}$ pointwise. Then $\mathbf{G}$ is homeomorphic to the group of even permutations of A_{k+1}. Let $H = \{\pi \in \mathbf{G} : \pi(x) = x\}$. Then H is a subgroup of $\mathbf{G}$. Furthermore, we claim that H is a normal subgroup of $\mathbf{G}$. To prove our assertion we need to show that for all $\phi \in \mathbf{G}$, $\phi H = H\phi$. To this end, fix a permutation $\phi \in \mathbf{G}$. If $\phi \in H$, then for all $\pi \in H$, $\phi\pi = (\phi\pi\phi^{-1})\phi$ and since H is a group we have that $\phi\pi\phi^{-1} \in H$, hence $\phi H \subseteq H\phi$ and similarly $H\phi \subseteq \phi H$. So we may assume that $\phi \in \mathbf{G} - H$. Let $X_i = \{x, y\}$. Then $\phi(x) = y$ (since $\phi \notin H$) and $\phi(y) = x$. Therefore, $\phi^{-1}(x)$ is also equal to y. Fix a permutation $\pi \in H$. Then $\pi(x) = x$, hence $\pi(y) = y$. Now, we have that $\phi\pi\phi^{-1}(x) = \phi\pi(y) = \phi(y) = x$. Thus, $\phi\pi\phi^{-1} \in H$ and consequently $\phi\pi = (\phi\pi\phi^{-1})\phi \in H\phi$ meaning that $\phi H \subseteq H\phi$. Similarly, we may prove that $H\phi \subseteq \phi H$, and so H is a normal subgroup of $\mathbf{G}$.

From group theory we know (see [2]) that for $n = 3$ or for $n \geq 5$ the group of even permutations on n elements has no normal subgroups other than the whole group and the trivial one, namely $\{\mathrm{id}\}$ where id is the identity mapping. Since $H \neq \mathbf{G}$ (for $\psi \in \mathbf{G} - H$; see above for the properties of ψ) we infer that $H = \{\mathrm{id}\}$. It follows that $\forall\phi \in \mathbf{G} - H$, $\forall\rho \in \mathbf{G} - H$, if $\phi \neq \rho$, then $\phi(x) = \rho(x)$ $(= y)$. Now $\mathbf{G}$ has at least 3 elements (since $|\mathbf{G}| = \frac{n!}{2} \geq 3$ since either $n = 3$ or $n \geq 5$) so it has at least two distinct elements ϕ and ρ such that $\phi, \rho \notin H$. Then $\rho^{-1}\phi(x) = x$, so $\rho^{-1}\phi \in H$, hence $\phi = \rho$, a contradiction.

From the above we conclude that whenever a permutation ϕ fixes E pointwise, then ϕ fixes X pointwise, hence X is well orderable contradicting the fact that X is a Russell set. Therefore, the model $\mathcal{N}$ does not have any Russell sets as required.

Case 2. $n = 4$. The suitable Fraenkel-Mostowski model $\mathcal{N}$ is defined as in cases $n = 3$ or $n \geq 5$. We show that $\mathcal{N}$ has no Russell sets. Assume the contrary and let $X = \bigcup_{i\in\omega} X_i$ be a Russell set in the model $\mathcal{N}$. Let E, $x \in X_i$, E_x, ψ, $\mathbf{G}$, and H be as in cases $n = 3$ or $n \geq 5$. Since $\mathbf{G}$ is homeomorphic to the group of even permutations of A_{k+1} and $|A_{k+1}| = 4$, it follows that $|\mathbf{G}| = 12$. Furthermore, since X_i is a two-element set and $\psi(x) \neq x$, it is easy to see that the index of H in $\mathbf{G}$ is 2. Thus, $|H| = 6$. But this contradicts the fact that the group of even permutations on 4 objects does not have any subgroup of cardinality 6 (see [2]). Therefore, any permutation which fixes E pointwise also fixes X pointwise meaning that X is well-orderable. This contradicts the fact that X is a Russell set and completes the proof of Case 2 and of the theorem. □

4. Partitions of generalized Russell sets

Definition 4. A *generalized Russell set* is a set X which can be written as $X = \bigcup_{i \in I} X_i$ where

1. for each $i \in I$, $|X_i| = 2$;
2. I is infinite;
3. for i and j in I, if $i \neq j$ then $X_i \cap X_j = \emptyset$;
4. no infinite subset of $\{X_i \mid i \in I\}$ has a choice function.

A *generalized Russell cardinal* is the cardinal number of a generalized Russell set.

Proposition 3. *Generalized Russell sets are Dedekind sets.*

Proof: This follows immediately from Definition 4. □

In Theorem 2 we showed that if a is a Russell cardinal which has a 3-ary partition, then the Russell cardinal $a + 2$ fails to have one. It is natural to ask whether this holds also for generalized Russell cardinals or Dedekind cardinals in general. We show next that it is relatively consistent with ZFA that there exists a generalized Russell cardinal a, hence a Dedekind cardinal a, such that a, $a + 1$ and $a + 2$ all have 3-ary partitions. Moreover, we prove that the existence of a generalized Russell set $X = \bigcup_{i \in I} X_i$ such that $|X| < |I|$ is consistent with ZFA. Note that in view of [4] this cannot happen for Russell sets (considered as generalized Russell sets by rearranging its elements into pairs).

Theorem 12. *There exists a model of* ZFA *in which there is a generalized Russell set $X = \bigcup_{i \in I} X_i$, hence a Dedekind set X, such that $|X| < |I|$ and such that $|X|$, $|X| + 1$ and $|X| + 2$ all have ternary partitions.*

Proof: We shall use the Fraenkel-Mostowski permutation model defined in the proof of [1, Theorem 3.1]. Similarly to the observation by the authors in [1] (see [1, Section 2]) the result can be transferred to ZF using the Jech-Sochor theorem which provides embeddings of arbitrary long initial segments of ZFA models into ZF models. Thus, we also obtain consistency with ZF.

The atoms are identified (for simplicity's sake) with the elements of $2^{<\omega}$, i.e., with finite non-empty sequences of 0s and 1s. Let A be the set of the atoms. We may view A as two infinite binary trees, the one having $\langle 0 \rangle$ as its root and the other having $\langle 1\rangle$ as its root. The set A is partially ordered by the extension of sequences, i.e., for $t, s \in A$, $t \leq s$ if and only if t is an initial segment of s. Let G be the group of all order automorphisms of $(A, \leq)$, i.e., if $t \in A$ and $\phi \in G$, then t and $\phi(t)$ have the same length and if $s \in A$ and $t \leq s$, then $\phi(t) \leq \phi(s)$. The normal ideal of supports is the set of all finite subsets of A. Let $\mathcal{N}$ be the resulting permutation model.

For each $t \in A$, let $P_t = \{t\hat{}0, t\hat{}1\}$, where $t\hat{}0$ is the sequence t with 0 adjoined as a last element and similarly for $t\hat{}1$. Put $P = \{P_t : t \in A\} \cup \{\{\langle 0\rangle, \langle 1\rangle\}\}$ where $\langle 0\rangle$ and $\langle 1\rangle$ are the sequences of length 1, i.e. the two roots. Then P is a collection of 2-element sets which belongs to the model since it has empty support, i.e., every permutation in G fixes P. Furthermore, the family P has no partial choice function in the model $\mathcal{N}$. To see this, assume on the contrary that P has an infinite subset P' with a choice function, say f, and let E be a support for f. Since P' is infinite, there is an element $t \in A - E$ such that t is not the initial segment of any element of E and $P_t = \{t\hat{}0, t\hat{}1\} \in P'$. Consider a permutation $\psi \in G$ which fixes E pointwise but interchanges the elements of P_t. Since E is a support of f we have that $\psi(f) = f$. However, $(P_t, f(P_t)) \in f \to (P_t, \psi(f(P_t))) \in f$ and $\psi(f(P_t)) \neq f(P_t)$, a contradiction. Therefore, P has no infinite subfamily with a choice function and consequently $A = \bigcup P$ is a generalized Russell set in the model $\mathcal{N}$.

Furthermore, $|A| \leq |P|$ in $\mathcal{N}$ since the function $f : A \to P$ defined by $f(t) = P_t$ for all $t \in A$, is injective and belongs to the model since it has empty support.

We assert that there is no injective function $g : P \to A$ in $\mathcal{N}$. Assume the contrary and let g be such a function with support E. For each $t \in A$ we denote the length of (the sequence indexing) t by $\ln(t)$ and we note

$$\text{(3)} \quad \begin{array}{l} \forall n \in \omega,\, n > 0,\, \forall t \in A \text{ such that } \ln(t) \geq n,\, \exists \phi \in G \text{ which fixes the set} \\ \{s \in A : \ln(s) < n\} \text{ pointwise and such that } \phi(t) \neq t. \end{array}$$

Choose an $n_0 \in \omega$ such that $n_0 > 0$ and $\forall t \in E$, $\ln(t) < n_0$. We make two assertions about g.

Lemma 7. (1) $\ln(g(\{\langle 0\rangle, \langle 1\rangle\})) < n_0$.
(2) $\forall t \in A$, *if* $\ln(t) < n_0$ *then* $\ln(g(P_t)) < n_0$.

Proof: We prove part 2. The proof of 1 is similar and is left to the reader. Assume $t \in A$, that $\ln(t) < n_0$ and, toward a proof by contradiction, that $\ln(g(P_t)) \geq n_0$. By equation (3) there is a $\phi \in G$ such that $\phi(s) = s$ for all $s \in A$ with $\ln(s) < n_0$ and such that $\phi(g(P_t)) \neq g(P_t)$. Since $\ln(t) < n_0$, $\phi(t) = t$. Since the function $r \mapsto P_r$ is in the model with empty support we may also conclude that $\phi(P_t) = P_t$. By our choice of n_0, ϕ also fixes E pointwise and therefore fixes g. This is a contradiction since if ϕ fixes g and P_t it must fix $g(P_t)$. $\square$

By the lemma g restricted to the set $\{\{\langle 0\rangle, \langle 1\rangle\}\} \cup \{P_t : \ln(t) < n_0\}$ has range included in the set $\{t : \ln(t) < n_0\}$. Since the first of these two sets has one more element than the second we arrive at the contradiction that g is not injective. Therefore, $|A| < |P|$ in the model $\mathcal{N}$.

For the second assertion of the theorem, we see that due to the definition of the group G of permutations of the set A of atoms, $\mathcal{P} = \{\{t, t\hat{}0, t\hat{}1\} : t \in A$ and $\ln(t)$ is odd$\}$ is a 3-ary partition of A which lives in the model since every permutation in G fixes (not pointwise) the family $\mathcal{P}$. Now if we discard from A the two roots, namely $\langle 0\rangle$ and $\langle 1\rangle$, we obtain again a generalized Russell set which, similarly to the case of A, also has a ternary partition, namely $\mathcal{Q} = \{\{t, t\hat{}0, t\hat{}1\} : t \in A$ and $\ln(t)$ is even$\}$. If we discard from A one of the two roots, then again we easily see that the resulting Dedekind set also has a ternary partition. This completes the proof of the theorem. $\square$

Remark 2. In [1, Theorem 3.1] it is shown that it is consistent with ZF that there exists a Dedekind set X such that for all natural numbers n, the set Y obtained from X by removing (or adding) n elements from X has a ternary partition. Theorem 12 above also yields the result of Theorem 3.1 in [1]. However, since it also establishes the existence (in some model of ZF) of a generalized Russell set $X = \bigcup_{i\in I} X_i$ such that $|X| < |I|$, the result of Theorem 12 is stronger.

Theorem 13. *There is a model of* ZFA *in which there exists a generalized Russell set A such that A has a 3-ary partition but $|A|$ is not divisible by* 3.

Proof: Let $\mathcal{N}$ be the permutation model defined in the proof of Theorem 12 and let A be its set of atoms. According to the latter proof, A is a generalized Russell set which has a 3-ary partition in $\mathcal{N}$. Hence, we only need to show that $|A|$ is not divisible by 3. Assume the contrary and let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a partition of A consisting of infinite pairwise equipollent sets. Let $f_{1,2} : P_1 \to P_2$ be a bijection in the model with support E and let $n_0 = \max\{\ln(t) : t \in E\}$. Without loss of generality we may assume that E contains all atoms of length less than or equal to n_0 (therefore E contains both roots $\langle 0\rangle$ and $\langle 1\rangle$).

Let $a = t\hat{}0 \in P_1$, $t \in A$, with length $m > n_0$, i.e. $a \notin E$, and suppose that a belongs to the subtree having $\langle 0\rangle$ as its root. By the fact that E is a support of the function $f_{1,2}$ and by the definition of the group G we may conclude that $b = t\hat{}1 \in P_1 = \mathrm{Dom}(f_{1,2})$. (Let $\psi \in \mathrm{fix}(E)$ such that ψ swaps a and b. Then $\psi(f_{1,2}) = f_{1,2}$ and $(a, f_{1,2}(a)) \in f_{1,2} \to (b, \psi(f_{1,2}(a))) \in f_{1,2}$, hence $b \in \mathrm{Dom}(f_{1,2})$). Let $f_{1,2}(a) = c$. Then $c \neq a, b$ since $c \in P_2$, $a, b \in P_1$ and $P_1 \cap P_2 = \emptyset$. We necessarily have that either a or b is a proper initial segment of c. Otherwise, considering the permutation ψ which fixes E pointwise, swaps the atoms a and b and fixes pointwise all the branches which contain neither a nor b, we obtain that $\psi(c) = c$ and $\psi(f_{1,2}) = f_{1,2}$ hence $f_{1,2}(a) = f_{1,2}(b) = c$ meaning that $f_{1,2}$ is not injective, a contradiction. Without loss of generality assume that $c = r\hat{}0$ for some atom r.

1. If a is an initial segment of c, then consider a permutation $\psi \in G$ which swaps c and $d = r\hat{}1$ and moves only c, d and their descendants. Then $\psi \in \mathrm{fix}(E)$, hence $\psi(f_{1,2}) = f_{1,2}$, and $\psi(a) = a$. But then $(a, c) \in f_{1,2} \to$

$\psi(a,c) \in \psi(f_{1,2}) \to (a,d) \in f_{1,2}$ meaning that $f_{1,2}$ is not a function, a contradiction.

2. If b is an initial segment of c, then a is neither a descendant of c nor of d. Working exactly as in (1) we arrive at a contradiction.

Therefore, $|A|$ is not divisible by 3 in $\mathcal{N}$ and the proof of the theorem is complete. $\square$

Acknowledgment. We would like to thank the anonymous referee for bringing Problem 1 to our attention which provided a focus for our results and for several other suggestions which greatly improved the exposition of our paper.

References

[1] Blair D., Blass A., Howard P., *Divisibility of Dedekind finite sets*, J. Math. Log. **5** (2005), no. 1, 49–85.

[2] Fraleigh J.B., *A First Course in Abstract Algebra*, Addison-Wesley Publ. Co., Reading, Mass.-London-Don Mills, Ont., 1967.

[3] Herrlich H., *Axiom of Choice*, Springer Lecture Notes in Mathematics, 1876, Springer, New York, 2006.

[4] Herrlich H., *Binary partitions in the absence of choice or rearranging Russell's socks*, Quaest. Math. **30** (2007), no. 4, 465–470.

[5] Herrlich H., Howard P., Tachtsis E., *The cardinal inequality* $\alpha^2 < 2^\alpha$, Quaest. Math. **34** (2011), no. 1, 35–66.

[6] Herrlich H., Keremedis K., Tachtsis E., *On Russell and anti Russell–cardinals*, Quaest. Math. **33** (2010), 1–9.

[7] Herrlich H., Tachtsis E., *On the number of Russell's socks or* $2+2+2+\cdots=?$, Comment. Math. Univ. Carolin. **47** (2006), 707–717.

[8] Herrlich H., Tachtsis E., *Odd-sized partitions of Russell-sets*, Math. Logic Quart. **56** (2010), no. 2, 185–190.

[9] Howard P., Rubin J.E., *Consequences of the Axiom of Choice*, Mathematical Surveys and Monographs, 59, American Mathematical Society, Providence, RI, 1998; (`http://consequences.emich.edu/conseq.htm`).

[10] Jech T.J., *The Axiom of Choice*, Studies in Logic and the Foundations of Mathematics, 75, North-Holland, Amsterdam, 1973; Reprint: Dover Publications, Inc., New York, 2008.

[11] Tarski A., *Cancellation laws in the arithmetic of cardinals*, Fund. Math. **36** (1949), 77-92.

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(*Received* February 8, 2011, *revised* June 6, 2011)