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Singular points of order k of Clarke regular and arbitrary functions

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Abstract. Let X be a separable Banach space and f a locally Lipschitz real function on X. For $k \in \mathbb{N}$, let $\Sigma_k(f)$ be the set of points $x \in X$, at which the Clarke subdifferential $\partial^C f(x)$ is at least k-dimensional. It is well-known that if f is convex or semiconvex (semiconcave), then $\Sigma_k(f)$ can be covered by countably many Lipschitz surfaces of codimension k. We show that this result holds even for each Clarke regular function (and so also for each approximately convex function). Motivated by a resent result of A.D. Ioffe, we prove also two results on arbitrary functions, which work with Hadamard directional derivatives and can be considered as generalizations of our theorem on $\Sigma_k(f)$ of Clarke regular functions (since each of them easily implies this theorem).

Keywords: Clarke regular functions, singularities, Hadamard derivative

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1. Introduction

If X is a Banach space and f a real function on X, then by the *singular set* $\Sigma(f)$ of f we mean the set of all points $x \in X$ at which f is not Gateâux differentiable. If f is a continuous convex (or semiconvex, or semiconcave) function, then $\Sigma(f) = \{x \in X : \dim \partial f(x) \geq 1\}$, where $\partial f(x)$ is the subdifferential (or the Clarke subdifferential) of f at x.

It is natural and useful to consider also the set $\Sigma_k(f) = \{x \in X : \dim \partial f(x) \geq k\}$ (where $k \in \mathbb{N}$) of *singular points of order* k (or of magnitude k by [4]). For convex functions, the smallness of sets $\Sigma_k(f)$ was considered (using formally different definition) e.g. in [3], [12] and [11], and for semiconvex (resp. semiconcave) functions in [2], [1] and [4].

For continuous convex functions in separable Banach spaces, the best possible result on smallness of sets $\Sigma_k(f)$ is the following theorem which is a reformulation (via Lemma 2.4) of results of [12].

Theorem A. *Let f be a continuous convex function defined on an open convex subset C of a separable Banach space X. Let $\dim X > k \in \mathbb{N}$. Then the set $\Sigma_k(f)$ can be covered by countably many DC surfaces of codimension k.*

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(For a complete characterization of singular sets $\Sigma(f) = \Sigma_1(f)$ for convex functions f in $\mathbb{R}^n$ see [10].)

For continuous semiconvex (resp. semiconcave) functions, the following analogous result was factually proved in [1]. Indeed, the proof in [1] works, if we work with Lipschitz surfaces of codimension k (see Definition 2.2 below) instead with "$\infty - k$ rectifiable sets" of [1].

Theorem B. *Let f be a continuous locally semiconvex (or semiconcave) function defined on an open subset G of a separable Banach space X. Let* $\dim X > k \in \mathbb{N}$. *Then the set $\Sigma_k(f)$ can be covered by countably many Lipschitz surfaces of codimension k.*

Note that this result can be improved ([5]) in the spirit of Theorem A. Namely, in superreflexive spaces, the Lipschitz surfaces of Theorem B can be "parametrized by differences of Lipschitz semiconvex functions".

In the present article we prove the following generalization of Theorem B.

Theorem 1.1. *Let f be a locally Lipschitz Clarke regular function defined on an open convex subset G of a separable Banach space X. Let* $\dim X > k \in \mathbb{N}$. *Then the set $\Sigma_k(f)$ can be covered by countably many Lipschitz surfaces of codimension k.*

In particular Theorem B holds with the weaker assumption that f is approximately convex in the sense of [8]. (For the fact that each approximately convex function is Clarke regular see [8, Corollary 3.5 and Theorem 3.6].)

Theorem 1.1 is an easy consequence of the following Proposition 1.2 on locally Lipschitz mappings. So, this proposition, which is perhaps of an independent interest, can be considered as a generalization of Theorem 1.1.

Proposition 1.2. *Let X be a separable Banach space, Y a Banach spaces, $G \subset X$ an open set, and $f : G \to Y$ a locally Lipschitz mapping. Let* $\dim X > k \in \mathbb{N}$. *Denote by $\Sigma_k^*(f)$ the set of those $x \in G$, for which there exists a k-dimensional space $V_x \subset X$, such that, for each $0 \neq v \in V_x$, the one-sided directional derivative $f'_+(x,v)$ exists but the (two-sided) directional derivative $f'(x,v)$ does not exist. Then the set $\Sigma_k^*(f)$ can be covered by countably many Lipschitz surfaces of codimension k.*

Note that, if f is a Clarke regular function, then it is easy to show that $\Sigma_k(f) = \Sigma_k^*(f)$ (see Lemma 2.4), and so Proposition 1.2 yields Theorem 1.1.

We will obtain also another generalization of Theorem 1.1 which concern lower directional derivatives $d^- f(x,v)$ of arbitrary locally Lipschitz functions:

Proposition 1.3. *Let X be a separable Banach space, $G \subset X$ an open set, and f a locally Lipschitz function on G. Let* $\dim X > k \in \mathbb{N}$. *Denote by $\tilde{\Sigma}_k(f)$ the set of those $x \in G$, for which there exists a k-dimensional space $V_x \subset X$, such that $d^- f(x,v) + d^- f(x,-v) > 0$ for each $0 \neq v \in V_x$. Then the set $\tilde{\Sigma}_k(f)$ can be covered by countably many Lipschitz surfaces of codimension k.*

Note that, if f is a Clarke regular function, then (2.6) easily implies $\tilde{\Sigma}_k(f) = \Sigma_k^*(f)$ and so $\tilde{\Sigma}_k(f) = \Sigma_k(f)$.

In a recent article [6], A.D. Ioffe has shown that some results concerning (Dini) derivatives of Lipschitz function can be generalized to results concerning Hadamard derivatives of arbitrary functions. Following this idea, we will prove the following theorems on Hadamard derivatives of arbitrary mappings (functions), which clearly imply (via (2.3) and (2.4)) Propositions 1.2 and 1.3.

Theorem 1.4. *Let X be a separable Banach space, Y a Banach space, $G \subset X$ an open set, and $f : G \to Y$ an arbitrary mapping. Let $\dim X > k \in \mathbb{N}$. Denote by $\Sigma^*_{H,k}(f)$ the set of those $x \in G$, for which there exists a k-dimensional space $V_x \subset X$, such that, for each $0 \neq v \in V_x$, the Hadamard one-sided directional derivative $f'_{H+}(x,v)$ exists but the (two-sided) Hadamard directional derivative $f'_H(x,v)$ does not exist. Then the set $\Sigma^*_{H,k}(f)$ can be covered by countably many Lipschitz surfaces of codimension k.*

Theorem 1.5. *Let X be a separable Banach space, $G \subset X$ an open set, and f an arbitrary real function on G. Let $\dim X > k \in \mathbb{N}$. Denote by $\tilde{\Sigma}_{H,k}(f)$ the set of those $x \in G$, for which there exists a k-dimensional space $V_x \subset X$, such that $f^-_H(x,v) + f^-_H(x,-v) > 0$ for each $0 \neq v \in V_x$. Then the set $\tilde{\Sigma}_{H,k}(f)$ can be covered by countably many Lipschitz surfaces of codimension k.*

Note that, in the case $k = 1$, Proposition 1.3 easily follows from Lemma 2 of [13] and, for $k = 1$, Theorem 1.5 follows from [6, Theorem 1.3(a)] (which deal with functions $f : X \to [-\infty, \infty]$).

Further note that our results do not deal with the case $\dim X = k$, but a slight modification of our proofs give that exceptional sets ($\Sigma_k(f), \dots$) from the above results are countable in this case.

In Section 2, we will recall some definitions and well-known facts. In Section 3 and Section 4 we prove Theorem 1.4 and Theorem 1.5, respectively.

2. Preliminaries

In the following, if it is not said otherwise, X will be a real Banach space. By $\operatorname{span} M$ we denote the linear span of $M \subset X$. If $X = E \oplus F$, then we denote by $\pi_{E,F}$ the projection of X on E along the space F. The symbol $B(x,r)$ denotes the open ball with center x and radius r.

For a Banach space Y, we set $S_Y := \{y \in Y : \|y\| = 1\}$. If $C \subset Y$ is a nonempty convex set, then we set $\dim C := \dim(\operatorname{span}(C - C)) \in (\mathbb{N} \cup \{0\} \cup \{\infty\})$. It is well-known that, for $k \in \mathbb{N}$, the inequality $\dim C \geq k$ holds iff there exists a k-dimensional space $V \subset Y$, $c \in Y$ and $r > 0$ such that $(c + V) \cap B(c,r) \subset C$.

Let X, Y be Banach spaces, $\emptyset \neq G \subset X$ an open set, and $f : G \to Y$ a mapping. The directional and one-sided directional derivatives of f at $x \in G$ in the direction $v \in X$ are defined respectively by

$$f'(x,v) := \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t},$$

$$f'_+(x,v) := \lim_{t\to 0+} \frac{f(x+tv)-f(x)}{t}\,.$$

The *Hadamard directional and one-sided directional derivatives* of f at $x \in G$ in the direction $v \in X$ are defined respectively by

$$f'_H(x,v) := \lim_{z\to v, t\to 0} \frac{f(x+tz)-f(x)}{t}\,,$$
$$f'_{H+}(x,v) := \lim_{z\to v, t\to 0+} \frac{f(x+tz)-f(x)}{t}\,.$$

It is easy to see that $f'(x,v)$ exists if and only if $f'_+(x,-v) = -f'_+(x,v)$ and

(2.1) $\qquad f'_H(x,v)$ exists if and only if $f'_{H+}(x,-v) = -f'_{H+}(x,v)$.

It is easy to prove that, if $f'_H(x,v)$ exists for each $v \in X$, then

(2.2) $\qquad f'_H(x,\cdot)$ is a continuous function.

It is well-known and easy to prove that, if f is locally Lipschitz on G, then

(2.3) $\qquad f'(x,v) = f'_H(x,v)$ (resp. $f'_+(x,v) = f'_{H+}(x,v)$)

whenever one of these two derivatives exists.

Now we suppose that f is a real function defined on an open subset G of X. The upper and lower (Dini) one-sided directional derivative of f at x in the direction v are defined respectively by

$$d^+f(x,v) := \limsup_{t\to 0+}\, (f(x+tv)-f(x))t^{-1},$$
$$d^-f(x,v) := \liminf_{t\to 0+}\, (f(x+tv)-f(x))t^{-1}.$$

Following [6] we denote the *upper and lower Hadamard one-sided directional derivatives* of f at x in the direction v by

$$f^+_H(x,v) := \limsup_{z\to v, t\to 0+} (f(x+tz)-f(x))t^{-1},$$
$$f^-_H(x,v) := \liminf_{z\to v, t\to 0+} (f(x+tz)-f(x))t^{-1}.$$

It is well-known (cf. [6]) and easy to prove that, if f is locally Lipschitz on G, then

(2.4) $\qquad d^+f(x,v) = f^+_H(x,v)$ and $d^-f(x,v) = f^-_H(x,v)$

for each $x \in G$ and $v \in X$.

It is also well-known (cf. [6, p. 1021]) and easy to prove that

(2.5) $\qquad f^-_H(x,\cdot)$ is a lower semicontinuous function.

Further we suppose that f is locally Lipschitz on G. Then

$$f^0(a,v) := \limsup_{z\to a, t\to 0+} \frac{f(z+tv)-f(z)}{t}$$

is the *Clarke derivative* of f at a in the direction v and

$$\partial^C f(a) := \{x^* \in X^* :\ x^*(v) \le f^0(a,v) \ \text{ for all } \ v \in X\}$$

is the *Clarke subdifferential* of f at a. Recall that

(2.6) $$f^0(a,\cdot) \text{ is a convex function,}$$

$\partial^C f(a)$ is a nonempty convex set, and

(2.7) $$f^0(a,v) = \max\{x^*(v) :\ x^* \in \partial^C f(a)\}.$$

We say that f is *Clarke regular* at $x \in G$ if $f^0(x,v) = f'_+(x,v)$ for each $v \in X$. We say that f is Clarke regular on G, if f is Clarke regular at each point of G.

Let X be a Banach space and Y, Z be closed non-trivial (i.e., different from $\{0\}$) subspaces of X. Then the gap between Y and Z (called also the opening or the deviation of Y and Z) is defined by

$$\gamma(Y,Z) = \max\{\sup_{y\in Y\cap S(X)} \operatorname{dist}(y,Z),\ \sup_{z\in Z\cap S(X)} \operatorname{dist}(z,Y)\}.$$

The gap need not be a metric on the set of all non-trivial subspaces of X; this property has the *distance* $\rho(Y,Z)$ (called also the spherical opening) between Y and Z defined as the Hausdorff distance between $Y \cap S(X)$ and $Z \cap S(X)$. It is well-known (see e.g., [7]) that always

(2.8) $$\rho(Y,Z)/2 \le \gamma(Y,Z) \le \rho(Y,Z).$$

For each $k \in \mathbb{N}$, we will denote by $\mathcal{V}_k(X)$ *the set of all k-dimensional subspaces of X equipped with the metric ρ*. We will need the following fact; because of the lack of a reference we supply a proof.

Lemma 2.1. *Let X be a separable Banach space and $k \in \mathbb{N}$. Then $\mathcal{V}_k(X)$ is a separable metric space.*

Proof: Let C be a countable dense subset of X. Denote by $\mathcal{D}_k$ the set of all spaces from $\mathcal{V}_k(X)$ which has a basis formed by elements of C. Then $\mathcal{D}_k$ is clearly countable. To prove that $\mathcal{D}_k$ is dense in $\mathcal{V}_k(X)$, consider an arbitrary $V \in \mathcal{V}_k(X)$ with the basis $v_1,\dots,v_k$. For each $\varepsilon > 0$, we can by [14, Lemma 2.4] find $\delta > 0$ such that the inequalities $\|w_1 - v_1\| < \delta, \dots, \|w_k - v_k\| < \delta$ imply that $W := \operatorname{span}\{w_1,\dots,w_k\} \in \mathcal{V}_k(X)$ and $\gamma(V,W) < \varepsilon$. So, using (2.8) and the density of C in X, we easily obtain that V belongs to the closure of $\mathcal{D}_k$. $\square$

Definition 2.2. Let X be a Banach space and $A \subset X$.

(i) Let F be a closed subspace of X. We say that A is an *F-Lipschitz surface* if there exists a topological complement E of F and a Lipschitz mapping $\varphi : E \to F$ such that $A = \{x + \varphi(x) : \ x \in E\}$.

(ii) Let $1 \leq n < \dim X$ be a natural number. We say that $A \subset X$ is a *Lipschitz surface of codimension n* if A is an F-Lipschitz surface for some n-dimensional space $F \subset X$. A Lipschitz surface of codimension 1 is said to be a *Lipschitz hypersurface*.

Note that the sets which can be covered by countably many Lipschitz hypersurfaces are sometimes called sparse sets (see [13] and [6]).

If X is a Banach space, $A \subset X$ and $x \in X$, then we denote by $T(A,x)$ the Bouligand's tangent (or contingent) cone of A at x. Recall that $v \in T(A,x)$ if and only if there exist sequences $v_n \to v$ and $t_n \to 0+$ such that $x + t_n v_n \in A$, $n \in \mathbb{N}$. We will need the following fact which is essentially a reformulation of [9, Lemma 2.10].

Lemma 2.3. *Let X be a Banach space and V a finite dimensional subspace of X with $\dim X > \dim V \geq 1$. Let $A \subset X$ and let $T(A,a) \cap V = \{0\}$ for each $a \in A$. Then A can be covered by countably many V-Lipschitz surfaces.*

Recall that, if $A \subset X^*$, then

$$A_{\perp} := \{x \in X : \ x^*(x) = 0 \ \text{ for each } \ x^* \in A\}$$

is a closed linear subspace of X. It is well-known and easy to prove that if $Z \subset X^*$ is a linear space and $k \in \mathbb{N}$, then

$$\dim Z \geq k \Leftrightarrow \operatorname{codim} Z_{\perp} \geq k. \tag{2.9}$$

Now suppose that f is a Clarke regular function on an open set $G \subset X$ and $a \in G$. Consider the closed linear space $L_a := (\partial f(a) - \partial f(a))_{\perp} = (\operatorname{span}(\partial f(a) - \partial f(a)))_{\perp}$. Then

$$L_a = \{v \in X : \ f'(a,v) \ \text{ exists}\}. \tag{2.10}$$

This fact immediately follows from the observation that $f'(a,v)$ exists iff

$$\begin{aligned} 0 = f'_+(a,v) + f'_+(a,-v) &= f^0(a,v) + f^0(a,-v) \\ &= \sup\{y_1(v) : \ y_1 \in \partial f(a)\} + \sup\{y_2(-v) : \ y_2 \in \partial f(a)\} \\ &= \sup\{y_1(v) : \ y_1 \in \partial f(a)\} - \inf\{y_2(v) : \ y_2 \in \partial f(a)\} \\ &= \sup\{(y_1 - y_2)(v) : \ y_1 \in \partial f(a), y_2 \in \partial f(a)\}. \end{aligned}$$

Using (2.10), it is easy to show that, for each $k \in \mathbb{N}$,

$$a \in \Sigma^*_k(f) \ \text{ if and only if } \ \operatorname{codim}(L_a) \geq k. \tag{2.11}$$

After these observations, we easily obtain the following fact.

Lemma 2.4. *Let f be a Clarke regular function on an open set $G \subset X$ and $k \in \mathbb{N}$. Then $\Sigma_k(f) = \Sigma_k^*(f)$.*

Proof: Let $a \in G$. Set $Z := \operatorname{span}(\partial f(a) - \partial f(a))$; so $a \in \Sigma_k(f)$ if and only if $\dim(Z) \geq k$. By (2.9), the latter condition is equivalent to $\operatorname{codim} Z_\perp = \operatorname{codim}(L_a) \geq k$, which is equivalent to $a \in \Sigma_k^*(f)$ by (2.11). □

3. Proof of Theorem 1.4

We will first prove two lemmas and then show that they easily imply Theorem 1.4.

Lemma 3.1. *Let X and Y be Banach spaces and $f : X \to Y$ an arbitrary mapping. For each finitely dimensional space $V \subset X$ and $n, p \in \mathbb{N}$, denote*

(a) *by B_V the set of all $a \in X$ such that, for each $0 \neq v \in V$, the one-sided Hadamard derivative $f'_{H+}(a, v)$ exists, but the (bilateral) Hadamard derivative $f'_H(a, v)$ does not exist, and*

(b) *by $B_{V,n,p}$ the set of all $a \in X$ such that*

$$\left\| \frac{f(a+\tau w) - f(a)}{\tau} - \frac{f(a+tz) - f(a)}{t} \right\| < 1/4n,$$
$$\left\| \frac{f(a+\tau w) - f(a)}{\tau} + \frac{f(a-tz) - f(a)}{t} \right\| > 1/n, \tag{3.1}$$

whenever $t, \tau \in (0, 1/p)$, $v \in S_V$, and $w, z \in X$ fulfil $\|v - w\| < 1/p$, $\|v - z\| < 1/p$.

Let $k \in \mathbb{N}$, and let $\mathcal{D}_k$ be a dense countable subset of $\mathcal{V}_k := \mathcal{V}_k(X)$ (see Lemma 2.1). Then

(i)

$$B_V \subset \bigcup_{n,p=1}^{\infty} B_{V,n,p} \quad \text{for each} \quad V \in \mathcal{V}_k, \quad \text{and}$$

(ii)

$$\bigcup_{V \in \mathcal{V}_k} B_V \subset \bigcup_{U \in \mathcal{D}_k, n \in \mathbb{N}, p \in \mathbb{N}} B_{U,n,p}.$$

Proof: Consider arbitrary $k \in \mathbb{N}$, $V \in \mathcal{V}_k$ and $a \in B_V$. For $v \in S_V$, denote $e(v) := \|f'_{H+}(a, v) + f'_{H+}(a, -v)\|$. Since $f'_{H+}(a, \cdot)$ is continuous on X (see (2.2)), the function e is continuous as well. Since S_V is compact and e is strictly positive on S_V by (2.1), we can find $n \in \mathbb{N}$ such that $e(v) > 2/n$ for each $v \in S_V$. Using the definition of $f'_{H+}(a, v)$ and $f'_{H+}(a, -v)$, we can find, for each $v \in S_V$, a number

$\delta_v > 0$ such that

$$\left\| \frac{f(a+tz)-f(a)}{t} - f'_{H+}(a,v) \right\| < \frac{1}{8n}, \tag{3.2}$$
$$\left\| \frac{f(a-tz)-f(a)}{t} - f'_{H+}(a,-v) \right\| < \frac{1}{8n}$$

whenever $\|z-v\| < \delta_v$ and $0 < t < \delta_v$.

Since S_V is compact, we can find a finite set $F \subset S_V$ such that

$$S_V \subset \bigcup_{v \in F} B(v, \delta_v/4). \tag{3.3}$$

Choose $p \in \mathbb{N}$ such that $1/p < \min\{\delta_v/4 : \ v \in F\}$.

Now we will show that

$$\text{if } \ U \in \mathcal{V}_k \ \text{ and } \ \rho(U,V) < 1/p, \ \text{ then } \ a \in B_{U,n,p}. \tag{3.4}$$

The condition (3.4) clearly implies (i) and (since $\mathcal{D}_k$ is dense in $\mathcal{V}_k$) also (ii).

So suppose that $U \in \mathcal{V}_k$ with $\rho(U,V) < 1/p$ be given. To prove $a \in B_{U,n,p}$, consider arbitrary $t, \tau \in (0, 1/p)$, $u \in S_U$, and $w, z \in X$ fulfilling $\|u-w\| < 1/p$, $\|u-z\| < 1/p$. Since $\rho(U,V) < 1/p$, we can find $\tilde{v} \in S_V$ with $\|u-\tilde{v}\| < 1/p$. By (3.3) we choose $v \in F$ such that $\|v-\tilde{v}\| < \delta_v/4$. Then $\|v-z\| \le \|v-\tilde{v}\| + \|\tilde{v}-u\| + \|u-z\| < \delta_v$, and so (3.2) implies

$$\left\| \frac{f(a+tz)-f(a)}{t} - f'_{H+}(a,v) \right\| < \frac{1}{8n},$$
$$\left\| \frac{f(a-tz)-f(a)}{t} - f'_{H+}(a,-v) \right\| < \frac{1}{8n}.$$

Similarly we obtain $\|v-w\| < \delta_v$, and so (3.2) implies

$$\left\| \frac{f(a+\tau w)-f(a)}{\tau} - f'_{H+}(a,w) \right\| < \frac{1}{8n}.$$

Therefore

$$\left\| \frac{f(a+\tau w)-f(a)}{\tau} - \frac{f(a+tz)-f(a)}{t} \right\| < \frac{1}{4n}$$

and

$$\begin{aligned} \left\| \frac{f(a+tw)-f(a)}{\tau} + \frac{f(a-tz)-f(a)}{t} \right\| &> \|f^-_H(a,v) + f^-_H(a,-v)\| - \frac{1}{4n} \\ &= e(v) - \frac{1}{4n} > \frac{1}{n}. \end{aligned}$$

So $a \in B_{U,n,p}$. □

Lemma 3.2. *Let X and Y be Banach spaces and $f : X \to Y$ an arbitrary mapping. Let $\dim X > k \in \mathbb{N}$ and let $V \subset X$ be a space of dimension k. Let $B = B_V$ or $B = B_{V,n,p}$, where B_V and $B_{V,n,p}$ are as in Lemma 3.1. Then B can be covered by countably many V-Lipschitz surfaces.*

Proof: By Lemma 3.1(i), it is sufficient to prove the statement for $B := B_{V,n,p}$ (where $n, p \in \mathbb{N}$). By Lemma 2.3, it is sufficient to prove that $T(B,x) \cap V = \{0\}$ for each $x \in B$. Suppose to the contrary that there exist $x \in B$ and $v \in T(B,x) \cap V$. We can and will suppose that $\|v\| = 1$. Set $\tau := \frac{1}{2p}$, $y := x + \tau v$ and $C := \frac{\|f(y)-f(x)\|}{\tau}$. Since $v \in T(B,x)$, we can clearly find $z \in X$ with $\|z - v\| < 1/p$ and $0 < t < \frac{1}{4np^2(C+1)}$ such that $a := x + tz \in B$. Since $x \in B = B_{V,n,p}$, we have

$$\left\| \frac{f(y)-f(x)}{\tau} - \frac{f(a)-f(x)}{t} \right\| < \frac{1}{4n}. \tag{3.5}$$

Set $w := 2p(y-a)$. Then $t, \tau \in (0, 1/p)$, $\|v - z\| < 1/p$ and also

$$\|v - w\| = \|2p(y-x) - 2p(y-a)\| = 2p\|a - x\| = 2pt\|z\| < 2p \cdot \frac{1}{4p^2} \cdot 2 = 1/p.$$

Consequently, since $x = a - tz$, $y = a + \tau w$ and $a \in B = B_{V,n,p}$, we have by (3.1)

$$\left\| \frac{f(y)-f(a)}{\tau} + \frac{f(x)-f(a)}{t} \right\| > \frac{1}{n}. \tag{3.6}$$

Using (3.5) and (3.6), we obtain

$$\begin{aligned} \left\| \frac{f(x)-f(a)}{\tau} \right\| \geq & \left\| \frac{f(y)-f(a)}{\tau} + \frac{f(x)-f(a)}{t} \right\| \\ & - \left\| \frac{f(y)-f(x)}{\tau} - \frac{f(a)-f(x)}{t} \right\| > \frac{1}{2n}. \end{aligned}$$

Consequently, $\|f(x) - f(a)\| > \frac{1}{4np}$. On the other hand, by (3.5) and the definitions of C and t, we obtain

$$\|f(x) - f(a)\| = t \left\| \frac{f(x)-f(a)}{t} \right\| \leq t \left(C + \frac{1}{4n} \right) < \frac{1}{4np},$$

which is a contradiction. $\square$

Proof of Theorem 1.4: By definition of $\Sigma^*_{H,k}(f)$ and Lemma 3.1(ii), we have

$$\Sigma^*_{H,k}(f) = \bigcup_{V \in \mathcal{V}_k} B_V \subset \bigcup_{U \in \mathcal{D}_k, n \in \mathbb{N}, p \in \mathbb{N}} B_{U,n,p}.$$

Since $\mathcal{D}_k$ is countable, the assertion of Theorem 1.4 follows from Lemma 3.2. $\square$

4. Proof of Theorem 1.5

We will first prove two lemmas and then show that they easily imply Theorem 1.5.

Lemma 4.1. *Let X be a Banach space and f a real function on X. For each finitely dimensional space $V \subset X$ and $n \in \mathbb{N}$, denote*

(a) *by A_V the set of all $a \in X$ such that*

$$f_H^-(a,v) + f_H^-(a,-v) > 0 \quad \text{for each} \quad 0 \neq v \in V \tag{4.1}$$

and

(b) *by $A_{V,n}$ the set of all $a \in A_V$ such that*

$$\frac{f(a+\tau w) - f(a)}{\tau} + \frac{f(a-tz) - f(a)}{t} > 1/n, \tag{4.2}$$

whenever $t, \tau \in (0, 1/n)$, $v \in S_V$, and $w, z \in X$ fulfil $\|v - w\| < 1/n$, $\|v - z\| < 1/n$.

Let $k \in \mathbb{N}$, and let $\mathcal{D}_k$ be a dense countable subset of $\mathcal{V}_k := \mathcal{V}_k(X)$ (see Lemma 2.1). Then

(i)

$$A_V = \bigcup_{n=1}^{\infty} A_{V,n} \quad \text{for each} \quad V \in \mathcal{V}_k, \quad \text{and}$$

(ii)

$$\bigcup_{V \in \mathcal{V}_k} A_V = \bigcup_{U \in \mathcal{D}_k} A_U.$$

Proof: Consider arbitrary $k \in \mathbb{N}$, $V \in \mathcal{V}_k$ and $a \in A_V$. For $0 \neq v \in V$, denote $e(v) := f_H^-(a,v) + f_H^-(a,-v)$. Since $f_H^-(a,\cdot)$ is lower semicontinuous on X (see (2.5)), the function e is lower semicontinuous as well. Since S_V is compact and e is strictly positive on S_X, we can find $\varepsilon > 0$ such that $e(v) > \varepsilon$, $v \in S_V$. Using the lower semicontinuity of $f_H^-(a,\cdot)$ and the definition of $f_H^-(a,v)$, we can find, for each $v \in S_V$, a number $\delta_v > 0$ such that

$$\begin{aligned} &f_H^-(a,u) > f_H^-(a,v) - \frac{\varepsilon}{8} \ \text{ and } \ f_H^-(a,-u) > f_H^-(a,-v) - \frac{\varepsilon}{8}, \\ &\text{whenever} \ \ \|u - v\| < \delta_v \end{aligned} \tag{4.3}$$

and

(4.4)

$$\begin{aligned} &\frac{f(a+tz) - f(a)}{t} > f_H^-(a,v) - \frac{\varepsilon}{8} \ \text{ and } \ \frac{f(a-tz) - f(a)}{t} > f_H^-(a,-v) - \frac{\varepsilon}{8} \\ &\text{whenever} \ \ \|z - v\| < \delta_v \ \text{ and } \ 0 < t < \delta_v. \end{aligned}$$

Since S_V is compact, we can find a finite set $F \subset S_V$ such that

$$S_V \subset \bigcup_{v\in F} B(v,\delta_v/4). \tag{4.5}$$

Choose $n \in \mathbb{N}$ such that $1/n < \min(\{\delta_v/4 :\ v \in F\} \cup \{\varepsilon/2\})$. To prove (i), it is sufficient to show that $a \in A_{V,n}$. To this end, suppose that $t,\tau \in (0,1/n)$, $v \in S_V$, and $w,z \in X$ fulfilling $\|v-w\| < 1/n$, $\|v-z\| < 1/n$ are given. By (4.5), we can choose $v^* \in F$ such that $\|v-v^*\| < \delta_{v^*}/4$. Then $\|w-v^*\| \le \|v-w\| + \|v-v^*\| < 1/n + \delta_{v^*}/4 < \delta_{v^*}$, and similarly $\|z-v^*\| < \delta_{v^*}$. Since also $t,\tau \in (0,1/n) \subset (0,\delta_{v^*})$, we obtain by (4.4)

$$\begin{aligned}\frac{f(a+tw)-f(a)}{\tau} + \frac{f(a-tz)-f(a)}{t} &> f_H^-(a,v^*) - \frac{\varepsilon}{8} + f_H^-(a,-v^*) - \frac{\varepsilon}{8}\\ &> \varepsilon - \frac{\varepsilon}{4} > \frac{1}{n}.\end{aligned}$$

To prove (ii), it is sufficient to prove that if $U \in \mathcal{V}_k$ and $\rho(V,U) < 1/n$, then $a \in A_U$. To this end, choose an arbitrary $u \in S_U$. Since $\rho(V,U) < 1/n$, we can find $\tilde{v} \in S_V$ with $\|\tilde{v}-u\| < 1/n$ and by (4.5) $v \in F$ with $\|\tilde{v}-v\| < \delta_v/4$. Then $\|v-u\| < 1/n + \delta_v/4 < \delta_v$ and (4.3) implies

$$f_H^-(a,u) + f_H^-(a,-u) > f_H^-(a,v) - \frac{\varepsilon}{8} + f_H^-(a,-v) - \frac{\varepsilon}{8} > e(v) - \frac{\varepsilon}{4} > 0.$$

So clearly $a \in A_U$. $\square$

Lemma 4.2. *Let X be a Banach space and f a real function on X. Let $\dim X > k \in \mathbb{N}$ and let $V \subset X$ be a space of dimension k. Let A_V be as in Lemma 4.1. Then A_V can be covered by countably many V-Lipschitz surfaces.*

Proof: By Lemma 4.1(i), it is sufficient to prove that each $A := A_{V,n}$ (where $n \in \mathbb{N}$) can be covered by countably many V-Lipschitz surfaces. By Lemma 2.3, it is sufficient to prove that $T(A,x) \cap V = \{0\}$ for each $x \in A$.

Suppose to the contrary that there exist $x \in A$ and $v \in T(A,x) \cap V$. We can and will suppose that $\|v\| = 1$. To infer a contradiction, we will distinguish the cases $f_H^-(x,v) = \infty$ and $f_H^-(x,v) < \infty$.

If $f_H^-(x,v) = \infty$, set $\tau := \frac{1}{n}$, $y := x + \tau v$ and $p := |f(y) - f(x)|$. Since $v \in T(A,x)$ and $f_H^-(x,v) = \infty$, we can clearly find $z \in X$ with $\|z-v\| < 1/n$ and $0 < t < \frac{1}{4n^2}$ such that $a := x + tz \in A$ and $\frac{f(a)-f(x)}{t} > 2np$. Set $w := 2n(y-a)$. Then $\tau, t \in (0,1/n)$, $\|v-z\| < 1/n$ and also

$$\|v-w\| = \|2n(y-x) - 2n(y-a)\| = 2n\|a-x\| = 2nt\|z\| < 2n\cdot\frac{1}{4n^2}\cdot 2 = \frac{1}{n}.$$

Consequently, since $x = a - tz$, $y = a + \tau w$ and $a \in A = A_{V,n}$, we have by (4.2)

$$\frac{f(y)-f(a)}{\tau} > \frac{f(a)-f(x)}{t} + \frac{1}{n} > 2np.$$

Therefore

$$p \geq f(y)-f(x) = (f(y)-f(a))+(f(a)-f(x)) > \tau \cdot \frac{f(y)-f(a)}{\tau} + 0 \geq \frac{1}{2n} \cdot 2np = p,$$

which is a contradiction.

If $D := f_H^-(x,v) < \infty$, then the definition of $A = A_{V,n}$ yields $D \in \mathbb{R}$. By the definition of $f_H^-(x,v)$, we can find $\tilde{u} \in X$ with $\|\tilde{u}\| < \frac{1}{4n}$ and $0 < \tau < \frac{1}{4n}$ such that $y := x + \tau(v + \tilde{u})$ satisfies

$$\frac{f(y)-f(x)}{\tau} < D + \frac{1}{4n}\,. \tag{4.6}$$

Since $v \in T(A,x)$ and $f_H^-(x,v) = D \in \mathbb{R}$, we can clearly find $z \in X$ with $\|z - v\| < \frac{1}{4n}$ and $0 < t < \frac{\tau}{4n\,(|D-\frac{1}{4n}|+1)}$ such that $a := x + tz \in A$ and

$$\frac{f(a)-f(x)}{t} > D - \frac{1}{4n}\,. \tag{4.7}$$

Set $w := \frac{y-a}{\tau}$. Then $\tau, t \in (0, 1/n)$, $\|v - z\| < 1/n$ and also

$$\|w - v\| \leq \left\| \frac{y-a}{\tau} - \frac{y-x}{\tau} \right\| + \|\tilde{u}\| \leq \frac{\|a-x\|}{\tau} + \frac{1}{4n} \leq \frac{2t}{\tau} + \frac{1}{4n} < \frac{1}{n}\,.$$

Consequently, since $x = a - tz$, $y = a + \tau w$ and $a \in A = A_{V,n}$, we have by (4.2)

$$\frac{f(y)-f(a)}{\tau} > \frac{f(a)-f(x)}{t} + \frac{1}{n} > D - \frac{1}{4n} + \frac{1}{n} > D + \frac{1}{2n}\,. \tag{4.8}$$

Using (4.6), (4.8) and (4.7), we obtain

$$\begin{aligned} \tau\left(D + \frac{1}{4n}\right) &> f(y) - f(x) = (f(y)-f(a)) + (f(a)-f(x)) \\ &> \tau\left(D + \frac{1}{2n}\right) + t\left(D - \frac{1}{4n}\right). \end{aligned}$$

Consequently $t \cdot |D - \frac{1}{4n}| > \tau(\frac{1}{4n})$, which contradicts $t < \frac{\tau}{4n\,(|D-\frac{1}{4n}|+1)}$. □

Proof of Theorem 1.5: By definition of $\tilde{\Sigma}_{H,k}(f)$ and Lemma 4.1(ii), we have

$$\tilde{\Sigma}_{H,k}(f) = \bigcup_{V \in \mathcal{V}_k} A_V = \bigcup_{U \in \mathcal{D}_k} A_U\,.$$

Since $\mathcal{D}_k$ is countable, the assertion of Theorem 1.5 follows from Lemma 4.2. □

References

[1] Albano P., Cannarsa P., *Singularities of semiconcave functions in Banach spaces*, in: Stochastic Analysis, Control, Optimization and Applications, W.M. McEneaney, G.G. Yin, Q. Zhang (eds.), Birkhäuser, Boston, 1999, pp. 171–190.

[2] Alberti G., Ambrosio L., Cannarsa P., *On the singularities of convex functions*, Manuscripta Math. **76** (1992), 421–435.

[3] Anderson R.D., Klee V.L., Jr., *Convex functions and upper semi-continuous collections*, Duke Math. J. **19** (1952), 349–357.

[4] Cannarsa P., Sinestrari C., *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, Progress in Nonlinear Differential Equations and their Applications, 58, Birkhäuser, Boston, MA, 2004.

[5] Duda J., Zajíček L., *Smallness of singular sets of semiconvex functions in separable Banach spaces*, submitted.

[6] Ioffe A.D., *Typical convexity* (*concavity*) *of Dini-Hadamard upper* (*lower*) *directional derivatives of functions on separable Banach spaces*, J. Convex Anal. **17** (2010), 1019–1032.

[7] Kato T., *Perturbation Theory for Linear Operators*, Springer, Berlin, 1976.

[8] Nagai H.V., Luc D.T., Théra M., *Approximate convex functions*, J. Nonlinear Convex Anal. **1** (2000), 155–176.

[9] Nekvinda A., Zajíček L., *Gâteaux differentiability of Lipschitz functions via directional derivatives*, Real Anal. Exchange **28** (2002-2003), 287–320.

[10] Pavlica D., *On the points of non-differentiability of convex functions*, Comment. Math. Univ. Carolin. **45** (2004), 727–734.

[11] Preiss D., *Almost differentiability of convex functions in Banach spaces and determination of measures by their values on balls*, Collection: Geometry of Banach spaces (Strobl, 1989), London Math. Soc. Lecture Note Ser. 158, Cambridge University Press, Cambridge, 1990, pp. 237–244.

[12] Zajíček L., *On the differentiation of convex functions in finite and infinite dimensional spaces*, Czechoslovak Math. J. **29** (1979), 340–348.

[13] Zajíček L., *Differentiability of the distance function and points of multi-valuedness of the metric projection in Banach space*, Czechoslovak Math. J. **33 (108)** (1983), 292–308.

[14] Zajíček L., *On Lipschitz and D.C. surfaces of finite codimension in a Banach space*, Czechoslovak Math. J. **58 (133)** (2008), 849–864.

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