Mesocompactness and selection theory

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Abstract. A topological space X is called *mesocompact* (*sequentially mesocompact*) if for every open cover $\mathcal{U}$ of X, there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\{V \in \mathcal{V} : V \cap K \neq \emptyset\}$ is finite for every compact set (converging sequence including its limit point) K in X. In this paper, we give some characterizations of mesocompact (sequentially mesocompact) spaces using selection theory.

Keywords: selections, l.s.c. set-valued maps, mesocompact, sequentially mesocompact, persevering compact set-valued maps

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1. Introduction

Let X and Y be topological spaces, and 2^Y stand for the family of non-empty subsets of Y. Let

$F(Y) = \{S \in 2^Y : S \text{ is closed}\}$;
$C(Y) = \{S \in F(Y) : S \text{ is compact}\}$;
$K(Y) = \{S \in F(Y) : S \text{ is finite}\}$, and
$S(Y) = \{S \in F(Y) : S \text{ is separable}\}$.

A set-valued map $\Phi : X \to 2^Y$ is *lower semi-continuous* (*upper semi-continuous*) or *l.s.c.* (*u.s.c.*), if the set

$$\Phi^{-1}(U) = \{x \in X : \Phi(x) \cap U \neq \emptyset\}$$

is open (closed) in X for every open (closed) subset U of X. A family $\mathcal{F}$ of subsets of a space X is *compact finite* (*sequential finite*), if $\{F \in \mathcal{F} : F \cap K \neq \emptyset\}$ is finite for every compact subset (converging sequence including its limit point) K of X. A topological space X is called *mesocompact* (*sequentially mesocompact*) [1], if every open cover of X has a compact finite (sequential finite) open refinement. There is a series of results which characterize separation and covering properties (like paracompactness, metacompactness, collectionwise normality and so on) by means of the existence of selections for l.s.c. maps. The following are two ones among those results.

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Theorem A ([4]). *For a Hausdorff space X, the following statements are equivalent.*

(1) *X is paracompact.*

(2) *For every complete metric space Y and l.s.c. set-valued map $\Phi : X \to F(Y)$, there exists an l.s.c. map $\varphi : X \to C(Y)$ and a u.s.c. map $\phi : X \to C(Y)$ such that $\varphi(x) \subset \phi(x) \subset \Phi(x)$ for every $x \in X$. (φ, ϕ) is called a Michael's pair of Φ.*

Theorem B ([2]). *For a regular space X, the following statements are equivalent.*

(1) *X is metacompact.*

(2) *For every complete metric space Y and l.s.c. set-valued map $\Phi : X \to F(Y)$, there exists an l.s.c. map $\varphi : X \to C(Y)$ such that $\varphi(x) \subset \Phi(x)$ for every $x \in X$.*

The first author of the present paper obtained the following result.

Theorem C ([8]). *For a regular space X, the following conditions are equivalent.*

(1) *X is metalindelöf.*

(2) *For every complete metric space Y and l.s.c. set-valued map $\Phi : X \to F(Y)$, there exists an l.s.c. map $\varphi : X \to S(Y)$ such that $\varphi(x) \subset \Phi(x)$ for every $x \in X$.*

In the present paper we shall give some characterizations of mesocompactness and sequential mesocompactness which are similar to the theorems above. For convenience, we introduce a new concept. A set-valued map $\Phi : X \to 2^Y$ is said to be *persevering compact* (*weakly persevering compact*), if $\Phi(K) = \bigcup\{\Phi(x) : x \in K\}$ is compact for every compact subset (converging sequence including its limit point) K of X.

Throughout this paper, all spaces are assumed to be Hausdorff. Let $\mathbb{N}$ be the set of all natural numbers. All undefined topological concepts are taken in the sense given Engelking [3]. In particular, $\overline{(\cdot)}$ is the closure operator.

2. Main results

We at first give a lemma.

Lemma 1. *Let X be a regular and mesocompact (sequentially mesocompact) space and (Y, ρ) be a metric space. Then for every l.s.c. map $\Phi : X \to F(Y)$, there exist a sequence $\{\mathcal{V}_n = \{V^n_\alpha : \alpha \in A_n\}\}$ of locally finite open covers of Y, two sequences $\{\mathcal{W}_n = \{W^n_\beta : \beta \in B_n\}\}$ and $\{\mathcal{U}_n = \{U^n_\alpha : \alpha \in A_n\}\}$ of compact (sequential) finite open covers of X, and two sequences $\{\pi_n : A_{n+1} \to A_n\}$ and $\{\sigma_n : B_n \to A_n\}$ of maps satisfying the following conditions:*

(a) $U^{n-1}_\alpha = \bigcup_{\beta \in \pi^{-1}_{n-1}(\alpha)} U^n_\beta$, $V^{n-1}_\alpha = \bigcup_{\beta \in \pi^{-1}_{n-1}(\alpha)} V^n_\beta$;

(b) $U^n_\alpha \subset \Phi^{-1}(V^n_\alpha)$;

(c) $\overline{W^n_\beta} \subset \Phi^{-1}(V^n_{\sigma_n(\beta)})$, $U^n_\alpha = \bigcup_{\beta \in \sigma^{-1}_n(\alpha)} W^n_\beta$;

(d) $\operatorname{diam} V_\alpha^n < 2^{-n}$, *where* $\operatorname{diam} V_\alpha^n$ *is the diametric of* V_α^n,

for every $n \in \mathbb{N}$ *and* $\alpha \in A_n$, $\beta \in B_n$.

Proof: For each $n \in \mathbb{N}$, let $\mathcal{V}_n' = \{G_\beta^n : \beta \in \Lambda_n\}$ be a locally finite open cover of Y such that $\operatorname{diam} G_\alpha^n < 2^{-n}$ for each $\alpha \in \Lambda_n$. We inductively construct the sequences above.

Since $\{\Phi^{-1}(G_\beta^1) : \beta \in \Lambda_1\}$ is an open cover of X, by regularity and mesocompactness of X, there exists a compact (sequential) finite open cover $\mathcal{W}_1 = \{W_b^1 : b \in B_1\}$ as its closure refinement (*i.e.* $\{\overline{W_b^1} : b \in B_1\}$ refines $\{\Phi^{-1}(G_\beta^1) : \beta \in \Lambda_1\}$). Define $\sigma_1 : B_1 \to \Lambda_1$ to be a refinement map, that is, $\overline{W_b^1} \subset \Phi^{-1}(G_{\sigma_1(b)}^1)$ for every $b \in B_1$. Let $U_\alpha^1 = \bigcup_{b \in \sigma_1^{-1}(\alpha)} W_b^1, A_1 = \Lambda_1, V_\beta^1 = G_\beta^1$. Then $\mathcal{U}_1 = \{U_\alpha^1 : \alpha \in A_1\}$, $\mathcal{V}_1 = \{V_\alpha^1 : \alpha \in A_1\}$, $\mathcal{W}_1$ and σ_1 satisfy the conditions (a)-(d) for $n = 1$.

For each $(\alpha, \beta) \in A_1 \times \Lambda_2$, take $V_{(\alpha,\beta)}^2 = V_\alpha^1 \cap G_\beta^2$. Then $\mathcal{V}_\alpha^2 = \{V_{(\alpha,\beta)}^2 : \beta \in \Lambda_2\}$ is an open cover of V_α^1. For each $b \in \sigma_1^{-1}(\alpha)$, $\Phi^{-1}(\mathcal{V}_\alpha^2)$ covers $\overline{W_b^1}$. Therefore there exists a compact (sequential) finite open cover $\mathcal{W}_{(\alpha,b)}' = \{W_{(\alpha,b,\delta)}' : \delta \in B_{(\alpha,b)}^2\}$ of $\overline{W_b^1}$ which is a closure refinement of $\Phi^{-1}(\mathcal{V}_\alpha^2)$. Let $\sigma_{(\alpha,b)}^2 : B_{(\alpha,b)}^2 \to \{\alpha\} \times \Lambda_2$ be a corresponding refinement map. Then $\mathcal{W}_{(\alpha,b)} = \{W_{(\alpha,b,\delta)}' \cap W_b^1 : \delta \in B_{(\alpha,b)}^2\}$ is an open cover of W_b^1 satisfying $\overline{W_{(\alpha,b,\delta)}' \cap W_b^1} \subset \Phi^{-1}(V_{\sigma_{(\alpha,b)}^2(\delta)}^2)$. Let $A_2 = A_1 \times \Lambda_2, B_2 = \bigcup \{B_{(\alpha,b)}^2 : \alpha \in A_1, b \in \sigma_1^{-1}(\alpha)\}$, where we may think that $\{B_{(\alpha,b)}^2\}$ is pair-disjoint. Define $\sigma_2 : B_2 \to A_2$ by $\sigma_2 \mid B_{(\alpha,b)}^2 = \sigma_{(\alpha,b)}^2$. Define $\pi_1 : A_2 \to A_1$ to be the projection to the first factor. For every $\delta \in B_2$, there exists an unique pair $(\alpha, b) \in A_1 \times B_1$ such that $\delta \in B_{(\alpha,b)}^2$. Let

$$W_\delta^2 = W_{(\alpha,b,\delta)}' \cap W_b^1.$$

For each $\gamma = (\alpha, \beta) \in A_2$, let

$$U_\gamma^2 = \bigcup \{W_\delta^2 : \sigma_2(\delta) = \gamma\}.$$

Then we may define families of open sets in X and Y, respectively, as follows

$$\mathcal{W}_2 = \{W_\delta^2 : \delta \in B_2\},$$
$$\mathcal{U}_2 = \{U_\gamma^2 : \gamma \in A_2\} \quad \text{and}$$
$$\mathcal{V}_2 = \{V_\gamma^2 : \gamma \in A_2\}.$$

It is not hard to verify that $\mathcal{U}_2$, $\mathcal{V}_2$, $\mathcal{W}_2$, π_1 and σ_2 satisfy the conditions (a)–(d) for $n = 2$.

Repeating the process above, we can obtain the sequences of covers and maps required. $\square$

For a metric space (Y, ρ), we may define the *Hausdorff metric* $\rho_H(E, F)$ between two compact sets E, F in Y as follows:

$$\rho_H(E, F) = \inf\{r > 0 : E \subset O_r(F) \ \text{ and } \ F \subset O_r(E)\},$$

where $O_r(E) = \{y \in Y : \rho(x, y) < r$ for some $x \in E\}$. In [7], the following lemma was proved:

Lemma 2. *Let X be a topological space and (Y, ρ) be a complete metric space. For each $i \in \mathbb{N}$, $\Phi_i : X \to C(Y)$ is an l.s.c. map which satisfies that $\rho_H(\Phi_i(x), \Phi_{i+1}(x)) < 2^{-i}$ for every $x \in X$. Define $\Phi : X \to 2^Y$ by $\Phi(x) = \{\lim y_i : y_i \in \Phi_i(x), \rho(y_i, y_{i+1}) \le 2^{-i}\}$. Then $\Phi(x) \in C(Y)$ for every $x \in X$ and $\Phi : X \to C(Y)$ is l.s.c.*

Using those lemmas, we show the following result.

Theorem 1. *For a regular space X, the following statements are equivalent.*

(1) *X is (sequentially) mesocompact.*
(2) *For every complete metric space (Y, ρ) and l.s.c. set-valued map $\Phi : X \to F(Y)$, there exists an l.s.c. map $\varphi : X \to C(Y)$ such that $\varphi(x) \subset \Phi(x)$ for every $x \in X$ and $\overline{\varphi(K)}$ is compact for every compact set (converging sequence including its limit point) K of X.*

PROOF: We only prove the result for mesocompactness.

(1)$\Rightarrow$(2). Let (Y, ρ) be a complete metric space and $\Phi : X \to F(Y)$ an l.s.c. map. Using Lemma 1, there exist some sequences satisfying the conditions in Lemma 1 and we use the same symbols to denote them as in Lemma 1. For every $\alpha \in A_n$, take $y_\alpha \in V_\alpha^n$. Let $\varphi_n(x) = \{y_\alpha : x \in U_\alpha^n, \alpha \in A_n\}$. Then $\varphi_n(x) \in K(Y)$ and, by the definition of φ_n, $\varphi_n^{-1}(y)$ is open in X for every $y \in Y$. Therefore, $\varphi_n : X \to K(Y)$ is l.s.c.

We shall prove the following:

(i) $\rho_H(\varphi_{n+1}(x), \varphi_n(x)) < 2^{-n}$ for every $x \in X$. In fact, for every $y_\alpha \in \varphi_{n+1}(x)$, we have $x \in U_\alpha^{n+1} \subset U_{\pi_n(\alpha)}^n$. Thus $y_{\pi_n(\alpha)} \in \varphi_n(x)$. It follows that $\rho(y_\alpha, \varphi_n(x)) \le \rho(y_\alpha, y_{\pi_n(\alpha)}) \le \operatorname{diam} V_{\pi_n(\alpha)}^n < 2^{-n}$. Therefore, $\varphi_{n+1}(x) \subset O_{2^{-n}}(\varphi_n(x))$. Similarly, $\varphi_n(x) \subset O_{2^{-n}}(\varphi_{n+1}(x))$.

(ii) $\varphi_n(x) \subset O_{2^{-n}}(\Phi(x))$. In fact, let $y_\alpha \in \varphi_n(x)$. Then $x \in U_\alpha^n \subset \Phi^{-1}(V_\alpha^n)$. Hence $\Phi(x) \cap V_\alpha^n \ne \emptyset$. Pick $y \in \Phi(x) \cap V_\alpha^n$. Then $\rho(y_\alpha, \Phi(x)) \le \rho(y_\alpha, y) \le \operatorname{diam} V_\alpha^n < 2^{-n}$. It follows that $\varphi_n(x) \subset O_{2^{-n}}(\Phi(x))$.

Define $\varphi : X \to 2^Y$ as follows:

$$\varphi(x) = \{\lim y_i : y_i \in \varphi_i(x), \ \rho(y_i, y_{i+1}) \le 2^{-i}\}.$$

By Lemma 2, $\varphi : X \to C(Y)$ is l.s.c. From (ii) and the closedness of $\Phi(x)$ it follows that $\varphi(x) \subset \Phi(x)$. It remains to prove that $\overline{\varphi(K)}$ is compact for each compact subset K of X.

Let K be a compact subset of X. For each $n \in \mathbb{N}$, since $\mathcal{U}_n$ is compact finite, $\varphi_n(K)$ is finite subset of Y. Note that the Hausdorff metric ρ_H on $C(Y)$ is complete, see [5]. It follows from $\rho_H(\varphi_{n+1}(K), \varphi_n(K)) < 2^{-n}$ that the sequence $\{\varphi_n(K)\}$ is a Cauchy sequence in this complete metric space $(C(Y), \rho_H)$. Hence this sequence converges to a compact subset of Y which contains $\varphi(K)$. Therefore, $\overline{\varphi(K)}$ is compact.

(2)$\Rightarrow$ (1). Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Let $Y = \Lambda$ be a discrete metric space. Then Y is complete. Define $\Phi : X \to F(Y)$ by $\Phi(x) = \{\alpha : x \in U_\alpha\}$. Φ is l.s.c. since $\Phi^{-1}(\alpha) = U_\alpha$. By (2), there exists an l.s.c. map $\varphi : X \to C(Y)$ such that $\varphi(x) \subset \Phi(x)$ for each $x \in X$ and $\overline{\varphi(K)}$ is compact for each compact subset K of X. Let K be a compact subset of X. Then $K \cap \varphi^{-1}(\alpha) \neq \emptyset$ if and only if $\alpha \in \varphi(K)$. Since $\overline{\varphi(K)}$ is compact in the discrete metric space Y, it is finite and hence so is $\varphi(K)$. It follows that $\{\varphi^{-1}(\alpha) : \alpha \in Y\}$ is compact finite open refinement of $\mathcal{U}$. This shows that X is mesocompact. □

Since the u.s.c. selection ϕ in Michael's pair is persevering compact, a natural problem is if (sequentially) mesocompactness can be characterized by replacing the u.s.c. selection with (weakly) persevering compact selection in Theorem A. The next theorem shows that one implication is true.

Theorem 2. *A topological space X is mesocompact (sequentially mesocompact) if, for every complete metric space Y and l.s.c. set-valued map $\Phi : X \to F(Y)$, there exist an l.s.c. map $\varphi : X \to C(Y)$ and a persevering compact (weakly persevering compact) map $\phi : X \to C(Y)$ such that $\varphi(x) \subset \phi(x) \subset \Phi(x)$ for every $x \in X$.*

Proof: Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Let $Y = \Lambda$ be equipped with the discrete metric. Then Y is complete. Define $\Phi : X \to F(Y)$ by $\Phi(x) = \{\alpha : x \in U_\alpha\}$. Φ is l.s.c. since $\Phi^{-1}(\alpha) = U_\alpha$. Thus, there exist an l.s.c. map $\varphi : X \to C(Y)$ and a persevering compact (weakly persevering compact) map $\phi : X \to C(Y)$ such that $\varphi(x) \subset \phi(x) \subset \Phi(x)$ for each $x \in X$. Note that $K \cap \phi^{-1}(\alpha) \neq \emptyset$ if and only if $\alpha \in \phi(K)$ for each compact (converging sequence including its limit) K of X. It follows from $\phi(K)$ being finite that $\{\phi^{-1}(\alpha) : \alpha \in Y\}$ is compact finite (sequential finite). For each $\alpha \in \Lambda$, $\varphi^{-1}(\alpha) \subset \phi^{-1}(\alpha)$. Thus $\{\varphi^{-1}(\alpha) : \alpha \in \Lambda\}$ is compact finite (sequential finite) open refinement of $\mathcal{U}$. Therefore, X is mesocompact (sequentially mesocompact). □

Moreover, for sequentially mesocompact spaces, we may show that another implication is also true for every normal space. To this end, we need three lemmas.

Lemma 3. *Let X be a normal and sequentially mesocompact space. Then, for every open cover $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ of X, there exists an open cover $\mathcal{V}=\{V_\alpha : \alpha \in A\}$ such that $\{\overline{V}_\alpha : \alpha \in A\}$ is sequential finite and $\overline{V}_\alpha \subset U_\alpha$ for each $\alpha \in A$.*

Proof: Without loss of generality, we can suppose that $\mathcal{U}$ is sequential finite. It follows from [3, Theorem 1.5.18] that there exists an open cover $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ such that $\overline{V}_\alpha \subset U_\alpha$. Then $\mathcal{V}$ satisfies the conditions. □

Lemma 4. *Let X be a topological space and (Y,ρ) be a complete metric space. For each $i \in \mathbb{N}$, $\Phi_i : X \to K(Y)$ is a weakly persevering compact map satisfying the following conditions:*

(a) $\Phi_i^{-1}(y) = \Phi_i^{-1}(\{y\})$ *is closed for each* $y \in Y$ *and* $i \in \mathbb{N}$;
(b) $\Phi_{i+1}(x) \subset O_{2^{-i}}(\Phi_i(x))$ *for each* $x \in X$ *and* $i \in \mathbb{N}$.

Define $\Phi : X \to 2^Y$ *by*

$$\Phi(x) = \{\lim y_i : y_i \in \Phi_i(x),\ \rho(y_i, y_{i+1}) \le 2^{-i}\}.$$

Then Φ is weakly persevering compact.

Proof: By the finiteness of $\Phi_i(x)$ and the completeness of (Y,ρ), $\Phi(x)$ is nonempty for every $x \in X$.

Let $L = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ be a converging sequence $\{x_n, n \in \mathbb{N}\}$ with its limit x in X. Denote $Z = \prod_{i\in\mathbb{N}} \Phi_i(L)$ with the product topology. Then Z is compact and metrizable as a countable product of compact metric spaces. Let

$$Z_0 = \{r = (y_1, y_2, \dots, y_i, \dots) : \exists\, l \in L \text{ such that } y_i \in \Phi_i(l) \text{ and } \rho(y_i, y_{i+1}) \le 2^{-i}\}.$$

Define $f : Z_0 \to Y$ by $f(y_1, y_2, \dots, y_i \dots,) = \lim y_i$. Trivially, $f(Z_0) = \Phi(L)$. We shall show that Z_0 is compact and f is continuous.

First we show that f is continuous. To this end, pick a sequence $\{r_n\}$ of elements of Z_0 which converges to some $r \in Z_0$. Let $r_n = (y_1^n, y_2^n, \dots)$ and $r = (y_1', y_2', \dots) \in Z_0$. We prove that $f(r_n) \to f(r)$. Suppose $\lim_{i\to\infty} y_i^n = y_n$ and $\lim_{i\to\infty} y_i' = y$. For $\varepsilon > 0$, take $i_0 \in \mathbb{N}$ such that $2^{-i_0+1} < \varepsilon/3$. Then $\rho(y_{i_0}^n, y_n) < \varepsilon/3$ and $\rho(y_{i_0}', y) < \varepsilon/3$. For i_0, take $n_0 \in \mathbb{N}$ such that $\rho(y_{i_0}^n, y_{i_0}') < \varepsilon/3$ if $n \ge n_0$. Therefore,

$$\rho(y_n, y) \le \rho(y_n, y_{i_0}^n) + \rho(y_{i_0}^n, y_{i_0}') + \rho(y_{i_0}', y) < \varepsilon$$

for $n \ge n_0$. It follows that f is continuous.

To show that Z_0 is compact, we can take a sequence $\{r_n\}$ in Z_0 which converges to r in Z, where $r_n = (y_1^n, y_2^n, \dots)$ and $r = (y_1, y_2, \dots) \in Z$. We shall prove that $r \in Z_0$, which implies that Z_0 is compact. At first, it is easy to see that $\rho(y_i, y_{i+1}) \le 2^{-i}$ for each $i \in \mathbb{N}$. We only need to prove that there exists $l_0 \in L$ such that $y_i \in \Phi_i(l_0)$ for each $i \in \mathbb{N}$. By the definition of L_0, for every n, there exists $l_n \in L$ such that $y_i^n \in \Phi_i(l_n)$. Then $\{l_n\}$ has a subsequence converging to some point $l_0 \in L$. Without loss of generality, we can suppose that $\{l_n\}$ converges to l_0. For every i and m, we have $y_i^n \to y_i$, $y_i^n \in \Phi_i(l_n)$ and $\Phi_i(\{l_n : n \in \mathbb{N} \text{ and } n > m\} \cup \{l_0\})$ is compact. It follows that $y_i \in \Phi_i(\{l_n : n \in \mathbb{N} \text{ and } n > m\} \cup \{l_0\})$. This implies that $y_i \in \Phi_i(l_0)$ or there are infinitely many $n \in \mathbb{N}$ satisfying $y_i \in \Phi_i(l_n)$. In the second case, there exists infinitely many $n \in \mathbb{N}$ such that $l_n \in \Phi_i^{-1}(y_i)$. Thus, $l_0 \in \Phi_i^{-1}(y_i)$ since $\Phi_i^{-1}(y_i)$ is closed. Therefore, we have also $y_i \in \Phi_i(l_0)$.

This means that $y_i \in \Phi_i(l_0)$ for any $i \in \mathbb{N}$, and hence $r \in Z_0$. Therefore Z_0 is compact and hence $\Phi(L) = f(Z_0)$ is compact. $\square$

The following lemma is similar to Lemma 1 with a simpler proof than that of Lemma 1.

Lemma 5. *Let X be a normal and sequentially mesocompact space and (Y, ρ) be a complete metric space. Then for every l.s.c. map $\Phi : X \to F(Y)$, there exist: a sequence $\{\mathcal{U}_n = \{U_\alpha : \alpha \in A_n\}\}$ of open covers of X with $\{\overline{U}_\alpha : \alpha \in A_n\}$ being sequentially finite for each $n \in \mathbb{N}$, a sequence $\{\mathcal{V}_n = \{V_\alpha : \alpha \in A_n\}\}$ of locally finite open covers of Y and a sequence $\{\pi_n : A_{n+1} \to A_n\}$ of maps such that, for each $\alpha \in A_n$ and $n \in \mathbb{N}$,*

(a) $U_\alpha = \bigcup_{\beta \in \pi_n^{-1}(\alpha)} U_\beta$, $V_\alpha = \bigcup_{\beta \in \pi_n^{-1}(\alpha)} V_\beta$;
(b) $\overline{U}_\alpha \subset \Phi^{-1}(V_\alpha)$, *and*
(c) $\operatorname{diam} V_\alpha < 2^{-n}$.

Proof: For each $n \in \mathbb{N}$, let $\mathcal{W}_n = \{W_\alpha : \alpha \in \Lambda_n\}$ be a locally finite open cover of Y such that $\operatorname{diam} W_\alpha < 2^{-n}$ for each $\alpha \in \Lambda_n$. We inductively construct the sequences above.

Since $\{\Phi^{-1}(W_\alpha) : \alpha \in \Lambda_1\}$ is an open cover of X, by Lemma 3, there exists an open cover $\mathcal{U}_1 = \{U_\alpha : \alpha \in \Lambda_1\}$ of X such that $\{\overline{U}_\alpha : \alpha \in \Lambda_1\}$ is sequentially finite and $\overline{U}_\alpha \subset \Phi^{-1}(W_\alpha)$ for each $\alpha \in \Lambda_1$. Let $A_1 = \Lambda_1$, $V_\alpha = W_\alpha$ for each $\alpha \in A_1$.

Denote $\mathcal{W}_\alpha = \{V_\alpha \cap W_\beta : \beta \in \Lambda_2\}$ for each $\alpha \in A_1$. Then $\Phi^{-1}(\mathcal{W}_\alpha)$ is an open cover of $\overline{U}_\alpha$. In the subspace $\overline{U}_\alpha$, by Lemma 3, there exists an open refinement $\{U'_{(\alpha,\beta)} : \beta \in \Lambda_2\}$ of $\Phi^{-1}(\mathcal{W}_\alpha)$ such that $\{\overline{U'_{(\alpha,\beta)}} : \beta \in \Lambda_2\}$ is sequentially finite and $\overline{U'_{(\alpha,\beta)}} \subset \Phi^{-1}(V_\alpha \cap W_\beta)$ for each $\beta \in \Lambda_2$. Let $U_{(\alpha,\beta)} = U'_{(\alpha,\beta)} \cap U_\alpha$, $V_{(\alpha,\beta)} = V_\alpha \cap W_\beta$, $A_2 = A_1 \times \Lambda_2$, $\mathcal{U}_2 = \{U_{(\alpha,\beta)} : (\alpha, \beta) \in A_2\}$, $\mathcal{V}_2 = \{V_{(\alpha,\beta)} : (\alpha, \beta) \in A_2\}$, let $\pi_1 : A_2 \to A_1$ be the project map. Note that $U_{(\alpha,\beta)}$ is open in X. Moreover, trivially, they satisfy (a)–(c) for $n = 2$.

Repeating the process above, we can obtain the required sequences. $\square$

We are in a position now to prove the following theorem.

Theorem 3. *For a normal space X, the following conditions are equivalent.*

(1) *X is sequentially mesocompact.*
(2) *For every complete metric space (Y, ρ) and l.s.c. set-valued map $\Phi : X \to F(Y)$, there exist an l.s.c. map $\varphi : X \to C(Y)$ and a weakly persevering compact map $\phi : X \to C(Y)$ such that $\varphi(x) \subset \phi(x) \subset \Phi(x)$ for every $x \in X$.*

Proof: (1)$\Rightarrow$(2). Let $\mathcal{U}_n$ and $\mathcal{V}_n$ be the sequence of covers such as in Lemma 5. For every $\alpha \in A_n$, take $y_\alpha \in V_\alpha$. Define $\varphi_n(x) = \{y_\alpha : x \in U_\alpha, \alpha \in A_n\}$ and $\phi_n(x) = \{y_\alpha : x \in \overline{U}_\alpha, \alpha \in A_n\}$. Then $\phi_n : X \to K(Y), \varphi_n : X \to K(Y)$. Moreover, from the definitions of ϕ_n and φ_n it follows that $\varphi_n^{-1}(y)$ is open and $\phi_n^{-1}(y)$ is closed in X for every $y \in Y$. Therefore, φ_n is l.s.c. and $\phi_n^{-1}(y)$ is closed for every $y \in Y$. We show that ϕ_n is weakly persevering compact. In fact, for every converging sequence L of X, $y_\alpha \in \phi_n(L)$ if and only if $\overline{U}_\alpha \cap L \neq \emptyset$. Thus $\phi_n(L)$ is finite and hence ϕ_n is weakly persevering compact.

For every $x \in X$ and $n \in \mathbb{N}$, we shall prove the following:

(i) $\rho_H(\varphi_{n+1}(x), \varphi_n(x)) < 2^{-n}$. In fact, let $y_\alpha \in \varphi_{n+1}(x)$, then $x \in U_\alpha \subset U_{\pi_n(\alpha)}$. Thus $y_{\pi_n(\alpha)} \in \varphi_n(x)$, and $\rho(y_\alpha, \varphi_n(x)) \leq \rho(y_\alpha, y_{\pi_n(\alpha)}) \leq \operatorname{diam} V_{\pi_n(\alpha)} < 2^{-n}$. Therefore $\varphi_{n+1}(x) \subset O_{2^{-n}}(\varphi_n(x))$. Similarly, $\varphi_n(x) \subset O_{2^{-n}}(\varphi_{n+1}(x))$.

(ii) $\phi_{n+1}(x) \subset O_{2^{-n}}(\phi_n(x))$. Indeed, for every $y_\alpha \in \phi_{n+1}(x)$, we have $x \in \overline{U}_\alpha \subset \overline{U}_{\pi_n(\alpha)}$ and hence $y_{\pi_n(\alpha)} \in \phi_n(x)$. Thus $\rho(y_\alpha, \phi_n(x)) \leq \rho(y_\alpha, y_{\pi_n(\alpha)}) \leq \operatorname{diam} V_{\pi_n(\alpha)} < 2^{-n}$. Therefore, $\phi_{n+1}(x) \subset O_{2^{-n}}(\varphi_n(x))$.

(iii) $\varphi_n(x) \subset O_{2^{-n}}(\Phi(x))$ and $\phi_n(x) \subset O_{2^{-n}}(\Phi(x))$. We only prove the second statement. For every $y_\alpha \in \phi_n(x)$, $x \in \overline{U}_\alpha \subset \Phi^{-1}(V_\alpha)$. It follows that $\Phi(x) \cap V_\alpha \neq \emptyset$. Pick $y \in \Phi(x) \cap V_\alpha$. Then $\rho(y_\alpha, \Phi(x)) \leq \rho(y_\alpha, y) \leq \operatorname{diam} V_\alpha < 2^{-n}$ and hence $\phi_n(x) \subset O_{2^{-n}}(\Phi(x))$.

Now we define $\varphi : X \to 2^Y$ and $\phi : X \to 2^Y$ as follows:

$$\varphi(x) = \{\lim y_i : y_i \in \varphi_i(x), \rho(y_i, y_{i+1}) \leq 2^{-i}\} \qquad \text{and}$$
$$\phi(x) = \{\lim y_i : y_i \in \phi_i(x), \rho(y_i, y_{i+1}) \leq 2^{-i}\}.$$

By Lemmas 2 and 4, $\varphi : X \to C(Y)$ is l.s.c. and $\phi : X \to C(Y)$ is weakly persevering compact. Moreover, $\varphi(x) \subset \phi(x) \subset \Phi(x)$ from (iii). Therefore, φ and ϕ satisfy the conditions in (2).

(2)$\Rightarrow$(1) By Theorem 2. $\square$

The following problems remain open.

Question 1. *Is there an analogue of Theorem 3 for mesocompact spaces?*

Question 2. *Can the normality of X in Theorem 3 be weakened to the regularity?*

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References

[1] Boone J.R., *Some characterization of paracompactness in χ-space*, Fund. Math. **72** (1971), 145–155.
[2] Choban M., *Many-valued mappings and Borel sets, II*, Trans. Moscow Math. Soc. **23** (1970), 286–310.
[3] Engelking R., *General Topology*, Revised and completed edition, Heldermann, Berlin, 1989.
[4] Michael E., *A theorem on semicontinuous set-valued funtions*, Duke Math. **26** (1956), 647–652.
[5] Michael E., *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–182.
[6] Miyazaki K., *Characterizations of paracompact-like properties by means of set-valued semi-continuous selections*, Proc. Amer. Math. Soc. **129** (2001), 2777–2782.

[7] Nedev S., *Selection and factorization theorems for set-valued mapings*, Serdica **6** (1980), 291–317.
[8] Yan P.-F., τ *selections and its applictions on* BCO, J. Math. (in Chinese) **17** (1997), 547–551.

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