H-closed extensions with countable remainder

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Abstract. This paper investigates necessary and sufficient conditions for a space to have an H-closed extension with countable remainder. For countable spaces we are able to give two characterizations of those spaces admitting an H-closed extension with countable remainder.

The general case is more difficult, however, we arrive at a necessary condition — a generalization of Čech completeness, and several sufficient conditions for a space to have an H-closed extension with countable remainder. In particular, using the notation of Császár, we show that a space X is a Čech g-space if and only if X is G_δ in σX or equivalently if EX is Čech complete. An example of a space which is a Čech f-space but not a Čech g-space is given answering a couple of questions of Császár. We show that if X is a Čech g-space and $R(EX)$, the residue of EX, is Lindelöf, then X has an H-closed extension with countable remainder. Finally, we investigate some natural generalizations of the residue to the class of all Hausdorff spaces.

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In this paper we will concern ourselves with finding H-closed extensions with countable remainder, i.e. the smallest H-closed extensions. Our topic is a generalization of a question of Morita [11]: characterize those spaces which have compactifications with countable remainder — an area studied in depth by Henriksen [7], Hoshina [8], [9], [10], Terada [16] and Charalambous [1] but still not entirely resolved.

The question of which spaces allow H-closed extensions with countable remainder is an obvious generalization of the question of compactifications with countable remainder, and has been considered by Porter and Vermeer [13] and Tikoo [17]. Much of the background for this paper can be found in [13], [17] and [15].

Recall that the Iliadis absolute of a Hausdorff space X is the pair (EX, k) — where EX is a zero-dimensional, extremally disconnected Hausdorff space and $k : EX \to X$ is a perfect, irreducible and θ-continuous surjection. Also recall that the space σX is the largest strict H-closed extension of X.

The bulk of the results in this paper are informed by the following facts.

Theorem 1 ([12], [14], [15]). *Let X be a Hausdorff space.*

(1) *Then $\sigma X \setminus X$ is homeomorphic to $\beta EX \setminus EX$.*

(2) *For each H-closed extension hX of X, there is a θ-continuous function $f_h : \sigma X \to hX$ such that $f_h = \mathrm{id}_X$ and $\{f_h^{\leftarrow}(y) : y \in h_X \setminus X\}$ is a partition of compact subsets of $\sigma X \setminus X$.*

(3) *For each partition $\mathcal{P}$ of nonempty compact sets of $\sigma X \setminus X$, there is an H-closed extension hX of X such that $\mathcal{P} = \{f_h^{\leftarrow}(y) : y \in hX \setminus X\}$.*

(4) *Let η be a cardinal. There is an H-closed extension hX of X with $|hX \setminus X| = \eta$ iff $\sigma X \setminus X$ can be partitioned into η many compact sets.*

Corollary 2. *The space X has an H-closed extension with countable remainder iff $\sigma X \setminus X \cong \beta EX \setminus EX$ has a countable partition of compact sets.*

A few more facts about the Iliadis absolute will be useful in this paper. First recall the definition of the small image of a set.

Definition 3. Given a function $f : X \to Y$ where X and Y are sets, we define

$$f^{\#}[A] = \{y \in Y : f^{\leftarrow}(y) \subseteq A\}.$$

Fact 4. *Let X be a Hausdorff space and $k : EX \to X$ be the absolute map.*

(1) [15] *If $U \in \tau(X)$, $OU = O(\mathrm{int}_X \mathrm{cl}_X U)$, $k[OU] = \mathrm{cl}_X U$ and $\mathrm{cl}_{EX} k^{\leftarrow}[U] = OU$.*

(2) [15] *For $x \in X$ and $U \in \tau(X)$, $k^{\leftarrow}(x) \subseteq OU$ iff $x \in \mathrm{int}_X \mathrm{cl}_X U$, in particular, $k^{\#}[OU] = \mathrm{int}_X \mathrm{cl}_X U$.*

(3) *If T is clopen in EX then $T = O(k^{\#}[T])$.*

Proof: Since T is clopen in EX, $T = OU$ for some $U \in \tau(X)$. By the above $k^{\#}[T] = \mathrm{int}_X \mathrm{cl}_X U$ and so $T = OU = O(\mathrm{int}_X \mathrm{cl}_X U) = O(k^{\#}[T])$. □

1. Countable spaces

Our goal is to determine which spaces have H-closed extensions with a countable remainder. As a sub-goal we first consider which countable spaces have countable H-closed extensions.

Fact 5. *A countable space X with a countable H-closed extension is Katětov.*

Proof: By 1.4 of [13], it suffices to show X has an infinite closed discrete subspace. If X has no infinite closed discrete subspaces, then every infinite subset of X has a derived point. This means X is countably compact. As X is countable, it follows that X is compact — hence Katětov. □

The other direction is to determine which countable spaces have a countable H-closed extension. We start with a countable, first countable, semiregular, Katětov space X. We may also assume X is not countably compact; that is, X contains an infinite, closed discrete subspace A.

Theorem 6. *A countable Hausdorff space X has a countable H-closed extension iff X is Katětov and X_s is first countable.*

Proof: Suppose a countable space X is Katětov and X_s is first countable. We want to show X has an H-closed extension with countable remainder. By Theorem 1, it suffices to show $\beta EX \setminus EX$ has a countable partition of compact sets.

Let X' denote X with the coarser H-closed topology. So we have that the identity function $\mathrm{id}_X : X \to X'$ is continuous.

(1) By [3], there is a continuous function $f : EX \to EX'$ such that $k_{X'} \circ f = \mathrm{id}_X \circ k_X$. That is, the following diagram commutes:

$$\begin{tikzcd} EX \arrow[r, "f"] \arrow[d, "k_X"'] & EX' \arrow[d, "k_{X'}"] \\ X \arrow[r, "\mathrm{id}_X"] & X' \end{tikzcd}$$

As X' is H-closed, EX' is compact Hausdorff by 1. Also, there is a continuous extension $\beta f : \beta EX \to EX'$ and the following diagram commutes.

$$\begin{tikzcd} \beta EX \arrow[dr, "\beta f"] & \\ EX \arrow[u, hook] \arrow[r, "f"] \arrow[d, "k_X"'] & EX' \arrow[d, "k_{X'}"] \\ X \arrow[r, "\mathrm{id}_X"] & X' \end{tikzcd}$$

Let $X = \{p_n : n \in \omega\}$ and $X' = \{p'_n : n \in \omega\}$ where $\mathrm{id}_X(p_n) = p'_n$ for $n \in \omega$. Since k_X is perfect, we have that $\{k_X^{\leftarrow}(p_n) : n \in \omega\}$ is a partition of EX into compact subsets, $\{k_{X'}^{\leftarrow}(p'_n) : n \in \omega\}$ is a partition of EX' into compact subsets, and $\{(k_{X'} \circ \beta f)^{\leftarrow}(p'_n) : n \in \omega\}$ is a partition of βEX into compact subsets. By commutativity of the diagram, it follows that $k_X^{\leftarrow}(p_n) = (k_{X'} \circ f)^{\leftarrow}(p'_n) \subseteq (k_{X'} \circ \beta f)^{\leftarrow}(p'_n)$ and $(k_{X'} \circ \beta f)^{\leftarrow}(p'_n) \cap EX = k_X^{\leftarrow}(p_n)$ for $n \in \omega$.

(2) As X_s is first countable, for each $x \in X$ there is a countable neighborhood base $\{U_n\}_\omega$ of regular open sets for $x \in X_s$. We now show $\{\mathrm{cl}_{\beta EX} OU_n\}_\omega$ is a countable family of clopen sets for which if $k_X^{\leftarrow}(x) \subseteq T \in \tau(\beta EX)$ then there is some $m \in \omega$ such that $\mathrm{cl}_{\beta EX} OU_m \subseteq T$. Let T be an open set in βEX such that $k_X^{\leftarrow}(x) \subseteq T$. As the clopen family $\{\mathrm{cl}_{\beta EX} S : S \text{ is clopen in } EX\}$ is a base for βEX which is closed under finite unions and $k_X^{\leftarrow}(x)$ is compact, we can suppose $T = \mathrm{cl}_{\beta EX} S$ for some clopen set S of EX. By 4, $S = OU$ for some $U \in \tau(X)$. As $k_X^{\leftarrow}(x) \subseteq OU$, it follows that $x \in \mathrm{int}_X \mathrm{cl}_X U$ and so for some $n \in \omega$, $x \in U_n \subseteq \mathrm{int}_X \mathrm{cl}_X U$. Hence we have $k_X^{\leftarrow}(x) \subseteq OU_n \subseteq O(\mathrm{int}_X \mathrm{cl}_X U) = OU = S$ and $k_X^{\leftarrow}(x) \subseteq$

$\mathrm{cl}_{\beta EX}\, OU_n \subseteq T$. Thus, $k_X^{\leftarrow}(x) = \bigcap_\omega \mathrm{cl}_{\beta EX}\, OU_n$, and we can suppose

$$\mathrm{cl}_{\beta EX}\, OU_{n+1} \subseteq \mathrm{cl}_{\beta EX}\, OU_n$$

for $n \in \omega$.

(3) Using the notation of 1, for each $n \in \omega$ we have $k_X^{\leftarrow}(p_n) \subseteq (k_{X'} \circ \beta f)^{\leftarrow}(p'_n)$ and $(k_{X'} \circ \beta f)^{\leftarrow}(p'_n) \setminus k_X^{\leftarrow}(p_n) \subseteq \beta EX \setminus EX$ and finally

$$\bigcup_\omega ((k_{X'} \circ \beta f)^{\leftarrow}(p'_n) \setminus k_X^{\leftarrow}(p_n)) = \beta EX \setminus EX.$$

Note

$$[(k_{X'} \circ \beta f)^{\leftarrow}(p'_n) \setminus k_X^{\leftarrow}(p_n)] \cap [\mathrm{cl}_{\beta EX}\, OU_k \setminus \mathrm{cl}_{\beta EX}\, OU_{k+1}] = K_{nk}$$

is a compact subset of $\beta EX \setminus EX$. Now, $\bigcup_{k\in\omega} K_{nk} = (k_{X'} \circ \beta f)^{\leftarrow}(p'_n) \setminus k_X^{\leftarrow}(p_n)$, $\beta EX \setminus EX = \bigcup_{n,k\in\omega} K_{nk}$ and $\{K_{nk} : n, k \in \omega\}$ is a partition of $\beta EX \setminus EX$. By 1, as $\beta EX \setminus EX$ has a countable partition of compact subsets, both EX and X have H-closed extensions with countable remainder.

Conversely, suppose the countable Hausdorff space X has a countable H-closed extension hX. By 1, $\sigma X \setminus X$ has a countable partition of compact sets. If X is not countably compact, X has a countably infinite closed discrete subspace. By 5, X is Katětov. If the countable space X is countably compact, then X is also compact and hence Katětov. As hX is countable and H-closed, hX_s is a countable minimal Hausdorff extension of X_s. But countable minimal Hausdorff spaces are first countable. Thus, X_s is first countable as well. □

2. Generalizations of Čech completeness

We recall some basic definitions before considering the question of how generalizations of Čech completeness relate to finding H-closed extensions with countable remainder.

Definition 7. A Tychonoff space X is Čech complete if it is G_δ in every Hausdorff extension.

The following theorem is well-known and provides two important characterizations of Čech completeness. The first allows us a reduction in the number of compact Hausdorff extensions we must consider, and the second provides an internal characterization of the property.

Theorem 8 ([5], [4]). *The following are equivalent for a Tychonoff space X.*

(1) *The space X is Čech complete.*
(2) *The space X is G_δ in βX.*
(3) *There exists a sequence $(\mathcal{C}_n)_\omega$ of open covers of X such that every filter base of closed sets subordinate to $(\mathcal{C}_n)_\omega$ has non-empty intersection.*

The following corollary is immediate.

Corollary 9. *If a space X has an H-closed extension with countable remainder then EX is Čech complete.*

Proof: Recall from 1 that a space X has an H-closed extension with countable remainder iff $\beta EX \setminus EX$ has a countable partition of compact sets. Of course, a prerequisite for $\beta EX \setminus EX$ to be the countable partition of compact sets is that it actually be the union of countably many compact sets. So if $\beta EX \setminus EX = \bigcup_\omega K_n$ where K_n is compact, then $G_n = \beta EX \setminus K_n$ is a family of open sets of βEX and $EX \subseteq G_n$ for all $n \in \omega$. Since $\bigcup_\omega K_n = \beta EX \setminus EX$, we have $\bigcap_\omega G_n = EX$. Hence EX is Čech complete. □

Though Čech completeness of the absolute is a necessary condition for the existence of an H-closed extension with countable remainder, we will see that it is not sufficient — some additional property is required.

For metric space, restrictions related to the following definitions (along with Čech completeness) are sufficient to allow a compactification with countable remainder.

Notation 10 ([13]). For a Tychonoff space X, let $R(X) = [\mathrm{cl}_{\beta X}(\beta X \setminus X)] \cap X$. We call $R(X)$ the residue of X.

Definition 11. A space X called rim-compact (or semicompact) if X has a basis of open sets each of which has a compact boundary.

Definition 12. A space X is called Lindelöf if every open cover of X has a countable subfamily which covers.

The characterization of metric spaces allowing compactification with countable remainder is due to Hoshina.

Theorem 13 ([8]). *A metrizable space X has a compactification with countable remainder iff X is Čech complete, rim-compact and $R(X)$ is Lindelöf.*

For compactifications of Tychonoff spaces with countable remainder Hoshina also provides a sufficient condition.

Theorem 14 ([8]). *Let X be a Čech complete, rim-compact space. If $R(X)$ is separable metrizable then X has a compactification with countable remainder.*

We quote the following lemma of Hoshina [9], which is necessary for the next example.

Lemma 15. *If X has a compactification with countable remainder and $\mathcal{U}$ is a collection of pairwise disjoint open sets of X with $U \cap R(X) \neq \varnothing$ for each $U \in \mathcal{U}$, then $\mathcal{U}$ is countable.*

First we consider an example of Charalambous [1] showing that Čech completeness is not enough to guarantee that a space has a compact extension with countable remainder; moreover there exist two spaces X and X_1 with homeomorphic residues, $R(X) \cong R(X_1)$, one of which has a compactification with countable remainder — while the other does not.

Example 16 ([1]). The construction starts with the following setup due to Terada [16]. Note $X = \beta\mathbb{R} \setminus \mathbb{N}$ has a compactification with countable remainder, namely $\beta\mathbb{R}$, and $R(X) = \beta\mathbb{N} \setminus \mathbb{N}$.

Now let $Z = \mathbb{N} \cup \{\infty\}$, the one point compactification of $\mathbb{N}$, $Y = Z \times Z \times (\beta\mathbb{N} \setminus \mathbb{N})$ and $X_1 = Y \setminus [\{\infty\} \times \mathbb{N} \times (\beta\mathbb{N} \setminus \mathbb{N})]$. Since Y is compact and $Y \setminus X_1$ is σ-compact and zero-dimensional, then X_1 is Čech complete and rim-compact. In addition, $R(X_1) = \{\infty\} \times \{\infty\} \times (\beta\mathbb{N} \setminus \mathbb{N})$ is homeomorphic with $R(X)$. But X_1 has no compactification with countable remainder. For let $\mathcal{U}$ be an uncountable collection of pairwise disjoint nonempty open subsets of $\beta\mathbb{N} \setminus \mathbb{N}$. For each $U \in \mathcal{U}$ let $U' = Z \times Z \times U$, then $\{U' \cap X_1 : U \in \mathcal{U}\}$ is an uncountable collection of pairwise disjoint open sets of X_1 with $U' \cap X_1 \cap R(X_1) \neq \varnothing$ for each $U \in \mathcal{U}$. So by the lemma above, X_1 has no compactification with countable remainder.

We note here, however, that X_1 *does* have an H-closed extension with countable remainder, since $Y \setminus X_1 = \{\infty\} \times \mathbb{N} \times (\beta\mathbb{N} \setminus \mathbb{N})$ is zero-dimensional and the countable union of compact G_δ sets.

We now consider how it may be possible to partition the space $\beta EX \setminus EX$ into countably many compact sets — which would allow us to construct an H-closed extension of X with countable remainder. Since $\beta EX \setminus EX$ is zero-dimensional, the following proposition, communicated to Porter and Vermeer by F. Galvin, will be very useful.

Proposition 17 ([13]). *A zero-dimensional space Y can be partitioned into a countable number of compact sets iff Y is the countable union of compact G_δ-sets.*

Seeking to generalize Hoshina's characterization of metrizable spaces allowing compactifications with countable remainder, Porter and Vermeer found the following sufficient conditions for an H-closed extension with countable remainder.

Theorem 18 ([13]). *If cX is a zero-dimensional compactification of a Čech complete space X and $R(X)$ is Lindelöf, then $cX \setminus X$ has a countable partition of compact sets.*

Corollary 19 ([13]). *Let X be a space.*

(1) *If X is not countably compact, EX is Čech complete, and $R(EX)$ is Lindelöf, then X has an H-closed extension with countable remainder and is Katětov.*
(2) *If X is Tychonoff and Čech complete and $R(X)$ is Lindelöf, then X has an H-closed extension with a countable remainder.*

Noting that Čech completeness of the absolute is necessary for a space to have an H-closed extension with countable remainder — we seek a generalization of Čech completeness to Hausdorff spaces which we may be able use directly. K. Császár in [2] modifies the internal characterization of a Čech complete space to obtain three different generalizations, two of which we will consider in depth.

Before we begin we will need the following definition also due to Császár:

Definition 20. A subset A of a topological space X is said to regularly embedded in X if whenever $x \in A \subseteq G$ and G is open, then there exists an open set V such that $x \in V \subseteq \mathrm{cl}_X V \subseteq G$.

Proposition 21 ([2]). *Suppose $A \subseteq X \subseteq Y$ are spaces. If A is regularly embedded in Y, then A is regularly embedded in X.*

Theorem 22 ([2]). *If X is a Hausdorff space, then X is regularly embedded in σX.*

The following definitions generalize the internal characterization of Čech completeness for Tychonoff spaces to all Hausdorff spaces.

Definition 23. Let $(\mathcal{C}_n)_\omega$ be a sequence of families of sets of a set X and $\mathcal{A}$ a family of sets. The family $\mathcal{A}$ is subordinate to the sequence $(\mathcal{C}_n)_\omega$ if, for every $m \in \omega$, there is some set $A \in \mathcal{A}$ and also a set $C \in \mathcal{C}_m$ such that $A \subseteq C$.

Definition 24. Let X be a topological space. A Čech sequence (Čech f-sequence, Čech g-sequence) in X is a sequence $(\mathcal{C}_n)_\omega$ of open covers of X such that every filter base $\mathcal{A}$ (of closed sets, of open sets) subordinate to $(\mathcal{C}_n)_\omega$ has an adherent point.

Definition 25. A Hausdorff space X is a Čech space (Čech g-space, Čech f-space) if there is a Čech sequence (Čech g-sequence, Čech f-sequence) in X.

Notice that for a Tychonoff space the concepts of Čech space, Čech g-space, Čech f-space, and Čech complete space coincide.

Theorem 26 ([2]). *A regularly embedded open subspace of a Čech g-space is a Čech g-space.*

Theorem 27 ([2]). *A regularly embedded, dense G_δ subspace of a Čech g-space is a Čech g-space.*

Definition 28. A sequence of open covers $(\mathcal{C}_n)_\omega$ is said to be monotone if $\mathcal{C}_{n+1}$ refines $\mathcal{C}_n$.

Proposition 29 ([2]). *If there exists a Čech sequence (g-sequence, f-sequence) for a space X, then there exists a monotone Čech sequence (g-sequence, f-sequence).*

The following proposition provides an external characterization of a Čech g-space comparable to that of a Čech complete space.

Proposition 30 ([2])**.** *For a space X the following are equivalent.*

(1) *X is G_δ in every Hausdorff extension.*
(2) *X is G_δ in σX.*
(3) *X is a Čech g-space.*

With regard to finding H-closed extensions with countable remainder, the previous proposition indicates that Čech g-spaces may be the generalization of Čech complete spaces we should consider. The next proposition provides more support for this observation. We begin with the following lemma which generalizes a theorem appearing in [14].

Lemma 31. *Let X be a space. If $A \subseteq \sigma X \setminus X$ and A is closed in $\sigma X \setminus X$, then $\mathrm{cl}_{\sigma X} A$ is an H-set of σX.*

PROOF: Let $\mathcal{U}$ be an open cover of $\mathrm{cl}_{\sigma X} A$. Extend, and possibly refine, $\mathcal{U}$ to an open cover, $\mathcal{C}$, of all of σX with basic open sets of the form oU where $U \in \tau(X)$. Since σX is H-closed we can find a finite subfamily of $\mathcal{C}$ with the closures covering σX, and since $\mathrm{cl}_{\sigma X} oU = \mathrm{cl}_X U \cup oU$ we get a finite subfamily covering A, hence finite subfamily whose closures cover $\mathrm{cl}_{\sigma X} A$. □

Corollary 32 ([14])**.** *Let X be a space. If $A \subseteq \sigma X \setminus X$ and A is closed in σX, then A is compact.*

Proposition 33. *A space X is a Čech g-space iff EX is Čech complete.*

PROOF: The space X is a Čech g-space iff X is G_δ in σX, i.e. $X = \bigcap_\omega U_n$ where $U_n \in \tau(\sigma X)$. Let $K_n = \sigma X \setminus U_n$, so $\sigma X \setminus X = \bigcup_\omega K_n$ and each K_n is compact. Now recall $\sigma X \setminus X \cong \sigma EX \setminus EX$. Consider $K_n \subseteq \sigma EX \setminus EX$, and let $\hat{U}_n = \sigma EX \setminus K_n$. Note $EX \subseteq \hat{U}_n$, and since $\bigcup_\omega K_n = \sigma EX \setminus EX$, then $EX = \bigcap_\omega \hat{U}_n$ and EX is G_δ in σEX and hence Čech complete.

The argument can also be reversed. □

Corollary 34. *A space X is a Čech g-space iff X_s is a Čech g-space.*

PROOF: This follows from $EX = EX_s$. □

The following proposition is another characterization of countable spaces admitting an H-closed extension with countable remainder. First we note that if X is countable then EX is Lindelöf.

Lemma 35. *Let X be a countable space, then EX is Lindelöf.*

PROOF: Since $k : EX \to X$ is compact, $EX = \bigcup\{k^{\leftarrow}(x) : x \in X\}$ is the countable union of compact sets — hence Lindelöf. □

Proposition 36. *A countable space X admits an H-closed extension with countable remainder iff X is a Čech g-space.*

PROOF: Clearly if X admits an H-closed extension with countable remainder, then X is a Čech g-space.

Now suppose X is countable and a Čech g-space, then EX is Tychonoff and Čech complete. Also note since X is countable that X is Lindelöf. Therefore EX is Lindelöf. Since $R(EX)$ is a closed subset of EX, it is Lindelöf as well. By 18, EX has an H-closed extension with countable remainder. Therefore X does as well. □

Combining the above with 6 we have the following.

Theorem 37. *For a countable space X the following are equivalent.*

(1) *X has an H-closed extension with countable remainder.*
(2) *X is Katětov and X_s is first countable.*
(3) *X is a Čech g-space.*

The following provides a characterization of all Hausdorff spaces having an H-closed extension with countable remainder in terms of a special class of Čech g-sequences.

Proposition 38. *The space X has an H-closed extension with countable remainder iff X admits a Čech g-sequence $(\mathcal{C}_n)_\omega$ for which each free open ultrafilter p is not subordinate to $\mathcal{C}_m$ only for $m = N_p$ for some $N_p \in \omega$.*

Proof: Recall X has an H-closed extension with countable remainder iff $\sigma X \setminus X = \beta EX \setminus EX$ has a countable partition of compact sets $\{K_n\}$. Let $G_n = \sigma X \setminus K_n$, then G_n is open in σX and so $G_n = \bigcup oU$ where $oU \subseteq G_n$ and $U \in \tau(X)$. Since $X \subseteq G_n$ and $oU \cap X = U$, $X = \bigcup\{U : oU \subseteq G_n\}$, i.e. $\{U : oU \subseteq G_n\}$ is an open cover of X. Note for each $p \in \sigma X \setminus X$, $p \in K_n$ implies $p \notin K_m$ for $m \neq n$, i.e. $p \notin \sigma X \setminus G_n$ implies $p \in \sigma X \setminus G_m$ for $m \neq n$. Finally we get $U \notin p$ for all U such that $oU \subseteq G_m$ implies $V \in p$ for all V such that $oV \subseteq G_m$ for $m \neq n$. Let $\mathcal{C}_n = \{U : oU \subseteq G_n\}$, then $(\mathcal{C}_n)_\omega$ is a sequence of open covers of X. Also, for each $p \in \sigma X \setminus X$ there is an $N \in \omega$ such that $U \notin p$ for all $U \in \mathcal{C}_N$ (i.e. $p \in K_N$). In addition, for all $p \in \sigma X \setminus X$, p (as an open filter) is subordinate to all $\mathcal{C}_n$ where $n \neq N$. Hence no free open ultrafilter on X is subordinate to $(\mathcal{C}_n)$ and $(\mathcal{C}_n)$ is a Čech g-sequence on X — one in which each open ultrafilter is excluded at exactly one level.

The argument above can be reversed. That is given a special Čech g-sequence $(\mathcal{C}_n)_\omega$, we simply notice that $\{K_n : K_n = \sigma X \setminus \bigcup\{oU : U \in \mathcal{C}_n\}\}$ is a countable compact partition of $\sigma X \setminus X$. □

Császár [2] gives an example showing not all Čech g-spaces are Čech f-spaces, a somewhat simpler example is provided by the following.

Example 39. Let X be the unit interval with the topology generated by open sets of the form $I \setminus M$ where I is an interval and M is countable. Then X is a Hausdorff Čech g-space which is not a Čech f-space.

Proof: Since X is H-closed, it is a Čech g-space.

To show X is not a Čech f-space, let $(\mathcal{C}_n)$ be a sequence of open covers of X. Select $C_n \in \mathcal{C}_n$ such that $0 \in C_n$ and then I_n and M_n such that $0 \in I_n \setminus M_n \subseteq C_n$.

Define

$$M_0 = \bigcup_1^\infty M_n \cup \{0\},$$

find some

$$x_k \in \left(\left(\bigcap_1^\infty I_n\right) \cap \left[0, \frac{1}{k}\right)\right) \setminus M_0,$$

and finally let

$$A_n = \{x_k : k \geq n\}.$$

After noting that A_n is closed by virtue of being countable, by $A_n \subseteq I_n \setminus M_0 \subseteq I_n \setminus M_n \subseteq C_n$ the system $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ is a closed filter base subordinate to $(\mathcal{C}_n)$. So since $\bigcap A_n = \varnothing$, X is not a Čech f-space. □

Császár goes on to ask whether every Čech f-space is also a Čech g-space. This is not the case.

Theorem 40. *There is a space which is a Čech f-space but not a Čech g-space.*

The following lemma is well known and can be found in Chapter 9 of [6].

Lemma 41. *If X is locally compact and realcompact, then every infinite closed subset of $\beta X \setminus X$ has cardinality at least $2^{\mathfrak{c}}$.*

We now construct a special subset of $\beta\omega \setminus \omega$.

Lemma 42. *There is a set $D \subseteq \beta\omega \setminus \omega = \omega^*$ for which D intersects every infinite compact subset of ω^* and $\omega^* \setminus D$ also intersects every infinite compact subset of ω^*.*

Proof: Note any infinite compact subset of ω^* has a countably infinite subset. We consider the family of sets $\mathcal{C} = \{C : C \text{ is a countably infinite subset of } \omega^*\}$. Note $|\mathcal{C}| = (2^{\mathfrak{c}})^\omega = 2^{\mathfrak{c}}$. Hence if $\mathcal{K} = \{K : K = \mathrm{cl}_{\beta\omega} C \text{ for some } C \in \mathcal{C}\}$, then $|\mathcal{K}| \leq 2^{\mathfrak{c}}$. We construct D recursively; begin by well-ordering $\mathcal{K} = \{K_\beta : \beta < 2^{\mathfrak{c}}\}$. Let $p \in D_0$ and $q \in E_0$ where $p, q \in K_0$ and $p \neq q$.

For $\alpha + 1$ a successor ordinal, let $D_{\alpha+1} = D_\alpha \cup \{p\}$ and $E_{\alpha+1} = E_\alpha \cup \{q\}$ where $p, q \in K_{\alpha+1} \setminus (D_\alpha \cup E_\alpha)$ and $p \neq q$. Note $K_{\alpha+1} \setminus (D_\alpha \cup E_\alpha) \neq \varnothing$ since $|K_{\alpha+1}| = 2^{\mathfrak{c}}$ but $|D_\alpha \cup E_\alpha| < 2^{\mathfrak{c}}$.

For α a limit ordinal, let $D_\alpha = \bigcup_{\beta<\alpha} D_\beta \cup \{p\}$ and $E_\alpha = \bigcup_{\beta<\alpha} E_\beta \cup \{q\}$ where $p, q \in K_\alpha \setminus (\bigcup_{\beta<\alpha} D_\beta \cup \bigcup_{\beta<\alpha} E_\beta)$ and $p \neq q$. Note $K_\alpha \setminus (\bigcup_{\beta<\alpha} D_\beta \cup \bigcup_{\beta<\alpha} E_\beta) \neq \varnothing$ since $|K_\alpha| = 2^{\mathfrak{c}}$ but still $|\bigcup_{\beta<\alpha} D_\beta \cup \bigcup_{\beta<\alpha} E_\beta| < 2^{\mathfrak{c}}$.

Let $D = \bigcup_{2^{\mathfrak{c}}} D_\alpha$ and $E = \bigcup_{2^{\mathfrak{c}}} E_\alpha$. Note $D \cap E = \varnothing$ and for each infinite compact subset K of ω^*, $K \cap D \neq \varnothing$ and $K \cap E \neq \varnothing$. □

Proof of 40: Consider the set D constructed above as a subset of $\kappa\omega$. Let $X = \kappa\omega \setminus D$, then X is a Čech f-space but not a Čech g-space.

To show X is a Čech f-space we must find a sequence of open covers $(\mathcal{C}_n)_\omega$ of X for which every subordinate closed filter base has nonempty adherence. The sequence $(\mathcal{C}_n)_\omega$ where $\mathcal{C}_n = \mathcal{C} = \{\{p\} \cup \omega : p \in X \setminus \omega\}$ suffices. For suppose $\mathcal{F}$ is a subordinate closed filter base, then there is some $F \in \mathcal{F}$ and $U \in \mathcal{C}$ for which $F \subseteq U$. Now F cannot contain an infinite subset V of ω because then $oV \cap X \subseteq \mathrm{cl}_X V \subseteq F$, but $oV \cap X \nsubseteq U$. So $F \cap \omega$ is finite, and hence F is finite. Now $\mathcal{F}$ contains a compact set and hence has nonempty adherence.

To show X is not a Čech g-space we consider the following diagram:

$$\begin{tikzcd}
\omega = E\omega \arrow[r, hook] \arrow[d] & EX = X_s \arrow[r, hook] \arrow[d] & E(\kappa\omega) = \beta\omega \arrow[d] \\
\omega \arrow[r, hook] & X \arrow[r, hook] & \kappa\omega.
\end{tikzcd}$$

In this case if X is a Čech g-space then $EX = X_s$ is Čech complete. But then EX is G_δ in every Hausdorff extension, in particular $\beta\omega$ — contradicting the construction of D. □

From the above a space must be a Čech g-space if it is to have an H-closed extension with countable remainder. By 18, if we also have that the residue of EX, $R(EX)$, is Lindelöf, then this is sufficient to guarantee an H-closed extension of the space with countable remainder. Hence we have the following corollary.

Corollary 43. *If a space X is a Čech g-space and $R(EX)$ is Lindelöf, then X has an H-closed extension with countable remainder.*

It seems that the next step would be to generalize the condition on $R(EX)$ to a condition on the original space X. What follows are several theorems and examples obtained while trying to find conditions both necessary and sufficient for a space to have an H-closed extension with countable remainder.

Lemma 44. *The countable intersection of σ-compact subspaces in a regular space is Lindelöf.*

PROOF: Let X be a regular space, $B_n \subseteq X$ where B_n is σ-compact for $n \in \omega$, and $A = \bigcap_\omega B_n$. Note $\prod_\omega B_n$ is Lindelöf. The function $e : A \to \prod_\omega B_n$ defined by $e(x)(n) = x$ is an embedding and $e[A]$ is closed in the product. Therefore A is Lindelöf. □

Proposition 45 ([13]). *Let X be a Tychonoff, nowhere locally compact space. If X has an H-closed extension with countable remainder, then X has a dense Lindelöf subspace.*

Fact 46 ([13]). *A complete metric space is Katětov.*

Example 47 ([13]). Let D be the discrete space of cardinality $\aleph_1$, and $\mathbb{P}$ be the irrationals. Note both D and $\mathbb{P}$ have compact extensions with countable remainder. Also, the space $D \times \mathbb{P}$ is locally Lindelöf and a complete metric space — hence Čech complete, first countable and Katětov. Recall $\mathbb{P}$ has a coarser compact Hausdorff topology. In particular, $\mathbb{P} \cong \prod_\omega \omega$, and there is a continuous bijection

$f : \prod_\omega \omega \to \prod_\omega(\omega \cup \{\infty\})$. Let $\mathbb{P}'$ denote $\mathbb{P}$ with this coarser compact Hausdorff topology, then $D \times \mathbb{P}'$ is locally compact and Hausdorff. Thus, $D \times \mathbb{P}$ has a coarser compact Hausdorff topology. However, since the space is nowhere locally compact and has no dense Lindelöf subspace, $D \times \mathbb{P}$ has no H-closed extension with countable remainder.

The converse of 45 is false, for consider the space $\mathbb{Q}$. Also consider the following example, which has a dense subspace admitting an H-closed extension with countable remainder, but has none itself.

Example 48. Again let D be the discrete space of cardinality $\aleph_1$ and let D^* be the one point compactification of D. Let $\mathbb{R}$ denote the real numbers with the usual topology and let $\mathbb{R}^+$ denote the two point compactification of $\mathbb{R}$. Let $X = \mathbb{P} \times D^* \times \mathbb{R}^+$ and note that $cX = \mathbb{R}^+ \times D^* \times \mathbb{R}^+$ is a compactification of X where $cX \setminus X = \mathbb{Q} \times D^* \times \mathbb{R}^+$ has a countable partition into compact sets. So X has an H-closed extension with countable remainder. Let $Y = X \cup (\mathbb{Q} \times D \times \mathbb{P})$, then cX is also a compactification of Y. However $cX \setminus Y = \mathbb{Q} \times [(D^* \times \mathbb{R}^+) \setminus (D \times \mathbb{P})]$ does not have a countable partition of compact sets, so Y has no H-closed extension with countable remainder. This is despite the fact Y is nowhere locally compact, X is a dense Lindelöf subspace of Y, and X itself has an H-closed extension with countable remainder.

Example 49. The space $X = \mathbb{P} \times 2$ with the lexicographic order has an H-closed extension with countable remainder, namely $Y = \mathbb{R}^+ \times 2$ with the lexicographic order, since X is both a Čech g-space and Lindelöf. The space X^2 also has an H-closed extension with countable remainder, though X^2 is not Lindelöf. In particular, notice Y^2 is a zero-dimensional compactification of X^2, which has a remainder that can be expressed as the countable union of compact G_δ sets. Namely,

$$Y^2 \setminus X^2 = \bigcup_{q \in \mathbb{R}^+ \setminus \mathbb{P}} [(\{q\} \times 2) \times (\mathbb{R} \times 2)] \cup \bigcup_{q' \in \mathbb{R}^+ \setminus \mathbb{P}} [(\mathbb{R} \times 2) \times (\{q'\} \times 2)].$$

Consider the following fact.

Fact 50. *Let a Tychonoff space X have an H-closed extension hX with a countable remainder. If $\mathcal{U}$ is a family of pairwise disjoint open sets in X, then $\{U \in \mathcal{U} : U \cap R(X) \neq \varnothing\}$ is countable.*

PROOF: If U is an open set of X we denote by $o_h U$ the largest open set in hX such that $o_h U \cap X = U$. By the denseness of X in hX, $\{o_h U : U \in \mathcal{U}\}$ is a family of pairwise disjoint open sets in hX. If $U \cap R(X) \neq \varnothing$, then $o_h U \setminus X \neq \varnothing$. As $hX \setminus X$ is countable, $\{U \in \mathcal{U} : U \cap R(X) \neq \varnothing\}$ is countable. □

We define the *relative cellularity* of a space X relative to a subspace A as follows: $c(A, X) = \sup\{\mathcal{U} : \mathcal{U}$ is a family of pairwise disjoint nonempty open subsets of X such that $U \cap A \neq \varnothing$ for all $U \in \mathcal{U}\}$.

Thus by the fact above, if X is a Tychonoff space with an H-closed extension with countable remainder, then $c(R(X), X) = \omega$.

Corollary 51. *If X is Tychonoff, nowhere locally compact and has an H-closed extension with a countable remainder then $c(X) = \omega$.*

Remark 52. As the space $D \times \mathbb{P}$ described in 47 is nowhere locally compact and $c(X) = \omega_1$, it follows from the above that X has no H-closed extension with a countable remainder.

The next result extends a result of Hoshina [9] which states that if a paracompact space X has a compactification with a countable remainder then $R(X)$ is Lindelöf, and answers a question of Porter and Vermeer [13].

Proposition 53. *Let X be a paracompact Tychonoff space which has an H-closed extension hX with a countable remainder, then $R(X)$ is Lindelöf.*

Proof: Let $\mathcal{C}$ be an open cover of $R(X)$. Extend each $C \in \mathcal{C}$ to an open set C' of X such that $C' \cap R(X) = C$. Now $\{C' : C \in \mathcal{C}\} \cup \{X \setminus R(X)\}$ is an open cover of X and has an open refinement $\{\mathcal{U}_n\}_\omega$, where each $\mathcal{U}_n$ is a pairwise disjoint family. Also, $\{U \cap R(X) : U \in \mathcal{U}_n, n \in \omega, U \cap R(X) \neq \varnothing\}$ is a refinement of $\mathcal{C}$. By 50, for each $n \in \omega$, $\{U \cap R(X) : U \in \mathcal{U}_n, U \cap R(X) \neq \varnothing\}$ is also countable. Hence $\mathcal{C}$ has a countable subcover. □

Considering the importance $R(X)$ seems to play in finding extension with countable remainder for Tychonoff spaces, we seek to generalize it all Hausdorff spaces. There are a few possibilities to consider. To begin we make the following notational definitions.

Definition 54. Given a space X set $R_\sigma(X) = X \cap \mathrm{cl}_{\sigma X}(\sigma X \setminus X)$.

Notice that $x \in R_\sigma(X)$ iff for every open neighborhood U of x in σX there is some $p \in \sigma X \setminus X$ such that $U \in p$.

Definition 55. Given a space X, set $R_{EX}(X) = k[R(EX)]$.

Another characterization of $R_{EX}(X)$ is: $x \in R_{EX}(X)$ iff for each $U \in \tau(X)$ with $x \in \mathrm{cl}_X U$ there is some $p \in \sigma X \setminus X$ such that $U \in p$.

Definition 56. Given a space X let

$$R_H(X) = \{x \in X : x \text{ has no H-closed neighborhood}\}.$$

Note that if $U \in \tau(X)$, A is an H-set of X and $U \subseteq A$ then $\mathrm{cl}_X U$ is H-closed, so replacing "H-closed" with "H-set" in the previous definition does not obtain a larger set.

Proposition 57. *For a space X, $R_{EX}(X) \subseteq R_\sigma(X) = R_H(X)$.*

Proof: Suppose $x \in R_{EX}(X)$, then there is some $p \in R(EX)$ such that $k(p) = x$. Now $p \in R(EX)$ iff for each $U \in p$ there is some $q \in \sigma EX \setminus EX$ such that $U \in q$.

Since $k(p) = x$ then $\mathcal{N}_p \subseteq p$. So for every open neighborhood U of there is some $q \in \sigma X \setminus X$ such that $U \in q$.

Now suppose $x \notin R_H(X)$, then there is some $U \in \mathcal{N}_x$ such that $\mathrm{cl}_X U$ is H-closed. Now if p is an open ultrafilter on X then $\mathrm{ad}(p) = \bigcap_p \mathrm{cl}_X V = \bigcap_p \mathrm{cl}_X(U \cap V) \neq \varnothing$. So every open ultrafilter containing U is fixed and $x \notin R_\sigma(X)$. Therefore $R_\sigma(X) \subseteq R_H(X)$.

Finally suppose $x \notin X \setminus R_\sigma(X)$, then there is some $U \in \mathcal{N}_x$ for which if p is a open ultrafilter and $U \in p$, then $\mathrm{ad}(p) \neq \varnothing$. This means every open filter on $\mathrm{cl}_X U$ has nonempty adherence and hence $\mathrm{cl}_X U$ is H-closed. $\square$

The next example shows that the containment in the previous proposition can be strict.

Example 58. Let $X = [0,1] \cup ([1,2] \cap \mathbb{Q})$ with the usual topology as a subspace of $\mathbb{R}$. Let $x = 1$, then x has no H-closed neighborhood so $1 \notin R_\sigma(X)$. But $1 \in \mathrm{cl}_X(0,1)$ so $1 \in R_{EX}(X)$.

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