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G-nilpotent units of commutative group rings

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Abstract. Suppose R is a commutative unital ring and G is an abelian group. We give a general criterion only in terms of R and G when all normalized units in the commutative group ring RG are G-nilpotent. This extends recent results published in [Extracta Math., 2008–2009] and [Ann. Sci. Math. Québec, 2009].

Keywords: group rings, normalized units, nilpotents, idempotents, decompositions, abelian groups

Classification: 16S34, 16U60, 20K10, 20K20, 20K21

1. Introduction

Throughout the present paper, let it be agreed that all groups are multiplicatively written and abelian as is customary when studying group rings, and all rings are commutative with identity 1 (further called commutative *unital*). For such a ring R and a group G, suppose $N(R)$ is the nil-radical of R and G_t is the torsion part of G with p-component G_p. Likewise, suppose RG is the group ring of G over R with group of normalized units $V(RG)$. Standardly, $I(LG;G)$ is the fundamental ideal of LG where $L \leq R$ and $I(RG;H)$ is the relative augmentation ideal of RG with respect to $H \leq G$. As usual, imitating [13], $\mathrm{id}(R) = \{e \in R \mid e^2 = e\}$, $\mathrm{inv}(R) = \{p \mid p \cdot 1 \in U(R)\}$, where p is a prime number, $U(R)$ is the unit group of R, $\mathrm{zd}(R) = \{p \mid \exists r \in R \setminus \{0\} : pr = 0\}$, and $\mathrm{supp}(G) = \{p \mid G_p \neq 1_G\}$.

Following [8], [9] we define the idempotent subgroup $\mathrm{Id}(RG)$ as follows:

$$\mathrm{Id}(RG) = \Big\{e_1 g_1 + \cdots + e_k g_k \mid e_i \in \mathrm{id}(R),$$
$$\sum_i e_i = 1, e_i e_j = 0 \ (i \neq j), g_i \in G; 1 \leq i, j \leq k\Big\}.$$

It is self-evident that $\mathrm{Id}(RG)$ is a group and that $\mathrm{Id}(RG) \leq V(RG)$.

All other notations and notions are standard and follow essentially those from the survey paper [9] and the classical monographs [8], [10], [11] and [12].

The purpose of this article is to establish a necessary and sufficient condition in terms associated only with R and G when all normalized units are G-nilpotent.

Definition. A normalized unit $v \in V(RG)$ is said to be G-nilpotent if v can be uniquely expressed as $v = gw$ where $g \in G$ and $w \in 1 + I(N(R)G; G)$.

This is tantamount to ask when the decomposition

$$V(RG) = G \times (1 + I(N(R)G; G))$$

holds; note that $G \cap (1 + I(N(R)G; G)) = 1$.

Some explorations that are closely related to this theme are given in [3], [4] and [5] (compare with Section 2). Here we shall amend our technique and, as a result, we will generalize the main assertions from these papers.

2. Preliminaries and main results

Before proving the chief statements, we need some technicalities.

Lemma 1. *For each ring R the following equality is fulfilled:*

$$U(R/N(R)) = \{r + N(R) \mid r \in U(R)\}.$$

Proof: Clearly the left hand-side contains the right one because there exist $r, f \in R$ with $rf = 1$ and hence $(r + N(R))(f + N(R)) = rf + N(R) = 1 + N(R)$.

As for the converse inclusion, let $x \in U(R/N(R))$ be given. Then, $x = r + N(R)$ for some $r \in R$ such that there exists $f \in R$ with $(r + N(R))(f + N(R)) = rf + N(R) = 1 + N(R)$. Consequently, $rf - 1 \in N(R)$ which means that $rf \in 1 + N(R) \subseteq U(R)$. Therefore, it is easily seen that $r \in U(R)$ as required. □

Lemma 2. *For any ring R the following equality holds:*

$$\mathrm{inv}(R) = \mathrm{inv}(R/N(R)).$$

Proof: Assume that $p \in \mathrm{inv}(R)$. Then $p{\cdot}1 \in U(R)$ and hence in view of Lemma 1 we have $p(1 + N(R)) = p \cdot 1 + N(R) \in U(R/N(R))$. Thus $p \in \mathrm{inv}(R/N(R))$ and the inclusion "$\subseteq$" is obtained.

As for the converse containment "$\supseteq$", choose $p \in \mathrm{inv}(R/N(R))$, whence $p(1 + N(R)) \in U(R/N(R))$. In accordance with Lemma 1 we may write $p \cdot 1 + N(R) = \alpha + N(R)$ where $\alpha \in U(R)$. Furthermore, $p \cdot 1 \in U(R) + N(R) = U(R)$ so that $p \in \mathrm{inv}(R)$, as required. □

Let R be a ring. Define $\mathrm{np}(R) = \{p \,|\, \exists r \in R \setminus N(R) : pr \in N(R)\}$. The following claim is useful.

Lemma 3. *For every ring R the following equality is true:*

$$\mathrm{zd}(R/N(R)) = \mathrm{np}(R).$$

PROOF: Given $p \in \mathrm{zd}(R/N(R))$, there is $r \notin N(R)$ such that $p(r + N(R)) = pr + N(R) = N(R)$. Thus $pr \in N(R)$ and $p \in \mathrm{np}(R)$.

Conversely, let $p \in \mathrm{np}(R)$. Then there is $r \in R \setminus N(R)$ with $pr \in N(R)$. Consequently, $p(r + N(R)) = N(R)$ and $r + N(R) \neq N(R)$ which implies that $p \in \mathrm{zd}(R/N(R))$. $\square$

Lemma 4. *Suppose R is a ring. Then*

$$\mathrm{id}(R) = \{0, 1\} \iff \mathrm{id}(R/N(R)) = \{0, 1\}.$$

PROOF: "$\Rightarrow$". Because of the classical fact that idempotents can always be lifted through $N(R)$ (see, e.g., [1]) if $R/N(R)$ has a non-trivial idempotent, then the same must be true of R, a contradiction.

"$\Leftarrow$". Choose an arbitrary element $r \in R$ with $r^2 = r$, hence $r + N(R) = r^2 + N(R) = (r + N(R))^2$. Therefore, either $r + N(R) = N(R)$, whence $r \in N(R)$ and thus $r = 0$, or $r + N(R) = 1 + N(R)$, whence $r \in 1 + N(R) \subseteq U(R)$. But then $r(1 - r) = 0$ ensures that $1 - r = 0$ that is $r = 1$, as required. $\square$

Another topological approach in proving the above can be based on the following two standard facts in commutative ring theory:

Let A be any commutative unital ring. Then the following are true (e.g., cf. [1]):

(i) A has no non-trivial idempotents if and only if $\mathrm{Spec}(A)$, the set of prime ideals of A equipped with the Zariski topology, is a connected topological space;

(ii) the canonical surjection from $\mathrm{Spec}(A/N(A))$ to $\mathrm{Spec}(A)$, sending $P + N(A)$ to P, is a homeomorphism (relative to the Zariski topology on each space).

Proposition 5. *Suppose R is a ring and $\phi : R \to R/N(R)$ is the natural map. Define $\Phi : RG \to (R/N(R))G$ and its restriction $\Phi_{V(RG)} : V(RG) \to V((R/N(R))G)$ by $\Phi(\sum_{g\in G} r_g g) = \sum_{g\in G} \phi(r_g)g = \sum_{g\in G}(r_g + N(R))g$. Then the following relations are valid:*

(a) *Φ is a surjective homomorphism;*

(b) $\ker \Phi = N(R)G$ *and* $\ker \Phi_{V(RG)} = 1 + I(N(R)G; G)$.

PROOF: (a) That Φ is a ring (and hence a group) homomorphism follows easily since so is ϕ.

As for the epimorphism (= surjection), we will restrict our attention only on $V(RG)$ because for RG this is evident. And so, choose $x \in V((R/N(R))G)$ whence there is $y \in RG$ with $\Phi(y) = x$. Moreover, there are $x' \in (R/N(R))G$ such that $xx' = 1$ and $y' \in RG$ such that $\Phi(y') = x'$. Therefore, $1 = \Phi(y)\Phi(y') = \Phi(yy')$, so that $\Phi(yy' - 1) = 0$ and point (b) below applies to write that $yy' - 1 \in N(RG)$. Finally, $yy' \in 1 + N(RG) \subseteq U(RG)$ and thus $y \in U(RG)$. Furthermore, since $U(RG) = V(RG) \times U(R)$, $U((R/N(R))G) = V((R/N(R))G) \times U(R/N(R))$ and $\Phi(V(RG)) \subseteq V((R/N(R))G)$, $\Phi(U(R)) \subseteq U(R/N(R))$, it easily follows now that $\Phi(V(RG)) = V((R/N(R))G)$, as expected.

(b) Clearly, $1 + I(N(R)G;G) \subseteq V(RG)$ because $I(N(R)G;G) \subseteq N(R)G \subseteq N(RG)$.

On the other hand, it is plainly seen that $\ker\Phi = N(R)G$. Moreover, one checks that $\ker\Phi_{V(RG)} = (1 + I(RG;G)) \cap (1 + N(R)G) = 1 + I(N(R)G;G)$ as asserted. $\square$

Remark 1. Actually, the pre-image y can be chosen with augmentation 1, and therefore $y \in U(RG)$ directly implies that $y \in V(RG)$. In fact, if $x = (r_1 + N(R))g_1 + \cdots + (r_s + N(R))g_s$ with $r_1 + \cdots + r_s - 1 = \alpha \in N(R)$, then $y = r_1g_1 + \cdots + r_sg_s - \alpha 1_G$ satisfies the required property that $\Phi(y) = x$ and $\mathrm{aug}(y) = 1$.

Proposition 6. *Suppose G is a group and R is a ring. Then the following equivalence holds:*

$$V(RG) = G \times (1 + I(N(R)G;G)) \iff V((R/N(R))G) = G.$$

PROOF: "$\Rightarrow$". Applying Proposition 5(a) and taking Φ in the both sides of the given equality, we derive that $\Phi(V(RG)) = \Phi(G)\Phi(1 + I(N(R)G;G))$. This is equivalent to $V((R/N(R))G) = G$ because $\Phi(G) = G$ and $\Phi(1 + I(N(R)G;G)) = 1$, as stated.

"$\Leftarrow$". Choose an arbitrary element $x \in V(RG)$. We have $\Phi(x) \in V((R/N(R))G) = G$. Thus we may write $\Phi(x) = g = \Phi(g)$ for some $g \in G$. Furthermore, $\Phi(x)[\Phi(g)]^{-1} = \Phi(x)\Phi(g^{-1}) = \Phi(xg^{-1}) = 1$. Hence $xg^{-1} \in \ker\Phi_{V(RG)} = 1 + I(N(R)G;G)$ utilizing Proposition 5(b). Finally, $x \in G \times (1 + I(N(R)G;G))$ as required. $\square$

The following statement is an amended version of [3, Proposition].

Proposition 7. *Suppose G is a group with $|G| = 3$ and R is a ring such that $3 \in \mathrm{inv}(R)$. Then $V(RG) = G$ if and only if $U(R) = 1$ and the equation $r^2 + f^2 + rf + r + f = 0$ has only trivial solutions in R.*

PROOF: "$\Rightarrow$". What we need to show is that $\mathrm{char}(R) = 2$. Assume the contrary, $2 \neq 0$. Then we observe that $\frac{2}{3} + \frac{2}{3}g - \frac{1}{3}g^2$ is a non-trivial unit with the inverse $\frac{2}{3} - \frac{1}{3}g + \frac{2}{3}g^2$. This contradiction allows us to conclude that $2 = 0$. Furthermore, we apply the proof of Proposition on p. 51 from [3] to deduce that $U(R) = 1$ and $r^2 + f^2 + rf + r + f = 0$ is possible unique when $r = 0$, $f = 0$ or $r = 1$, $f = 0$ or $r = 0$, $f = 1$.

"$\Leftarrow$". Certainly $U(R) = 1$ implies that $-1 = 1$, i.e., $2 = 0$. Thus $\mathrm{char}(R) = 2$ and the further argument follows as that in [3, p. 51, Proposition]. $\square$

Remark 2. Note also that $2 \notin U(R)$ since otherwise $\frac{1}{2} + \frac{1}{2}g \in V(RG)$ with the inverse $1 - g + g^2$. Moreover, we point out that the equations here and in [3] are the same, which follows via the substitutions $a = 1 + r$ and $b = 1 + f$.

Now we list the following criterion from [4] which will be useful in the sequel.

Theorem A. *Let R be a ring and G a group. Then $V(RG) = G$ if and only if $\mathrm{id}(R) = \{0, 1\}$, $N(R) = 0$, $V(RG_t) = G_t$ and precisely one of the following conditions is true:*

(1) $G = G_t$;
(2) $G \neq G_t$, $\mathrm{supp}(G) \cap (\mathrm{inv}(R) \cup \mathrm{zd}(R)) = \emptyset$.

Now we are planning to give a new, more conceptual, proof of the following result from [3].

Theorem B. *Suppose G is a group and R is a ring such that* $\mathrm{supp}(G) \cap \mathrm{inv}(R) \neq \emptyset$. *Then $V(RG) = G$ if and only if* $\mathrm{id}(R) = \{0, 1\}$, $N(R) = 0$ *and at most one of the following conditions holds:*

(1) $|G| = |U(R)| = 2$;
(2) $|G| = 3$, $U(R) = 1$ *and the equation* $a^2 + b^2 + ab + 1 = 0$ *has only trivial solutions in R for each pair (a, b).*

Proof: "$\Rightarrow$". If either the set $\mathrm{id}(R)$ contains a non-trivial idempotent e or the nil-ideal $N(R)$ contains a non-trivial nilpotent r, taking $g \in G$ we can construct one of the elements $x_e = eg + 1 - e$ or $x_r = 1 - r + rg$ — for each of them it is easily verified that $x_e \in V(RG) \setminus G$ with inverse $x_e^{-1} = eg^{-1} + 1 - e$, or $x_r \in V(RG) \setminus G$ as the sum of 1 and the nilpotent $-r + rg = r(g-1)$, a contradiction in each of the two situations. That is why both $\mathrm{id}(R)$ and $N(R)$ are trivial.

Claim that G is finite of order 2 or 3. In fact, assume in a way of contradiction that G is infinite. Since there is a prime, say q, such that $G_q \neq 1$ and $q \in \mathrm{inv}(R)$, it is well known that there exists an idempotent $e \in RF$ where $F \leq G_q$ is a finite subgroup. Choose $g \notin F$ (this choice is possible since G is infinite while F is finite) and in the same manner as above one can construct the element $x_e = eg + 1 - e \in V(RG) \setminus G$. Thus G is necessarily finite. By the same reason, it follows that G does not contain proper subgroups, that is, G is of prime cardinality — thereby $|G|$ is a prime, say q. Furthermore, we claim that G has cardinality 2 or 3. To show this, we assume the contrary that $|G| \geq 5$ and consider the element $u = (1+g)^{q-1} - \frac{2^{q-1}-1}{q}(1 + g + \cdots + g^{q-1})$ where $G = \langle g \rangle$ with $g^q = 1$. It is well known that u is a unit with augmentation 1 which does not lie in G (see, e.g., [12]). This contradiction shows that $|G| \leq 4$. Finally, either $|G| = 2$ or $|G| = 3$ as claimed.

Moreover, another approach is to notice that there is a nontrivial idempotent $e = \frac{1}{2}(1+g)$ or $e = \frac{1}{3}(1 + g + g^2)$ where g is either of order 2 or 3. If $g' \notin \langle g \rangle$, then $1 - e + eg'$ is a nontrivial unit.

Next, we consider separately these two possibilities:

Case 1. G is cyclic of order 2.

Firstly, note that $2 \in U(R)$. We claim that if $r \in U(R)$ is an arbitrary element, then either $r = 1$ or $r = -1$; so $2 = -1$ and hence $3 = 0$ since $2 = 1$ does not hold. In fact, consider the element $x_r = \frac{1}{2} - \frac{r}{2} + (\frac{1}{2} + \frac{r}{2})g$. It is simple checked that $x_r \in V(RG)$ with the inverse $x_{r^{-1}} = \frac{1}{2} - \frac{r^{-1}}{2} + (\frac{1}{2} + \frac{r^{-1}}{2})g$. Since there exist only trivial units, it must be fulfilled that $\frac{r}{2} = \frac{1}{2}$ or $\frac{r}{2} = -\frac{1}{2}$, i.e., $r = 1$ or $r = -1$. Thus $U(R)$ has only two elements, as claimed.

Case 2. G is cyclic of order 3.

It follows immediately from Proposition 7.

"$\Leftarrow$". (1) First, note that $1 \neq -1$ and $\mathrm{char}(R) = 3$ because $2 \in U(R) = \{1, -1\}$ and thus $2 = -1$; the equality $2 = 1$ is impossible since it yields that $1 = 0$. Let $x_r = 1 - r + rg$. Then, there is $f \in R$ such that $(1 - r + rg)(1 - f + fg) = 1$. This is equivalent to $f(2r - 1) = r$. Since $2rf - r - f = 0$, we have $(2r - 1)(2f - 1) = 1$ and it must be that $2r - 1 \in U(R)$. Consequently, $2r - 1 = 1$ or $2r - 1 = -1$. Thus $2r = 2$, whence $r = 1$, or $2r = 0$, whence $r = 0$. Finally, either $x_r = 1$ or $x_r = g$. In both cases we observe that $V(RG) = G$, as expected.

(2) Follows by a direct application of Proposition 7. □

Remark 3. First, notice that in clause (2) we must have $\mathrm{char}(R) = 2$ if $\mathrm{char}(R)$ is a prime integer. In fact, always $-1 \in U(R)$ and since $U(R) = 1$, we have that $-1 = 1$ which is tantamount to $2 = 0$ as asserted.

Certainly, in the Main Theorem from [3], point (1) $G = 1$ is not realistic and cannot be happen since $\mathrm{supp}(G) \neq \emptyset$.

The question of the triviality of units in commutative group rings will be completely exhausted if the following can be settled:

Problem 1. Find a criterion only in terms associated with R and G when $V(RG) = G$ holds, provided that $G = G_t$ and $\mathrm{supp}(G) \cap \mathrm{inv}(R) = \emptyset$.

We have now at our disposal all the information needed to prove the following.

Theorem 8. *Suppose G is a group and R is a ring. Then $V(RG) = G \times (1 + I(N(R)G; G))$ if and only if $\mathrm{id}(R) = \{0, 1\}$, $V(RG_t) = G_t \times (1 + I(N(R)G_t; G_t))$ and at most one of the following conditions holds:*

(1) $G = G_t$;
(2) $G \neq G_t$, $\mathrm{supp}(G) \cap (\mathrm{inv}(R) \cup \mathrm{np}(R)) = \emptyset$.

Proof: Employing Proposition 6 we equivalently reduce the decomposition of $V(RG)$ to the equality $V((R/N(R))G) = G$. Next, we subsequently apply Theorem A combined with Lemmas 2, 3 and 4. □

Theorem 9. *Suppose G is a group and R is a ring such that $\mathrm{supp}(G) \cap \mathrm{inv}(R) \neq \emptyset$. Then $V(RG) = G \times (1 + I(N(R)G; G))$ if and only if $\mathrm{id}(R) = \{0, 1\}$ and exactly one of the following points is valid:*

(1) $|G| = |U(R/N(R))| = 2$;
(2) $|G| = 3$, *$U(R/N(R)) = 1$ and the relation $a^2 + b^2 + ab + 1 \in N(R)$ has only trivial solutions in $R/N(R)$ for every pair $(a, b) \in R$.*

Proof: By application of Proposition 6 we can write in an equivalent way that $V((R/N(R))G) = G$. Hereafter we subsequently employ Theorem B together with Lemma 2 and Lemma 4. □

As a consequence, we deduce

Corollary 10 ([5])**.** *Suppose $\mathrm{char}(R) = p$ is a prime integer and $G \neq 1$. Then $V(RG) = G \times (1 + I(N(R)G; G))$ if and only if $\mathrm{id}(R) = \{0, 1\}$ and at most one of the following holds:*

(a) $G_t = 1$;
(b) $|G| = p = 2$, $R = L + N(R)$ *with* $|L| = 2$;
(c) $p = 3$, $|G| = 2$ *and* $U(R) = \pm 1 + N(R)$;
(d) $p = 2$, $|G| = 3$, $U(R) = 1 + N(R)$ *and the equation* $X^2 + XY + Y^2 = 1 + N(R)$ *possesses only trivial solutions in* $R/N(R)$.

Proof: First of all, observe that $\mathrm{inv}(R)$ contains all primes but p. That R is indecomposable follows easily since $1-r+rg \in V(RG)$ is always a non-G-nilpotent unit whenever $r \in \mathrm{id}(R) \setminus \{0,1\}$ and $g \in G \setminus \{1\}$. Moreover, if G is torsion-free, everything was done in [6], [7] (see [8] and [9] as well). So, assume $G_t \neq 1$. Further, if $G_t \neq G_p$ we see that $\mathrm{supp}(G) \cap \mathrm{inv}(R) \neq \emptyset$ and hence Theorem 9 applies to get the result. If now G is p-mixed, i.e., $G_t = G_p$, it follows that $V(RG) = G(1 + I(RG; G_p) + I(N(R)G; G))$. Hereafter, the proof goes on by arguments similar to these from [5] considering the cases $G = G_t$ and $G \neq G_t$. The first one leads to $|G| = 2 = p$, while the second one is impossible. □

Finally, we will apply the results alluded to above to derive a recent achievement from [2]. First, we need the following technicality.

Lemma 11. *Let* $\mathrm{char}(R) = p$ *be a prime integer. Then*

$$V(RG) = GV_p(RG) \iff V(R(G/G_p)) = (G/G_p)V_p(R(G/G_p)).$$

Proof: Consider the natural map $\psi : G \to G/G_p$. It is well known that it can be linearly extended to the homomorphism $\Psi : V(RG) \to V(R(G/G_p))$ with kernel $1 + I(RG; G_p)$. Since $1 + I(RG; G_p) \subseteq V_p(RG)$, it easily follows by standard arguments that Ψ is actually an epimorphism (= surjective homomorphism). Moreover, it is also clear that $\Psi(V_p(RG)) = V_p(R(G/G_p))$. So, under the action of Ψ on the both sides of $V(RG) = GV_p(RG)$ we immediately obtain that $V(R(G/G_p)) = (G/G_p)V_p(R(G/G_p))$ holds, as stated.

As for the sufficiency, choose an arbitrary element $x \in V(RG)$ and observe that there is $y \in V(R(G/G_p))$ such that $\Psi(x) = y$. Write $y = g'v'$ where $g' \in G/G_p$ and $v' \in V_p(R(G/G_p))$. Since by what we have shown above there exist $g \in G$ and $v \in V_p(RG)$ such that $\Psi(g) = g'$ and $\Psi(v) = v'$, we get $\Psi(x) = \Psi(gv)$. Furthermore, $\Psi(xg^{-1}v^{-1}) = 1$ and thus $xg^{-1}v^{-1} \in \ker \Psi \subseteq V_p(RG)$ as previously noticed. This leads to $x \in GV_p(RG)$, as required. □

So, we are ready to prove the following affirmation.

Proposition 12 ([2]). *Suppose* $\mathrm{char}(R) = p$ *is a prime natural number. Then* $V(RG) = GV_p(RG)$ *if and only if*

(1) $G = G_p$ *or*
(2) $G \neq G_p$, R *is indecomposable and precisely one of the following points holds:*
  (2.1) $G_t = G_p$;
  (2.2) $p = 3$, $U(R) = \pm 1 + N(R)$ *and* $G = G_p \times C$ *with* $|C| = 2$;

(2.3) $p = 2$, $U(R) = 1 + N(R)$, *the equality* $X^2 + XY + Y^2 = 1 + N(R)$ *has only trivial solutions in* $R/N(R)$ *and* $G = G_p \times C$ *with* $|C| = 3$.

Proof: By virtue of Lemma 11, we may with no harm of generality assume that $G_p = 1$. Since it is plainly checked that then $V_p(RG) = 1 + I(N(R)G; G)$, we obviously deduce that $V(RG) = G \times (1+I(N(R)G;G))$ — see also [5]. Henceforth, we employ the main theorem from [5] or, respectively, Corollary 10. □

We close the work with the following:

Problem 2. Find a necessary and sufficient condition when the equality

$$V(RG) = G \times (1 + I(N(R)G; G))$$

holds, provided that $\mathrm{supp}(G) \cap \mathrm{inv}(R) = \emptyset$.

In particular, as an immediate consequences, we will extract the cases $G_t = 1$ (Karpilovsky) and $R = \mathbb{Z}$ (May).

In conclusion, one can expect that if $\mathrm{supp}(G) \cap \mathrm{zd}(R) \neq \emptyset$, then there is a non-$G$-nilpotent unit. However, this is not generally true. For instance, a counterexample may be obtained for rings of characteristic 4 by taking $R = \mathbb{Z}_4 = \mathbb{Z}/(4)$ (i.e., R to be the ring of all integers modulo 4) and G is of order 2. There are only four elements of augmentation 1, so that the computations are minimal. If now a counterexample of a ring of characteristic 0 is desired, let G be of order 2 again and let $R = \mathbb{Z}[x]$ be the polynomial ring of x over $\mathbb{Z}$ where the element x is subject to the relations $x^2 = 2x = 0$.

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