Reproducing kernels for Dunkl polyharmonic polynomials

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Abstract. In this paper, we compute explicitly the reproducing kernel of the space of homogeneous polynomials of degree n and Dunkl polyharmonic of degree m, i.e. $\Delta_k^m u = 0$, $m \in \mathbb{N} \setminus \{0\}$, where Δ_k is the Dunkl Laplacian and we study the convergence of the orthogonal series of Dunkl polyharmonic homogeneous polynomials.

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1. Introduction and main results

The Dunkl Laplacian Δ_k associated to a root system $\mathcal{R}$ and a weight function k is a generalisation of the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}\,.$$

In the last two decades Ch. Dunkl developed a beautiful theory based for the Dunkl Laplacian which generalizes the theory of spherical harmonics and leads to important applications in the theory of multivariate orthogonal polynomials. The aim of this paper is to generalize a recent result of H. Render in [6] about reproducing kernels for polyharmonic polynomials to the context of the Dunkl Laplacian.

We point out that this problem is studied by Ch. Dunkl and Y. Xu for Dunkl harmonic polynomials in [2]. In particular, they established the reproducing property by means of the Poisson kernel. Also, they add the Funk-Hecke formula by using the Gegenbauer polynomials.

First of all, we begin by giving some definitions and results concerning the Dunkl operators. For more details (see [1], [2] and [7]).

Let $\langle x, y\rangle$ be the euclidean scalar product on $\mathbb{R}^d$ and $|\cdot|$ the associated norm. We recall that for $v \in \mathbb{R}^d$ the reflection σ_v with respect to the hyperplane H_v orthogonal to v is given for $x \in \mathbb{R}^d$ by

$$\sigma_v(x) = x - 2\frac{\langle x, v\rangle}{|v|^2}\, v.$$

A finite set $\mathcal{R} \subseteq \mathbb{R}^d \setminus \{0\}$ is called a root system if $\mathcal{R} \cap \mathbb{R}v = \{\pm v\}$ and $\sigma_v(\mathcal{R}) = \mathcal{R}$ for all $v \in \mathcal{R}$. We assume that it is normalized by $|v|^2 = 2$ for all $v \in \mathcal{R}$.

For a given root system $\mathcal{R}$ the reflections σ_v, $v \in \mathcal{R}$, generate a finite group $W \subseteq O(d)$, the orthogonal group. We fix a positive root system $\mathcal{R}_+ = \{\alpha \in \mathcal{R} : \langle \alpha, \beta \rangle > 0\}$ for some $\beta \in \mathbb{R}^d \setminus \bigcup_{v \in \mathcal{R}} H_v$, then for each $v \in \mathcal{R}$ either $v \in \mathcal{R}_+$ or $-v \in \mathcal{R}_+$.

A W-invariant function $k : \mathcal{R}_+ \to \mathbb{C}$ is called a multiplicity function. We will assume throughout this paper that the multiplicity function k is nonnegative. For abbreviation, we introduce the index

$$\gamma = \gamma(k) := \sum_{v \in \mathcal{R}_+} k(v). \tag{1.1}$$

Moreover, let w_k denote the weight function

$$w_k(x) = \prod_{v \in \mathcal{R}_+} |\langle x, v \rangle|^{2k(v)}$$

which is W-invariant and homogeneous of degree 2γ and

$$c_k = \int_{\mathbb{S}^{d-1}} w_k(t)\, d\sigma(t),$$

where $\mathbb{S}^{d-1}$ is the unit sphere of $\mathbb{R}^d$ and $d\sigma$ is the surface measure on $\mathbb{S}^{d-1}$.

The Dunkl operators on $\mathbb{R}^d$ denoted D_j, $1 \le j \le d$, associated with the group W and the multiplicity function k are given for a function u of class $\mathcal{C}^1$ on $\mathbb{R}^d$ by

$$D_j u(x) = \partial_j u(x) + \sum_{v \in \mathcal{R}_+} k(v) v_j \frac{u(x) - u(\sigma_v(x))}{\langle x, v \rangle}.$$

The properties of these operators can be found in [1]. In particular, the operator D_j maps $\mathcal{P}_n^d$ to $\mathcal{P}_{n-1}^d$, where $\mathcal{P}_j^d$ is the space of homogeneous polynomials of degree j on $\mathbb{R}^d$. Furthermore the family $(D_j)_{1 \le j \le d}$ is commutative.

The Dunkl Laplacian Δ_k on $\mathbb{R}^d$ is given for $u \in \mathcal{C}^2(\mathbb{R}^d)$ by

$$\Delta_k u = \sum_{j=1}^{d} D_j^2 u.$$

A polynomial P is called Dunkl harmonic if $\Delta_k P(x) = 0$ for all $x \in \mathbb{R}^d$, and more general, Dunkl polyharmonic of order m if $\Delta_k^m P(x) = 0$ for all $x \in \mathbb{R}^d$. In [6] H. Render determined the reproducing kernel for the space of all polyharmonic polynomials of order m and degree n. In this paper, we show that a similar result holds in the context of Dunkl polyharmonic polynomials.

Let $\mathcal{P}^d$ be the space of polynomials on $\mathbb{R}^d$ equipped with the apolar inner scalar product

$$[P, Q]_k = [P(D)\overline{Q}](0),$$

where $D = (D_1, D_2, \ldots, D_d)$ and $P(D) = P(D_1, D_2, \ldots, D_d)$.

We denote by $\mathcal{PH}^k_{n,m}$ the space of homogeneous polynomials of degree n on $\mathbb{R}^d$ and Dunkl polyharmonic of order $m \in \mathbb{N} \setminus \{0\}$, (i.e. $\Delta_k^m P = 0$, $P \in \mathcal{P}_n^d$).

Suppose that $\{Q^k_{n,m,j}\}_{1\le j\le a_d(n,m)}$, $a_d(n,m) := \dim \mathcal{PH}^k_{n,m}$, is an orthonormal basis of $\mathcal{PH}^k_{n,m}$ with respect to the inner product $[\cdot,\cdot]_k$ and define the function $Z^k_{n,m}$ of $\mathcal{PH}^k_{n,m}$ by

$$Z^k_{n,m}(x,y) := \sum_{j=1}^{a_d(n,m)} Q^k_{n,m,j}(x)\overline{Q^k_{n,m,j}(y)}.$$

It is a well known result of Hilbert space theory that $Z^k_{n,m}$ is independent of the choice of the orthonormal basis $Q^k_{n,m,j}$ for $j = 1, \dots, a_d(n,m)$ and that

$$[P(x), Z^k_{n,m}(x,y)]_k = P(y)$$

for all $y \in \mathbb{R}^d$ and $P \in \mathcal{PH}^k_{n,m}$. For this reason, the function $Z^k_{n,m}$ is called the reproducing kernel of the space $\mathcal{PH}^k_{n,m}$ with respect to the inner product $[\cdot,\cdot]_k$.

In this paper, we show that the reproducing kernel $Z^k_{n,m}$ can be described explicitly as

(1.2)
$$Z^k_{n,m}(x,y) = \sum_{s=0}^{\min([\frac{n}{2}],m-1)} \frac{|x|^{2s}|y|^{2s} Z^k_{n-2s}(x,y)}{2^s s!(d+2\gamma)(d+2\gamma+2)\dots(d+2(\gamma+n-s-1))},$$

where γ is defined in formula (1.1) and $[\frac{n}{2}]$ is the integral part of $\frac{n}{2}$.

Note that the reproducing kernel Z^k_n is defined for the inner product

$$(P,Q)_k = \frac{1}{c_k}\int_{\mathbb{S}^{d-1}} P(x)\overline{Q(x)} w_k(x)\, d\sigma(x)$$

on the space $\mathcal{PH}^k_{n,1}$ while $Z^k_{n,1}$ is the reproducing kernel with respect to $[\cdot,\cdot]_k$. The dimension of $\mathcal{PH}^k_{j,1}$ is given (see [2]) by

$$a_d(j,1) = (2j+d-2)\frac{(j+d-3)!}{j!(d-2)!}.$$

Using (1.2) and the expression of the reproducing kernel of $\mathcal{PH}^k_{j,1}$, we prove that there exists a positive constant C depending only on d and γ such that for all $x, y \in \mathbb{S}^{d-1}$,

$$|Z^k_{n,m}(x,y)| \le C\, \frac{n^{m+d+2\gamma}}{2^n n!}. \tag{1.3}$$

Formula (1.3) allows to prove that the following orthogonal series

$$\sum_{n=0}^{+\infty} \sum_{j=1}^{a_d(n,m)} \mu_{n,j} Q^k_{n,m,j}$$

converges absolutely and uniformly on compact subsets of the open ball $B(\rho)$ centered at the origin and with radius

$$\rho = \frac{1}{\sqrt{2}} \limsup_{n\to+\infty} \left(\frac{\|\mu_n\|_2}{\sqrt{n!}} \right)^{\frac{-1}{n}},$$

where

$$\|\mu_n\|_2 = \left(\sum_{j=1}^{a_d(n,m)} |\mu_{n,j}|^2 \right)^{\frac{1}{2}}.$$

The paper is organized as follows. In the second section, we give a relation between the two scalar products $[\cdot,\cdot]_k$ and $(\cdot,\cdot)_k$ on $\mathcal{PH}^k_{n,1}$ and we describe explicitly the reproducing kernel $Z^k_{n,m}$ by means of Z^k_n. The third section contains a criterion for the convergence of the series $\sum_{n=0}^{+\infty} \sum_{j=1}^{a_d(n,m)} \mu_{n,j} Q^k_{n,m,j}$.

2. The reproducing kernel for Dunkl polyharmonic polynomials

Since Δ_k^m is a homogeneous operator of order $2m$, using arguments analogous as in the proof of the case $m = 1$, (see [2, Theorem 5.1.7, p. 178]) we find that

$$a_d(n,m) = \binom{d+n-1}{n} - \binom{d+n-1-2m}{n-2m}. \tag{2.1}$$

It is shown in [7] that the scalar product $[\cdot,\cdot]_k$ satisfies the following properties

$$[x_i P, Q]_k = [P, D_i Q]_k, \ i = 1, \dots, d,$$

for all $P, Q \in \mathcal{P}^d$. Using this relation, we see that, for all $P, Q, R \in \mathcal{P}^d$, we have

$$[P^\star(D)Q, R]_k = [Q, PR]_k, \tag{2.2}$$

where $P^\star(x)$ is the polynomial obtained by conjugation the coefficients of the polynomial P and $P^\star(D)$ is the operator associated to $P^\star(x)$. In addition, we note that for all homogeneous polynomials P and Q of degree n,

$$[P, Q]_k = P(D)\overline{Q}.$$

To establish a relation between $[\cdot,\cdot]_k$ and $(\cdot,\cdot)_k$ on $\mathcal{PH}^k_{n,1}$, we give the following technical lemmas. We mention that Lemma 2.1, in the classical case is due to Kuran, see [3, p. 19].

Lemma 2.1. *Let n be a positive integer. If h is Dunkl harmonic function in $\mathbb{R}^d$ (i.e. $\Delta_k h = 0$) and $P \in \mathcal{P}_n^d$, then*

$$\Delta_k^n(Ph) = 2^n n! P(D)h. \tag{2.3}$$

Proof: It is proved in [2, Lemma 5.1.10, p. 179], that for all $i \in \{1, \dots, d\}$ and $f \in \mathcal{C}^2(\mathbb{R}^d)$,

$$\Delta_k(x_i f) = x_i \Delta_k(f) + 2D_i(f). \tag{2.4}$$

Using (2.4) and the fact that Δ_k and D_i commute, we prove by induction that, if $\alpha \in \mathbb{N}^d$ and $i \in \{1, 2, \dots, d\}$ be fixed, then for all $p \in \mathbb{N} \setminus \{0\}$

$$\Delta_k^p(x^{\alpha+e_i} f) = x_i \Delta_k^p(x^\alpha f) + 2pD_i \Delta_k^{p-1}(x^\alpha f) \tag{2.5}$$

where e_i is the i-th standard basis vector.

We prove now (2.3) by induction for $P = x^\alpha$. The result is true for $|\alpha| = 1$. Suppose

$$\Delta_k^n(x^\alpha h) = 2^n n! D^\alpha(h),$$

for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| = n$, then by (2.5) and (2.4), we have for all $i = 1, \dots, d$

$$\begin{aligned}
\Delta_k^{n+1}(x^{\alpha+e_i} h) &= x_i \Delta_k^{n+1}(x^\alpha h) + 2(n+1) D_i \Delta_k^n(x^\alpha h) \\
&= x_i \Delta_k(\Delta_k^n(x^\alpha h)) + 2(n+1) D_i(2^n n! D^\alpha h) \\
&= x_i \Delta_k(2^n n! D^\alpha(h)) + 2^{n+1}(n+1)! D^{\alpha+e_i}(h) \\
&= 2^n n! x_i D^\alpha(\Delta_k h) + 2^{n+1}(n+1)! D^{\alpha+e_i}(h) \\
&= 2^{n+1}(n+1)! D^{\alpha+e_i}(h).
\end{aligned}$$

Hence, the result is true for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| = n+1$ and the induction is terminated. Finally, we conclude by linearity. □

Lemma 2.2. *Let P be a homogeneous polynomial of degree n. Then*

$$\int_{\mathbb{S}^{d-1}} \Delta_k P(y) w_k(y)\, d\sigma(y) = n(n+d+2\gamma-2) \int_{\mathbb{S}^{d-1}} P(y) w_k(y)\, d\sigma(y). \tag{2.6}$$

If further n is even, i.e. $n = 2s$, then

$$\Delta_k^s P = \frac{2^s s!}{c_k}\, u_{d,\gamma}(s) \int_{\mathbb{S}^{d-1}} P(y) w_k(y)\, d\sigma(y), \tag{2.7}$$

where

$$u_{d,\gamma}(s) = (d+2\gamma)(d+2\gamma+2)\dots(d+2\gamma+2s-2).$$

Proof: By Euler's identity,

$$nP=\sum_{i=1}^{d}x_i\frac{\partial P}{\partial x_i}=r\frac{\partial P}{\partial r},\ r=|x|.$$

Then, by Green's formula (see [4]), the homogeneity of w_k of degree 2γ and the homogeneity of $\Delta_k P$ of degree $n-2$, we have

$$\begin{aligned}n\int_{\mathbb{S}^{d-1}}P(y)w_k(y)\,d\sigma(y)&=\int_{\mathbb{S}^{d-1}}\frac{\partial P}{\partial r}(y)w_k(y)\,d\sigma(y)\\&=\int_B\Delta_kP(x)w_k(x)\,dx\\&=\int_0^1\int_{\mathbb{S}^{d-1}}\Delta_kP(rt)w_k(rt)r^{d-1}\,dr\,d\sigma(t)\\&=\int_0^1\int_{\mathbb{S}^{d-1}}\Delta_kP(t)w_k(t)r^{2\gamma+n+d-3}\,dr\,d\sigma(t)\\&=\frac{1}{(2\gamma+n+d-2)}\int_{\mathbb{S}^{d-1}}\Delta_kP(t)w_k(t)\,d\sigma(t),\end{aligned}$$

which gives (2.6). To prove (2.7), it is enough to note that Δ_k^sP is a constant (and hence) it is equal to its mean value on $\mathbb{S}^{d-1}$ (see [4]). We conclude by applying (2.6) s times. □

Theorem 2.3. *Let P,Q be homogeneous polynomials of degree n and Dunkl harmonic. Then*

$$[P,Q]_k=u_{d,\gamma}(n)(P,Q)_k. \tag{2.8}$$

Proof: Using Lemma 2.1 and 2.2 (the latter applied to $P\overline{Q}$ instead of P), we obtain

$$n!2^n[P,Q]_k=\Delta_k^n(P\overline{Q})=n!2^nu_{d,\gamma}(n)(P,Q)_k,$$

which gives (2.8). □

Lemma 2.4. *Let m be a positive integer and $P\in\mathcal{PH}_{n,1}^k$. Then for any integer s such that $m\le 2s$, we have*

$$\Delta_k^m(|x|^{2s}P)=2^m[s(s-1)\dots(s-(m-1))]v_{d,\gamma}(m,n,s)|x|^{2s-2m}P, \tag{2.9}$$

where

$$v_{d,\gamma}(m,n,s)=[(d+2(s+n+\gamma-1))\dots(d+2(s+n+\gamma-m))].$$

In particular,

$$\Delta_k^s(|x|^{2s}P)=2^ss!u_{d,\gamma+n}(s)P. \tag{2.10}$$

PROOF: It is a special case of Lemma 5.1.9, p. 178, in [2], which gives

$$\Delta_k(|x|^{2s}P) = 2s(2\gamma + d + 2s + 2n - 2)|x|^{2s-2}P.$$

A simple induction argument gives (2.9). It is clear that if $m = s$, then we obtain (2.10). □

In the following, we need some facts from the theory of spherical harmonics associated with the Dunkl Laplacian.

Let $\{Y^k_{n,j}\}_{1\leq j\leq a_d(n,1)}$ be an orthonormal basis of $\mathcal{PH}^k_{n,1}$ with respect to the scalar product $(\cdot,\cdot)_k$. Then

$$Z^k_n(x,y) = \sum_{j=1}^{a_d(n,1)} Y^k_{n,j}(x)\overline{Y^k_{n,j}(y)} \tag{2.11}$$

is the reproducing kernel of $\mathcal{PH}^k_{n,1}$ with respect to the scalar product $(\cdot,\cdot)_k$ and the function $x \mapsto Z^k_n(x,y)$ is called the zonal k-harmonic of degree n and with pole y.

Using (2.11), it is easy to show that

$$\frac{1}{c_k}\int_{\mathbb{S}^{d-1}} Z^k_n(x,x)w_k(x)\,d\sigma(x) = a_d(n,1).$$

Theorem 2.5. *Let $Y^k_{n,j}$, $n \in \mathbb{N}$, $j = 1,\ldots,a_d(n,1)$, be an orthonormal basis of $\mathcal{PH}^k_{n,1}$ with respect to the inner product $(\cdot,\cdot)_k$. Then the polynomials $|x|^{2s}Y^k_{n,j}$, $s \in \mathbb{N}$, are orthogonal with respect to the inner product $[\cdot,\cdot]_k$ and we have*

$$\||x|^{2s}Y^k_{n,j}\|^2_k = 2^s s!\,u_{d,\gamma}(s+n). \tag{2.12}$$

Here $\|\cdot\|_k$ is the norm associated to the inner product $[\cdot,\cdot]_k$.

PROOF: The proof is analogous to the one in [6, Theorem 2.2, p. 139] using (2.10) and (2.2). □

Proposition 2.6. *Let $m \in \mathbb{N}\setminus\{0\}$. The system $|x|^{2s}Y^k_{n-2s,j}$ for $s = 0,1,\ldots,\min\{[\frac{n}{2}],m-1\}$ and $j = 1,\ldots,a_d(n-2s,1)$, is an orthogonal basis of $\mathcal{PH}^k_{n,m}$.*

PROOF: By the Almansi theorem for Dunkl operators, (see [5]) and similarly to the proof of Proposition 2.3, p. 140 in [6], we obtain the desired result. □

Fix a point $y \in \mathbb{R}^d$, and consider the linear map

$$\Lambda : \mathcal{PH}^k_{n,m} \to \mathbb{C}$$

defined by

$$\Lambda(P) = P(y).$$

Because $\mathcal{PH}^k_{n,m}$ is a finite-dimensional inner-product space, there exists a unique function $Z^k_{n,m}(\cdot,y)\in\mathcal{PH}^k_{n,m}$ such that

$$P(y)=[P,Z^k_{n,m}(\cdot,y)]_k$$

for all $P\in\mathcal{PH}^k_{n,m}$.

$Z^k_{n,m}$ is called the reproducing kernel of $\mathcal{PH}^k_{n,m}$ endowed with the inner product $[\cdot,\cdot]_k$.

By standard Hilbert space theory, we have the following well known result.

Proposition 2.7. (a) *If $\{Q_{n,m,j}\}_{1\le j\le a_d(n,m)}$ is an orthonormal basis of $\mathcal{PH}^k_{n,m}$ with respect to the inner product $[\cdot,\cdot]_k$, then for all $x,y\in\mathbb{R}^d$,*

$$Z^k_{n,m}(x,y)=\sum_{j=1}^{a_d(n,m)}Q_{n,m,j}(x)\overline{Q_{n,m,j}}(y).$$

(b) *$Z^k_{n,m}$ is real valued.*

(c) *$Z^k_{n,m}(x,y)=Z^k_{n,m}(y,x)$, for all $x,y\in\mathbb{R}^d$.*

Theorem 2.8. *The reproducing kernel $Z^k_{n,m}$ of $\mathcal{PH}^k_{n,m}$ endowed with the scalar product $[\cdot,\cdot]_k$ is given by*

$$Z^k_{n,m}(x,y)=\sum_{s=0}^{\min\{[\frac{n}{2}],m-1\}}\frac{1}{2^s s!u_{d,\gamma}(n-s)}|x|^{2s}|y|^{2s}Z^k_{n-2s}(x,y). \tag{2.13}$$

Proof: Since the system

$$\frac{1}{\sqrt{2^s s!u_{d,\gamma}(n-s)}}|x|^{2s}Y^k_{n-2s,j}$$

for $s=0,1,\ldots,\min\{[\frac{n}{2}],m-1\}$ and $j=1,\ldots,a_d(n-2s,1)$ is an orthonormal basis of $\mathcal{PH}^k_{n,m}$ with respect to the scalar product $[\cdot,\cdot]_k$, by definition of the reproducing kernel of $\mathcal{PH}^k_{n,m}$,

$$\begin{aligned}
Z^k_{n,m}(x,y)&=\sum_{s=0}^{\min\{[\frac{n}{2}],m-1\}}\sum_{j=1}^{a_d(n-2s,1)}\frac{1}{2^s s!u_{d,\gamma}(n-s)}|x|^{2s}Y^k_{n-2s,j}(x)|y|^{2s}\overline{Y^k_{n-2s,j}(y)}\\
&=\sum_{s=0}^{\min\{[\frac{n}{2}],m-1\}}\frac{1}{2^s s!u_{d,\gamma}(n-s)}|x|^{2s}|y|^{2s}\sum_{j=1}^{a^k_d(n-2s,1)}Y^k_{n-2s,j}(x)\overline{Y^k_{n-2s,j}(y)}\\
&=\sum_{s=0}^{\min\{[\frac{n}{2}],m-1\}}\frac{1}{2^s s!u_{d,\gamma}(n-s)}|x|^{2s}|y|^{2s}Z^k_{n-2s}(x,y).
\end{aligned}$$

$\square$

Proposition 2.9. *Let $q_n \in \mathcal{PH}^k_{n,m}$. Then for all $x \in \mathbb{R}^d$,*

$$\frac{u_{d,\gamma}(n)}{c_k} \int_{\mathbb{S}^{d-1}} q_n(y) Z^k_{j,m}(x,y) w_k(y)\, d\sigma(y) = \delta_{nj} q_n(x). \tag{2.14}$$

(δ_{ij} being the Kronecker symbol).

Proof: Let $x \in \mathbb{S}^{d-1}$. Since $Z^k_{j,m}(x,\cdot)$ is a homogeneous polynomial of degree j, by [2, Theorem 5.1.6, p. 177], if $n \neq j$, the left hand side of the relation of the lemma vanishes. If $n = j$, then using (2.8), we obtain

$$\begin{aligned}\frac{1}{c_k} \int_{\mathbb{S}^{d-1}} q_n(y) Z^k_{j,m}(x,y) w_k(y)\, d\sigma(y) &= (q_n, Z^k_{n,m}(x,\cdot))_k \\ &= \frac{1}{u_{d,\gamma}(n)} [q_n, Z^k_{n,m}(x,\cdot)]_k \\ &= \frac{1}{u_{d,\gamma}(n)} q_n(x).\end{aligned}$$

□

We give now some results which are used to study the convergence of orthogonal series. Before stating these results, we recall the existence and the unicity of the so-called Dunkl intertwining operator denoted V_k (see [1]) which relates the Dunkl operators D_j, $1 \leq j \leq d$, with the usual partial derivatives ∂_j and is a linear isomorphism from the space $\mathcal{P}^d_n$ onto itself satisfying

$$V_k(1) = 1,\ D_j V_k = V_k \partial_j,\ 1 \leq j \leq d.$$

Moreover, it is proved in [9] that V_k can be extended to a topological isomorphism of the space $\mathcal{C}^\infty(\mathbb{R}^d)$ (the space of $\mathcal{C}^\infty$-functions on $\mathbb{R}^d$) onto itself and admits for all $x \in \mathbb{R}^d$ the following integral representation

$$V_k(f)(x) = \int_{\mathbb{R}^d} f(y)\, d\mu_x(y),\ f \in \mathcal{C}(\mathbb{R}^d), \tag{2.15}$$

where $d\mu_x$ is a probability measure on $\mathbb{R}^d$, with support in the closed ball $\overline{B(\|x\|)}$ centered at the origin and with radius $\|x\|$.

It can be easily seen that Δ_k satisfies

$$V_k \Delta^n = \Delta^n_k V_k,\ n \in \mathbb{N}.$$

Lemma 2.10. *Let $Z^k_n(x,y)$ be the reproducing kernel for the space $\mathcal{PH}^k_{n,1}$ with respect to $(\cdot,\cdot)_k$. Then the following estimate holds:*
For $x, y \in \mathbb{S}^{d-1}$, we have

$$|Z^k_n(x,y)| \leq \frac{(n+\gamma+\frac{d}{2}-1)(d+2\gamma-2)_n}{(\gamma+\frac{d}{2}-1)n!}\,. \tag{2.16}$$

Proof: For $x, y \in \mathbb{S}^{d-1}$, according to Corollary 5.3.2 in [2], the kernel $Z_n^k(x,y)$ can be written as

$$Z_n^k(x,y) = \frac{(n+\gamma+\frac{d}{2}-1)(d+2\gamma-2)_n}{(\gamma+\frac{d}{2}-1)n!}\; V_k \tilde{C}_n^{\gamma+\frac{d}{2}-1}(\langle x,\cdot\rangle)(y), \tag{2.17}$$

where $\tilde{C}_n^{\gamma+\frac{d}{2}-1}$ is the normalized Gegenbauer polynomial such that $\tilde{C}_n^{\gamma+\frac{d}{2}-1}(1) = 1$ and $(a)_n$ is the Pochammer symbol.

From (2.15) and the fact that $\tilde{C}_n^{\gamma+\frac{d}{2}-1}(t) \le 1$ for $|t| \le 1$, we obtain (2.16). □

Using (2.13) and (2.16), we obtain:

Lemma 2.11. *For $x, y \in \mathbb{S}^{d-1}$, we have*

$$|Z_{n,m}^k(x,y)| \le \lambda_{d,\gamma}(n,m), \tag{2.18}$$

where

$$\lambda_{d,\gamma}(n,m) = \sum_{s=0}^{\min\{[\frac{n}{2}],m-1\}} \frac{(n-2s+\gamma+\frac{d}{2}-1)(d+2\gamma-2)_{n-2s}}{2^s s!(\gamma+\frac{d}{2}-1)(n-2s)!u_{d,\gamma}(n-s)}\,.$$

Proposition 2.12. *For $x \in \mathbb{S}^{d-1}$, we have*

$$\|Z_{n,m}^k(x,\cdot)\|_k^2 \le \lambda_{d,\gamma}(n,m)$$

and

$$\|Z_{n,m}^k(x,\cdot)\|_{2,k}^2 \le \frac{\lambda_{d,\gamma}(n,m)}{u_{d,\gamma}(n)}\,.$$

Here $\|\cdot\|_{2,k}$ is the norm associated with the inner product $(\cdot,\cdot)_k$.

Proof: We have

$$\|Z_{n,m}^k(x,\cdot)\|_k^2 = [Z_{n,m}^k(x,\cdot), Z_{n,m}^k(x,\cdot)]_k = Z_{n,m}^k(x,x),$$

and

$$\|Z_{n,m}^k(x,\cdot)\|_{2,k}^2 = \frac{1}{u_{d,\gamma}(n)}\; \|Z_{n,m}^k(x,\cdot)\|_k^2.$$

Hence, (2.18) completes the proof. □

Theorem 2.13. *Let $q_n \in \mathcal{PH}_{n,m}^k(\mathbb{R}^d)$. For all $x \in \mathbb{S}^{d-1}$, we have*

$$|q_n(x)| \le \frac{u_{d,\gamma}(n)\lambda_{d,\gamma}(n,m)}{c_k} \int_{\mathbb{S}^{d-1}} |q_n(y)| w_k(y)\, d\sigma(y). \tag{2.19}$$

PROOF: By (2.14) and (2.18), we have

$$\begin{aligned} |q_n(x)| &= \left| \frac{u_{d,\gamma}(n)}{c_k} \int_{\mathbb{S}^{d-1}} q_n(y) Z^k_{n,m}(x,y) w_k(y)\, d\sigma(y) \right| \\ &\le \frac{u_{d,\gamma}(n)\lambda_{d,\gamma}(n,m)}{c_k} \int_{\mathbb{S}^{d-1}} |q_n(y)| w_k(y)\, d\sigma(y). \end{aligned}$$

□

Corollary 2.14. *Let* $q_n \in \mathcal{PH}^k_{n,m}$. *Then,*

$$\|q_n\|_{2,k} \le \|q_n\|_\infty \le u_{d,\gamma}(n)\lambda_{d,\gamma}(n,m)\|q_n\|_{2,k}, \tag{2.20}$$

where $\|f\|_\infty = \sup_{z\in\mathbb{S}^{d-1}} |f(z)|$.

PROOF: The first inequality is trivial. Using the Cauchy-Schwarz inequality and the fact that $\frac{1}{c_k} w_k d\sigma$ is a probability measure, we obtain

$$\frac{1}{c_k}\int_{\mathbb{S}^{d-1}} |q_n(y)| w_k(y) d\sigma(y) \le \left(\frac{1}{c_k}\int_{\mathbb{S}^{d-1}} |q_n(y)|^2 w_k(y)\, d\sigma(y) \right)^{\frac{1}{2}} = \|q_n\|_{2,k}.$$

Hence the second inequality is a consequence of (2.19). □

Lemma 2.15. *There exists a positive constant* C *which depends only on* d *and* γ *such that for all* $x, y \in \mathbb{S}^{d-1}$, *we have*

$$|Z^k_{n,m}(x,y)| \le C\, \frac{n^{d+2\gamma+m}}{2^n n!}\,. \tag{2.21}$$

PROOF: According to Lemma 2.11 the following estimate holds for $n \ge m$:

$$|Z^k_{n,m}(x,y)| \le \sum_{s=0}^{m-1} \frac{(n-2s+\gamma+\frac{d}{2}-1)(d+2\gamma-2)_{n-2s}}{2^s s!(\gamma+\frac{d}{2}-1)(n-2s)! u_{d,\gamma}(n-s)}\,.$$

Since $u_{d,\gamma}(n) = 2^n(\frac{d}{2}+\gamma)_n \ge 2^n n!$ one obtains

$$u_{d,\gamma}(n-s) \ge 2^{n-s}(n-s)! \ge 2^{n-s}\tfrac{n!}{n^s} \tag{2.22}$$

and therefore

$$|Z^k_{n,m}(x,y)| \le \frac{1}{2^n n!} \sum_{s=0}^{m-1} \frac{n^s(n-2s+\gamma+\frac{d}{2}-1)(d+2\gamma-2)_{n-2s}}{(\gamma+\frac{d}{2}-1)(n-2s)! s!}\,.$$

On the other hand, using Stirling formula:

$$\Gamma(t+1) \sim \sqrt{2\pi t}\; t^t e^{-t},\ t \to +\infty,$$

we find that there exists a positive constant C_1 which depends only on d and γ such that

$$\frac{(n+\gamma+\frac{d}{2}-1)(d+2\gamma-2)_n}{(\gamma+\frac{d}{2}-1)n!} \leq C_1 \ \ n^{d+2\gamma} \tag{2.23}$$

and so,

$$|Z^k_{n,m}(x,y)| \leq C_1 \ \frac{n^{d+2\gamma}}{2^n n!} \ \sum_{s=0}^{m-1} \frac{n^s}{s!} \leq \left[C_1 \sum_{s=0}^{m-1} \frac{1}{s!}\right] \ \frac{n^{m+d+2\gamma}}{2^n n!} \leq eC_1 \ \frac{n^{m+d+2\gamma}}{2^n n!}$$

which completes the proof. □

3. Convergence of orthogonal series

Using formula (2.21), we obtain the radius of the convergence of series of Dunkl polyharmonic homogeneous polynomials.

Theorem 3.1. *Suppose that $\{Q^k_{n,m,j}\}_{1\leq_n j\leq a_d(n,m)}$ is an orthonormal basis of $\mathcal{PH}^k_{n,m}$ for each $n \in \mathbb{N}$ and let $\mu_{n,j}$, $j = 1,\ldots,a_d(n,m)$ be complex numbers. Then the series*

$$\sum_{n=0}^{+\infty} \sum_{j=1}^{a_d(n,m)} \mu_{n,j} Q^k_{n,m,j}$$

converges absolutely and uniformly on compact subsets of the open ball $B(\rho)$ centered at the origin and with radius ρ such that

$$\frac{1}{\rho} = \frac{1}{\sqrt{2}} \limsup_{n\to+\infty} \left(\frac{\|\mu_n\|_2}{\sqrt{n!}}\right)^{\frac{1}{n}},$$

where

$$\|\mu_n\|_2 = \left(\sum_{j=1}^{a_d(n,m)} |\mu_{n,j}|^2\right)^{\frac{1}{2}}.$$

Proof: By Proposition 2.7, we have

$$Z^k_{n,m}(x,x) = \sum_{j=1}^{a_d(n,m)} |Q_{n,m,j}(x)|^2, \quad \text{for all} \quad x \in \mathbb{S}^{d-1}.$$

Since $Q^k_{n,m,j}$, $j=1,\dots,a_d(n,m)$, is homogeneous of degree n,

$$Q^k_{n,m,j}(x) = r^n Q^k_{n,m,j}\Big(\frac{x}{r}\Big),\ x\in\mathbb{R}^d,\ r=|x|.$$

Thus,

$$\begin{aligned}\Big|\sum_{n=0}^{+\infty}\sum_{j=1}^{a_d(n,m)}\mu_{n,j}Q^k_{n,m,j}(x)\Big| &= \Big|\sum_{n=0}^{+\infty} r^n\sum_{j=1}^{a_d(n,m)}\mu_{n,j}Q^k_{n,m,j}\Big(\frac{x}{r}\Big)\Big| \\ &\le \sum_{n=0}^{+\infty} r^n\Big|\sum_{j=1}^{a_d(n,m)}\mu_{n,j}Q^k_{n,m,j}\Big(\frac{x}{r}\Big)\Big|.\end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\Big|\sum_{j=1}^{a_d(n,m)}\mu_{n,j}Q^k_{n,m,j}\Big(\frac{x}{r}\Big)\Big| \le \|\mu_n\|_2\Big[Z^k_{n,m}\Big(\frac{x}{r},\frac{x}{r}\Big)\Big]^{\frac12}.$$

Applying the inequality (2.21), we find that there exists a positive constant C which depends only on d and γ such that

$$\Big[Z^k_{n,m}\Big(\frac{x}{r},\frac{x}{r}\Big)\Big]^{\frac12} \le C\,\frac{n^{\frac{d+m}{2}+\gamma}}{\sqrt{2^n n!}}$$

and so

$$\Big|\sum_{n=0}^{+\infty}\sum_{j=1}^{a_d(n,m)}\mu_{n,j}Q^k_{n,m,j}(x)\Big| \le C\sum_{n=0}^{+\infty}\|\mu_n\|_2\frac{n^{\frac{m+d}{2}+\gamma}}{\sqrt{2^n n!}}\, r^n$$

which finishes the proof. □

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