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Distinct equilateral triangle dissections of convex regions

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Abstract. We define a proper triangulation to be a dissection of an integer sided equilateral triangle into smaller, integer sided equilateral triangles such that no point is the vertex of more than three of the smaller triangles. In this paper we establish necessary and sufficient conditions for a proper triangulation of a convex region to exist. Moreover we establish precisely when at least two such equilateral triangle dissections exist.

We also provide necessary and sufficient conditions for some convex regions with up to four sides to have either one, or at least two, proper triangulations when an internal triangle is specified.

Keywords: equilateral triangle dissection, latin trade

Classification: 05B45

1. Introduction

The dissection of an integer sided equilateral triangle into smaller, integer sided equilateral triangles is a classic problem considered by Tutte [12]. He showed various properties of such a dissection, including the fact that some of the smaller triangles must have equal sides.

If we apply an extra restriction to such a dissection, namely that no point is the vertex of more than three of the smaller triangles, then the dissection gives rise to a *latin trade* within the addition table for the integers modulo n ([3]). We call such a dissection a proper triangulation. (It was Drápal, in [3], who first observed the connection between latin trades and proper triangulations, and as a consequence of this in some papers (see [2] and [11]) proper triangulations are also known as Drápal Triangulations.) More details about this connection and latin trades may be found in [1], [3], [4], [6] and [7].

This application of triangle dissections to latin trades is our key motivation. In particular, the results in this paper are applied to classify flaws in cryptographic applications of latin squares [2]. However, the results have some geometric interest in their own right.

It is conjectured that there exists a constant c such that for each integer n, there exists a non-trivial proper triangulation of an integer sided equilateral triangle containing at most $c \log p$ triangles, where p is the least prime that divides n ([3]). The results in this paper may provide insights into this question. A further possible application is the enumeration of proper triangulations (see [5]). Other, laterally related result on triangulations include [8], [9] and [10].

In Section 2 we introduce necessary terminology. In Section 3 we establish precisely when a convex regions has at least one or at least two proper triangulations (Theorem 3.6). In Section 4 we consider the same question when an internal triangle is specified; our results are restricted to convex region with at most four sides which are not rectangles.

2. Proper triangulations

For ease of notation we consider the equivalent problem of dissections of right-angled isosceles triangles into smaller such objects, where each triangle has hypotenuse of gradient -1. To see this equivalence, consider such a dissection with the large triangle lying in the first quadrant with its right angle at the origin. Then the linear transformation $T(x) = xA$ given by

$$A = \begin{bmatrix} 1 & 0 \\ 0.5 & \sqrt{3}/2 \end{bmatrix}$$

shows the equivalence to an equilateral triangle dissection.

Let $k, i, x_i, y_i \in \mathbb{Z}$ and $0 \le i \le k-1$. Let $R = (x_0, y_0), (x_1, y_1), \ldots, (x_i, y_i), \ldots, (x_{k-1}, y_{k-1})$ be a sequence of points which satisfies the following condition: for all $0 \le i \le k-1$,

$$x_i = x_{i+1 \,(\mathrm{mod}\ k)} \text{ or } y_i = y_{i+1 \,(\mathrm{mod}\ k)} \text{ or } x_i + y_i = x_{i+1 \,(\mathrm{mod}\ k)} + y_{i+1 \,(\mathrm{mod}\ k)}.$$

Then we say that R is a *region* in the plane $\mathbb{R}^2$. The *reduced form* R' of R is formed by successively deleting any points (x_i, y_i) from R whenever (x_{i-1}, y_{i-1}), (x_i, y_i) and (x_{i+1}, y_{i+1}) are collinear.

If the straight line segments between $(u_i, v_i) \in R'$ and $(u_{i+1 \,(\mathrm{mod}\ l)}, v_{i+1 \,(\mathrm{mod}\ l)}) \in R'$, $0 \le i \le l-1 = |R'|-1$, form the boundary of a convex polygon (where R' is the reduced form of R), then R is called a *convex region*. Furthermore, the region R is denoted by $R = (x_0, y_0) \to (x_1, y_1) \to \ldots \to (x_i, y_i) \to \ldots \to (x_{k-1}, y_{k-1})$, and if $1 < |R|$, we refer to the elements of the reduced form of R as the *corners* of R.

If the reduced form of R has precisely three corners, then R is said to be a *triangle*. Let $0 \le x$, denote the region

$$FT_x^{(z_1,z_2)} = (z_1, z_2) \to (z_1 + x, z_2) \to (z_1, z_2 + x) \text{ as a } \textit{forward triangle} \text{ and}$$

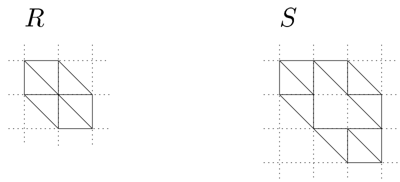
$$BT_x^{(z_1,z_2)} = (z_1, z_2) \to (z_1 - x, z_2) \to (z_1, z_2 - x) \text{ as a } \textit{backward triangle}.$$

Let R be the union of regions $R_1, R_2, \ldots, R_t$; that is, $R = \bigcup_{1 \le i \le t} R_i$. If for each $1 \le i < j \le t$, the regions R_i and R_j intersect in at most their respective boundaries, then $\{R_i \mid 1 \le i \le t\}$ is called a *tessellation* of R and each R_i is a *subregion* of R.

If each of the subregions R_i is a triangle, R is said to have a *triangulation*, namely $\{R_i \mid 1 \le i \le t\}$, furthermore each subregion, R_i, is referred to as a *subtriangle* of R. If, in addition, each element $(a, b) \in R$ is the corner of at most three distinct subtriangles, $\{R_i \mid 1 \le i \le t\}$ is called a *proper triangulation* of the region R. It is this property which makes the problem of finding proper triangulations of a specified region non-trivial.

Example 2.1. In Figure 1 we provide an example of a region, R, that has a triangulation but no possible proper triangulation and a region S that has a proper triangulation.

FIGURE 1. A triangulation and a proper triangulation



Consider the following group of matrices, isomorphic to the Dihedral group D_6:

$$G = \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \right\rangle.$$

Let $\lambda \in G$, $(p, q) \in \mathbb{R}^2$ and $S \subset \mathbb{R}^2$. In this paper the set $\{(m, n)\lambda + (p, q) \mid (m, n) \in S\}$ is denoted by $S\lambda + (p, q)$.

If there exists some $(i, j) \in \mathbb{Z}^2$ and some $\lambda \in G$ such that $R_2 = R_1\lambda + (i, j)$, then R_1 and R_2 are said to be *equivalent*. Observe that the property of possessing a proper triangulation is invariant under this equivalence, even though the gradients of lines may change. We frequently make use of this observation.

Recall that, for a proper triangulation, the condition that each vertex of a triangle is the vertex of at most three triangles must be satisfied. Suppose that $\{R_i \mid 1 \le i \le p\}$ is a tessellation of a region R and that each subregion R_i has a proper triangulation Q_i, where $1 \le i \le p$. Then the set of triangles

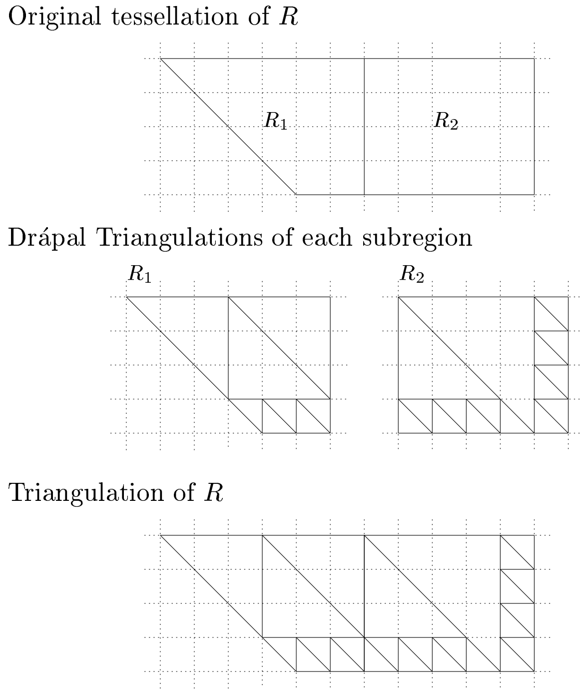
$$\bigcup_{1 \le i \le p} Q_i$$

does not necessarily form a proper triangulation of R.

Example 2.2. In Figure 2 we provide an example of a tessellation of a region R for which each region has a proper triangulation and the union of the subtriangles does not yield a proper triangulation of the region R.

To avoid this problem, whenever two distinct regions R_i and R_j in the tessellation of R both have a triangulation containing more than one triangle, then we ensure that their boundaries do not share a line segment of non zero length.

Figure 2. Failing to construct a proper triangulation



Example 2.3. In Figure 3 we provide an example of a tessellation of a region R for which each region has a proper triangulation and the union of the subtriangles yields a proper triangulation of the region R.

3. Proper triangulations of convex regions

For $\alpha, \beta \in \mathbb{Z}$, let:

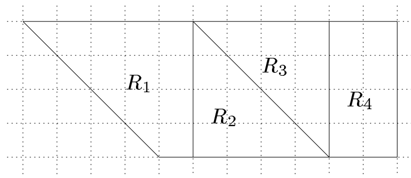
$Z_0 = (0,0)$,
$Z_1 = (0,0) \to (1,0) \to (1,\alpha) \to (0,\alpha)$ where $0 < \alpha$,
$Z_2 = (1,0) \to (\alpha,0) \to (\alpha,1) \to (0,1)$ where $0 < \alpha$,
$Z_3 = (\alpha,0) \to (\alpha,\beta) \to (0,\beta) \to (0,\beta-1) \to (\alpha-1,\beta-1) \to (\alpha-1,0)$ where $1 < \alpha, \beta$,
$Z_4 = (2,0) \to (2,2) \to (0,2) \to (0,1) \to (1,0)$,
$Z_5 = (3,0) \to (3,1) \to (1,3) \to (0,3) \to (0,1) \to (1,0)$ and
$Z_6 = (2,0) \to (2,1) \to (1,2) \to (0,2) \to (0,1) \to (1,0)$.

Let $\mathcal{Z}$ be the set of all regions equivalent to any Z_i, where $0 \leq i \leq 6$ (see the Appendix for an illustration of these regions).

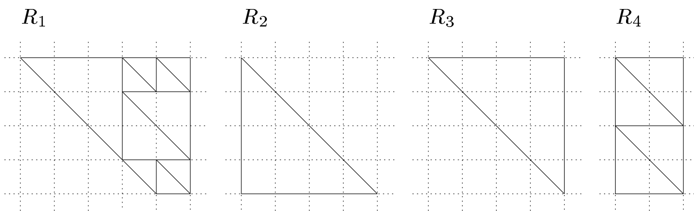
By inspection, the regions equivalent to Z_i where $0 \leq i \leq 5$ have a unique proper triangulation, while Z_6 has no proper triangulation. The aim of this

Figure 3. Constructing a proper triangulation

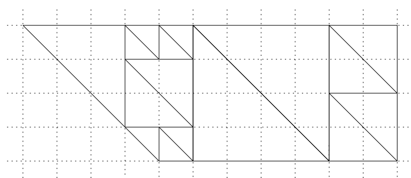
Original tessellation of R



Drápal Triangulations of each subregion



Drápal Triangulation of R



section is to show that any convex region not belonging to $\mathcal{Z}$ has at least two proper triangulations (Theorem 3.6).

We begin by investigating when it is possible for a region with three or four sides to have at least two distinct proper triangulations.

Lemma 3.1. *Let R be a region with three or four corners (sides). Thus R is equivalent to*

$$\begin{aligned} R_1 &= (\delta,0) \to (0,\delta) \to (0,0), \text{ or} \\ R_2 &= (\beta,0) \to (\beta,\alpha) \to (0,\alpha) \to (\gamma,0), \end{aligned}$$

where $0 < \delta$, and either $\gamma = 0$ and $0 < \alpha \leq \beta$, or $\gamma = \alpha$ and $0 < \alpha < \beta$. Then R has a proper triangulation. Further, if $1 < \delta$, there exists a second distinct proper triangulation of R_1 and, unless R_2 is equivalent to Z_1 or Z_2, there exists a second distinct proper triangulation of R_2.

Proof: Since R_1 is a triangle, a proper triangulation trivially exists. If $\delta = 1$, then, by inspection, $R_1 = FT_1^{(0,0)}$ has precisely one proper triangulation. However if $1 < \delta$, then

$$\{FT_{\delta-1}^{(1,0)}\} \cup \{FT_1^{(0,0)}\} \cup \bigcup_{1\le i\le \delta-1} \{FT_1^{(0,i)}, BT_1^{(1,i)}\}$$

is a second distinct proper triangulation of R_1.

There are two cases to consider for R_2: Case A, $\gamma = 0$ and Case B, $\gamma = \alpha$.

Case A: $\gamma = 0$.

Consider the tessellation $\{FT_\alpha^{(0,0)}, BT_\alpha^{(\alpha,\alpha)}\ R_3 = (\beta,0) \to (\beta,\alpha) \to (\alpha,\alpha) \to (\alpha,0)\}$ of R_2. If $\alpha = \beta$, then R_3 is empty and we are done. Otherwise, R_3 is a rectangular region with area strictly less than $\alpha\beta$. Thus, by recursion, R_1 has a proper triangulation.

If $\alpha = 1$, then R is equivalent to Z_1 and by inspection it has precisely one proper triangulation. When $1 < \alpha$, the second distinct proper triangulation is obtained by applying the argument given for R_1 to the triangle $FT_\alpha^{(0,0)}$.

Case B: $\gamma = \alpha$.

Let $1 < \alpha$. Consider the tessellation $\{BT_\alpha^{(\alpha,\alpha)}, S = (\beta,0) \to (\beta,\alpha) \to (\alpha,\alpha) \to (\alpha,0)\}$ of R_2. If $1 < \beta - \alpha$, the argument presented in Case A implies S has two distinct proper triangulations. If $\beta - \alpha = 1$, then the above gives one proper triangulation. Consider the proper triangulation $\{BT_\alpha^{(\beta,\alpha)}\} \cup \bigcup_{1\le i\le \alpha}\{FT_1^{(i,\alpha-i)}, BT_1^{(i,\alpha-i+1)}\}$ of R_2. This yields a second distinct proper triangulation of R_2.

If $\alpha = 1$, then R_2 is equivalent to Z_2 and by inspection it has precisely one proper triangulation. □

An L-region will be defined to be a region equivalent to

$$(\delta,0) \to (\delta,\beta) \to (0,\beta) \to (0,\alpha) \to (\gamma,\alpha) \to (\gamma,0),$$

where $0 < \alpha < \beta$ and $0 < \gamma < \delta$.

In order to obtain a similar result to Lemma 3.1 for convex regions with five sides we first prove the following result detailing when an L-region has at least two distinct proper triangulations.

Lemma 3.2. *Let $0 < \alpha < \beta$ and $0 < \gamma < \delta$. The L-region $\mathsf{L}_1 = (\delta,0) \to (\delta,\beta) \to (0,\beta) \to (0,\alpha) \to (\gamma,\alpha) \to (\gamma,0)$ has a proper triangulation, and a second distinct proper triangulation when L_1 is not equivalent to Z_3.*

Proof: Several cases are considered which, together with the associated conditions, are summarized in the following table.

Case A	**Case B**	**Case C**	**Case D**
$\alpha+\gamma \ge \delta,\beta$	$\delta \le \alpha+\gamma < \beta$	$\beta \le \alpha+\gamma < \delta$	$\alpha+\gamma < \beta,\delta$

Case A: $\alpha + \gamma \ge \delta, \beta$.

Consider the tessellation $R = \{BT^{(\delta,\beta)}_{\beta+\delta-\alpha-\gamma}, R_1 = (\delta, 0) \to (\delta, \alpha+\gamma-\delta) \to (\gamma, \alpha) \to (\gamma, 0), R_2 = (\gamma, \alpha) \to (\alpha+\gamma-\beta, \beta) \to (0, \beta) \to (0, \alpha)\}$ of L_1. By Lemma 3.1, L_1 has a proper triangulation.

Provided R_1 or R_2 are not both equivalent to Z_2 then Lemma 3.1 implies there exists a second distinct proper triangulation of L_1. When both R_1 and R_2 are equivalent to Z_2, then $\beta - \alpha = \delta - \gamma = 1$, L_1 is equivalent to Z_3 and by inspection it has precisely one proper triangulation.

Case B**:** $\delta \le \alpha + \gamma < \beta$**.**

Consider the tessellation $\{BT^{(\delta,\alpha+\gamma)}_{\delta}, FT^{(0,\alpha)}_{\gamma}, R_1 = (\delta, 0) \to (\delta, \alpha+\gamma-\delta) \to (\gamma, \alpha) \to (\gamma, 0), R_2 = (\delta, \alpha+\gamma) \to (\delta, \beta) \to (0, \beta) \to (0, \alpha+\gamma)\}$ of L_1. As $\alpha+\gamma < \beta$, R_2 is not equivalent to Z_0. By Lemma 3.1, L_1 has a proper triangulation. If at least one of R_1 and R_2 is equivalent to neither Z_1 nor Z_2, then L_1 has a second distinct proper triangulation.

If $1 < \gamma$, then, by Lemma 3.1, $FT^{(0,\alpha)}_{\gamma}$ (and hence L_1) has a second distinct proper triangulation.

If R_1 and R_2 are each equivalent to either Z_1 or Z_2 and $\gamma = 1$, then $\delta - \gamma = \gamma = \beta - \alpha - \gamma = 1$, hence, $\delta = 2$. Consider the tessellation $\{BT^{(2,\alpha+2)}_{2}, FT^{(0,\alpha)}_{2}, R_1 = (2, 0) \to (2, \alpha) \to (1, \alpha) \to (1, 0)\}$ of L_1. By Lemma 3.1 there exists a second distinct proper triangulation of L_1.

Case C**:** $\beta \le \alpha + \gamma < \delta$**.**

Via the transformation $\mathsf{L}_1 \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ this region is equivalent to the region in Case B.

Case D**:** $\alpha + \gamma < \beta, \delta$**.**

Consider the tessellation $\{BT^{(\alpha+\gamma,\alpha+\gamma)}_{\alpha+\gamma}, FT^{(0,\alpha)}_{\gamma}, FT^{(\gamma,0)}_{\alpha}, R_1 = (\delta, 0) \to (\delta, \beta) \to (0, \beta) \to (0, \alpha+\gamma) \to (\alpha+\gamma, \alpha+\gamma) \to (\alpha+\gamma, 0)\}$ of L_1. Observe that R_1 is an L-region, so is equivalent to one of the regions given in Case A, B, C or (recursively) D. For Cases B and C, there exists at least two distinct proper triangulations so we are done. Otherwise we have the following subcases:

Subcase $D.1$**:** Suppose that R_1 is equivalent to a region given in Case A. If either $\alpha \ne 1$, $\gamma \ne 1$, $\beta - \alpha - \gamma \ne 1$ or $\delta - \alpha - \gamma \ne 1$, then, by Lemma 3.1, there exist two distinct proper triangulations of L_1.

Otherwise, $\alpha = \gamma = \beta - \alpha - \gamma = \delta - \alpha - \gamma = 1$. Then $\alpha = \gamma = 1$ and $\beta = \delta = 3$. Consider the triangulation $\{BT^{(3,3)}_{3}, BT^{(2,1)}_{1}, FT^{(0,1)}_{2}, FT^{(1,0)}_{1}, FT^{(2,0)}_{1}\}$ of L_1. This is a second distinct (to the above) proper triangulation of L_1.

Subcase $D.2$**:**

Otherwise R_1 is equivalent to a region given in Case D. Note that R_1 has area strictly less than L_1, so by recursion there exists a proper triangulation of R_1. Moreover, the tessellation of R_2 contains the triangle $FT^{(0,\alpha+\gamma)}_{\alpha+\gamma}$. But $\alpha + \gamma \ge 2$, so by Lemma 3.1 there exists a second distinct proper triangulation. □

We will now make use of Lemmas 3.1 and 3.2 to determine when there exists precisely one and when there exists at least two distinct proper triangulations of convex regions with five sides.

Lemma 3.3. *Let R be a convex region with five corners (sides). Then R has a proper triangulation. Moreover, whenever R is not equivalent to Z_4 then R has a second distinct proper triangulation.*

Proof: Under the appropriate transformation, we may assume without loss of generality that $R = (\beta, 0) \to (\beta, \gamma) \to (0, \gamma) \to (0, \alpha) \to (\alpha, 0)$, where $0 < \alpha < \beta \leq \gamma$.

Consider the tessellation $\{BT_\alpha^{(\alpha,\alpha)}, R_1 = (\beta, 0) \to (\beta, \gamma) \to (0, \gamma) \to (0, \alpha) \to (\alpha, \alpha) \to (\alpha, 0)\}$ of R. By Lemma 3.2 the region R has a proper triangulation.

Provided R_1 is not equivalent to Z_3 ($\gamma - \alpha \neq 1$ or $\beta - \alpha \neq 1$), Lemma 3.1 and 3.2 imply a second triangulation of R_1, and so R has a second distinct proper triangulation.

Suppose that R_1 is equivalent to Z_3. Then $\gamma - \alpha = 1$ and $\gamma = \beta$. If in addition $\alpha = 1$, then $\beta = \gamma = 2$, and hence $R = Z_4$; otherwise $2 < \beta = \gamma$ and $\{BT_\gamma^{(\beta,\gamma)}\} \bigcup_{1 \leq i \leq \beta - 1} \{FT_1^{(i-1,\gamma-i)}, BT_1^{(i,\gamma-i)}\} \bigcup FT_1^{(\beta-1,0)}$ is a second distinct proper triangulation of R. □

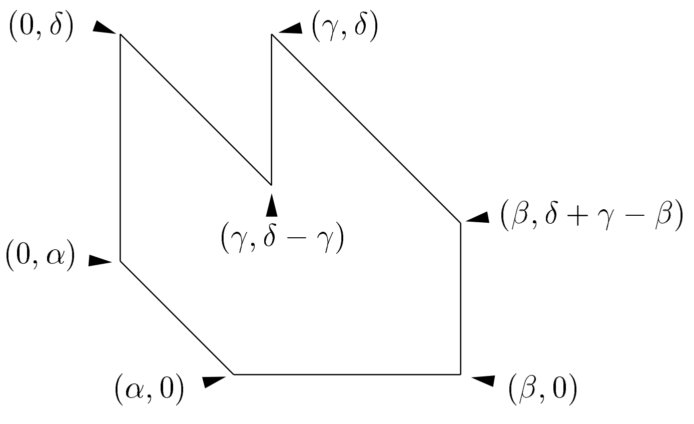
We will now prove another technical lemma which we will use to establish when a convex region with six sides has a proper triangulation and when it has at least two distinct proper triangulations.

Let R be a region equivalent to

$$(\beta, 0) \to (\beta, \delta + \gamma - \beta) \to (\gamma, \delta) \to (\gamma, \delta - \gamma) \to (0, \delta) \to (0, \alpha) \to (\alpha, 0)$$

with $0 \leq \alpha < \beta, \delta$; $0 < \gamma < \beta, \delta + 1$; and $0 \leq \delta + \gamma - \beta$ (this region is illustrated in Figure 4).

Figure 4. Illustration of the region R.



It will be shown that R possesses at least two distinct proper triangulations, except when R is equivalent to any of the following (see the Appendix for illustrations):

$X_1 = (\beta,0) \to (\beta,\beta-1) \to (\beta-1,\beta) \to (\beta-1,1) \to (0,\beta) \to (0,\beta-1) \to (\beta-1,0)$;
$X_2 = (3,0) \to (1,2) \to (1,1) \to (0,2) \to (0,1) \to (1,0)$;
$X_3 = (3,0) \to (3,1) \to (1,3) \to (1,2) \to (0,3) \to (0,1) \to (1,0)$;
$X_4 = (3,0) \to (3,2) \to (2,3) \to (2,1) \to (0,3) \to (0,1) \to (1,0)$;
$X_5 = (\beta,0) \to (\beta,\beta-2) \to (\beta-1,\beta-1) \to (\beta-1,0) \to (0,\beta-1) \to (0,\beta-2) \to (\beta-2,0)$.

Let $\mathcal{X}$ be the set of all regions equivalent to any X_i, where $1 \le i \le 5$.

By inspection the regions equivalent to X_i, where $2 \le i \le 5$, have a unique proper triangulation. Furthermore, by inspection, the region X_1 has no proper triangulation.

Lemma 3.4. *Let $0 \le \alpha < \beta,\delta$; $0 < \gamma < \beta, \delta+1$; and $0 \le \delta+\gamma-\beta$. The region $R = (\beta,0) \to (\beta,\gamma+\delta-\beta) \to (\gamma,\delta) \to (\gamma,\delta-\gamma) \to (0,\delta) \to (0,\alpha) \to (\alpha,0)$ has a proper triangulation if and only if $R \ne X_1$ and a second distinct proper triangulation if and only if R is not equivalent to any X_i, where $1 \le i \le 5$.*

PROOF: Consider the transformation $R\lambda + (\delta+\gamma,0)$, where $\lambda = \left[\begin{smallmatrix}-1 & 0\\ -1 & 1\end{smallmatrix}\right] \in G$. This transformation replaces β with $\delta+\gamma-\alpha$ and α with $\delta+\gamma-\beta$. Hence, without loss of generality, we may assume that $\delta+\gamma-\beta \le \alpha$.

Several cases are considered which, together with the associated conditions, are summarized in the following table.

Case A	**Case** B	**Case** C	**Case** D
$\gamma < \delta$; $\gamma+2 \le \beta$; $\alpha < \gamma$	$\gamma < \delta$; $\gamma+2 \le \beta$; $\alpha \ge \gamma$	$\gamma < \delta$; $\beta = \gamma+1$	$\delta = \gamma$

For Case B several additional subcases are considered which are summarized in the following table.

$B1$ $\alpha-\gamma \ge \gamma+\delta-\beta$			
$B2$ $\alpha-\gamma < \gamma+\delta-\beta$	$B2.1$ $\beta-\gamma > \delta-\alpha$		
	$B2.2$ $\beta-\gamma = \delta-\alpha$	$B2.2.1$ $\alpha \ge 2$	
		$B2.2.2$ $\alpha \le 1$	

Case A: $\gamma < \delta$, $\gamma+2 \le \beta$ and $\alpha < \gamma$.

The conditions for this case together with the assumption $\delta+\gamma-\beta \le \alpha$ imply $\delta < \beta$.

Consider the tessellation $\{FT_{\delta-\alpha}^{(\alpha,0)}, R_1 = (\beta,0) \to (\beta,\gamma+\delta-\beta) \to (\gamma,\delta) \to (\gamma,\delta-\gamma) \to (\delta,0), R_2 = (\alpha,0) \to (\alpha,\delta-\alpha) \to (0,\delta) \to (0,\alpha)\}$ of R. By Lemmas 3.1 and 3.3, there exists a proper triangulation of R.

Unless either R_1 is equivalent to one of Z_2 or Z_4 or R_2 is equivalent to one of Z_0 or Z_1, Lemmas 3.1 and 3.3 imply the existence of a second proper triangulation of R.

Suppose that R_1 is equivalent to Z_2.

Then $\beta-\delta = 1$ and $\delta+\gamma-\beta = 0$, so $\gamma = 1$.

Thus $\alpha = 0$ and R_2 is equivalent to Z_0. In this case Lemma 3.1 applied to R_3 in the tessellation $\{FT_\delta^{(1,0)}, R_3 = (1,0) \to (1, \delta-1) \to (0,\delta) \to (0,0)\}$ verifies the existence of a second proper triangulation of R.

Otherwise suppose R_1 is equivalent to Z_4. Then $\gamma = 2$, $\beta - \gamma = 2$, $\beta - \delta = 1$ and $\delta + \gamma - \beta = 1$. Hence $\beta = 4$, $\delta = 3$ and since $\delta + \gamma - \beta \leq \alpha$ and $\alpha < \gamma$ it follows that $\alpha = 1$. In which case $\{FT_2^{(0,1)}, FT_2^{(2,0)}, FT_1^{(1,0)}, FT_1^{(2,2)}, FT_1^{(3,1)}, BT_1^{(1,1)}, BT_1^{(2,1)}, BT_1^{(3,2)}, BT_1^{(4,1)}\}$ is a second distinct proper triangulation of R.

Subcase $B1$: $\gamma < \delta$, $\gamma + 2 \leq \beta$, $\gamma \leq \alpha$ and $\alpha - \gamma \geq \gamma + \delta - \beta$.

From the conditions for this subcase and the assumption that $0 < \gamma$ it follows that $\gamma + \delta - \beta < \alpha$ or equivalently $\gamma + \delta - \alpha < \beta$.

Consider the tessellation $\{FT_{\delta+\gamma-\alpha}^{(\gamma,\alpha-\gamma)}, R_1 = (\beta, 0) \to (\beta, \gamma+\delta-\beta) \to (2\gamma+\delta-\alpha, \alpha-\gamma) \to (\gamma, \alpha-\gamma) \to (\alpha, 0), R_2 = (\gamma, \alpha-\gamma) \to (\gamma, \delta-\gamma) \to (0,\delta) \to (0,\alpha)\}$ of R. By Lemmas 3.1 and 3.3, there exists a proper triangulation of R.

By Lemmas 3.1 and 3.3 either there exists a second proper triangulation of R or R_1 is equivalent to an element of $\{Z_0, Z_1, Z_2, Z_4\}$ and R_2 is equivalent to Z_1. Henceforth, assume the latter.

Suppose that $\delta = 2$. Then $\gamma = \alpha = 1$ and so R is equivalent to X_2 and by inspection there does not exist a second distinct proper triangulation of R. Thus, $\delta > 2$.

Since R_2 is equivalent to Z_1, either $\gamma = 1$ or $\delta - \alpha = 1$.

First suppose $\gamma = 1$ and $\delta - \alpha = 1$. The fact that $\delta > 2$ implies $\alpha > 1$. In this case consider the tessellation $\{FT_1^{(0,\delta-1)}, FT_1^{(1,\delta-1)}, BT_2^{(2,\delta-1)}, R_3 = (\beta,0) \to (\beta, \delta+\gamma-\beta) \to (2, \delta-1) \to (2, \alpha-2) \to (\alpha, 0)\}$.

Secondly, suppose $\gamma = 1$ and $\delta - \alpha > 1$ and so $\delta \geq \alpha + 2$. Furthermore $\delta - \alpha > 1$ implies that $\gamma + \delta - \alpha > 2$, so R_1 is not equivalent to Z_4; thus $\alpha \leq 2$. If $\alpha = 1$, then R_1 is equivalent to Z_0, and the conditions $0 \leq \delta + \gamma - \beta \leq \alpha - \gamma$ imply $\delta + \gamma - \beta = 0$. Here we take the tessellation $\{FT_{\delta-1}^{(0,1)}, FT_1^{(1,\delta-1)}, R_4 = (\beta, 0) \to (2, \delta-1) \to (1, \delta-1) \to (\delta-1, 1) \to (0,1) \to (1,0)\}$ of R. Note R_4 is equivalent to Z_3 and hence there exists a proper triangulation of R_4. Otherwise $\alpha = 2$ and we take the tessellation $\{FT_{\delta-2}^{(1,2)}, BT_2^{(2,2)}, R_5 = (\beta,0) \to (\beta, \gamma+\delta-\beta) \to (\delta-1, 2) \to (2,2) \to (2,0), R_6 = (1,2) \to (1, \delta-1) \to (0,\delta) \to (0,2)\}$.

Thirdly, suppose $\gamma > 1$ and $\delta - \alpha = 1$. Thus $\gamma + \delta - \alpha > 2$, so R_2 is not equivalent to Z_4. If $\alpha \neq \gamma$, R_1 is not equivalent to Z_0, thus $\delta - \gamma = (\delta-\alpha) + (\alpha-\gamma) = 2$; take the tessellation $\{FT_\gamma^{(\gamma,2)}, BT_2^{(\gamma+1,2)}, R_7 = (\beta,0) \to (\beta, \gamma+\delta-\beta) \to (\delta+\gamma-2, 2) \to (\gamma+1, 2) \to (\alpha, 0), R_8 = (\gamma, \delta-\gamma) \to (0,\delta) \to (0,\alpha) \to (\gamma-1, \delta-\gamma)\}$. Otherwise $\alpha = \gamma$. When $\beta = \gamma + 2$, from the conditions for this subcase $\gamma + \delta - \beta \leq \alpha - \gamma = 0$, so, $\delta \leq 2$, hence, as $\gamma < \delta$ for this case, $\gamma \leq 1$, a contradiction. Thus $\beta \geq \gamma + 3$; take the tessellation $\{FT_2^{(\alpha,0)}, R_2, R_9 = (\beta, 0) \to (\gamma, \delta) \to (\gamma, 2) \to (\gamma+2, 0)\}$.

In each of the above cases the given tessellation, together with Lemmas 3.1, 3.2 and 3.3, verify the existence of a second distinct proper triangulation of R.

Subcase $B2.1$: $\gamma < \delta$, $\gamma + 2 \leq \beta$, $\gamma \leq \alpha$, $\alpha - \gamma < \gamma + \delta - \beta$ and $\beta - \gamma > \delta - \alpha$.

Note that $\gamma > 0$ and $\alpha \geq \gamma$ imply $\alpha > 0$. Also since $\alpha - \gamma < \gamma + \delta - \beta$, then $\gamma + \delta - \beta \geq 1$.

Consider the tessellation $\{FT_{\beta-\gamma}^{(\gamma,\alpha-\gamma)}, R_1 = (\beta, \alpha - \gamma) \to (\beta, \gamma + \delta - \beta) \to (\gamma, \delta) \to (\gamma, \alpha + \beta - 2\gamma), R_2 = (\gamma, \alpha - \gamma) \to (\gamma, \delta - \gamma) \to (0, \delta) \to (0, \alpha), R_3 = (\beta, 0) \to (\beta, \alpha - \gamma) \to (\gamma, \alpha - \gamma) \to (\alpha, 0)\}$ of R. By Lemma 3.1, there exists a proper triangulation of R.

Lemma 3.1 verifies the existence of a second proper triangulation of R unless R_1 is equivalent to Z_1, R_2 is equivalent to Z_1 and R_3 is equivalent to one of Z_0 or Z_2.

Suppose that R_1 is equivalent to Z_1. This supposition together with the condition $\gamma + 2 \leq \beta$ imply that $\alpha - \gamma + 1 = \gamma + \delta - \beta$.

If $\alpha > \gamma$, then take the tessellation $\{FT_{\beta-\gamma}^{(\gamma,\alpha-\gamma+1)}, BT_2^{(\gamma+1,\alpha-\gamma+1)} R_4 = (\gamma, \alpha - \gamma + 1) \to (\gamma, \delta - \gamma) \to (0, \delta) \to (0, \alpha) \to (\gamma - 1, \alpha - \gamma + 1), R_5 = (\beta, 0) \to (\beta, \alpha - \gamma + 1) \to (\gamma + 1, \alpha - \gamma + 1) \to (\gamma + 1, \alpha - \gamma - 1) \to (\alpha, 0)\}$.

If $\alpha = \gamma$, then $R_3 = Z_0$ and $\delta + \gamma - \beta = 1$. In this case further suppose that R_2 is equivalent to Z_1. Under this supposition either $\gamma = 1$ or $\delta - \alpha = 1$. When $\gamma = 1$, $\alpha = 1$ and so the condition $\delta + \gamma - \beta = 1$ implies $\beta = \delta$, or equivalently $\beta - \gamma = \beta - 1 = \delta - 1 = \delta - \alpha$, contradicting the condition that $\beta - \gamma > \delta - \alpha$. Hence $\gamma > 1$. If $\delta - \alpha = 1$ and $\gamma + 2 = \beta$, the condition $\delta + \gamma - \beta = 1$ implies $\delta = 3$ and $\alpha = 2$, which in turn implies $\gamma = 2$; in this case R is equivalent to X_4 and there exists only one proper triangulation of R. So it is left to check the case where $\alpha = \gamma$, $\delta - \alpha = 1$ and $\gamma + 2 < \beta$. Under these conditions take the tessellation $\{FT_{\beta-\gamma-1}^{(\gamma,0)}, R_2, R_6 = (\beta, 0) \to (\beta, 1) \to (\gamma, \delta) \to (\gamma, \beta - \gamma - 1) \to (\beta - 1, 0)\}$.

In each of the above cases the given tessellation together with Lemmas 3.1 and 3.3, verify the existence of a second distinct proper triangulation of R.

Subcase $B2.2.1$: $\gamma < \delta$, $\gamma + 2 \leq \beta$, $\gamma \leq \alpha$, $\beta - \gamma = \delta - \alpha$ and $\alpha \geq 2$.

Note that $\delta + \gamma - \beta = \alpha$.

If $2 \leq \gamma$, then, as $\gamma = \alpha + \beta - \delta$, $2 \leq (\gamma + \delta - \beta) - (\alpha - \gamma)$, so, $\alpha - \gamma + 2 \leq \gamma + \delta - \beta$. So, consider the tessellation $\{FT_{\beta-\gamma}^{(\gamma-1,\alpha-\gamma+1)}, BT_2^{(\beta,\alpha-\gamma+2)}, R_1 = (\beta, \alpha - \gamma + 2) \to (\beta, \delta + \gamma - \beta) \to (\gamma, \delta) \to (\gamma, \delta - \gamma) \to (\beta - 2, \alpha - \gamma + 2), R_2 = (\beta, 0) \to (\beta, \alpha - \gamma) \to (\beta - 1, \alpha - \gamma + 1) \to (\gamma - 1, \alpha - \gamma + 1) \to (\alpha, 0), R_3 = (\gamma - 1, \alpha - \gamma + 1) \to (\gamma - 1, \delta - \gamma + 1) \to (0, \delta) \to (0, \alpha)\}$ of R. By Lemmas 3.1 and 3.3, there exists a proper triangulation of R. As $2 \leq \gamma$ and $\gamma + 2 \leq \beta$, there exists a second distinct proper triangulation of R_1 and hence there exists a second distinct proper triangulation of R.

If $\gamma = 1$, then the subcase condition $\alpha = \gamma + \delta - \beta$ implies $\alpha - \gamma + 1 = \gamma + \delta - \beta$. Consider the tessellation $\{FT_{\beta-\gamma}^{(0,\alpha)}, BT_2^{(\beta,\alpha)}, R_4 = (\beta, \alpha) \to (1, \delta) \to (1, \delta - 1) \to (\beta - 1, \alpha), R_5 = (\beta, 0) \to (\beta, \alpha - 2) \to (\beta - 2, \alpha) \to (0, \alpha) \to (\alpha, 0)\}$ of R. By Lemmas 3.1 and 3.3, there exists a proper triangulation of R.

Note that R_4 is equivalent to Z_2, so unless R_5 is equivalent to Z_1, Lemmas 3.1 and 3.3 verify the existence of a second distinct proper triangulation of R.

Suppose R_5 is equivalent to Z_1; then $\alpha = 2$ and $\beta = 3$. Since $\delta+\gamma-\beta = \alpha$, $\delta = 4$. In this case $R = (3,0) \to (3,2) \to (1,4) \to (1,3) \to (0,4) \to (0,2) \to (2,0)$ and a second distinct proper triangulation of R is $\{FT_2^{(1,2)}, FT_1^{(0,3)}, FT_1^{(0,2)}, FT_1^{(2,0)}, FT_1^{(2,1)}, BT_2^{(2,2)}, BT_1^{(1,3)}, BT_1^{(3,2)}, BT_1^{(3,1)}\}$.

Subcase $B2.2.2$: $\gamma < \delta$, $\gamma + 2 \leq \beta$, $\gamma \leq \alpha$, $\beta - \gamma = \delta - \alpha$ and $\alpha \leq 1$.

From the conditions for this subcase, $0 < \gamma \leq \alpha$; it follows that $\alpha = \gamma = 1$. In addition, $\gamma + 2 \leq \beta$, so $3 \leq \beta = \delta$. Consider the tessellation $\{FT_{\delta-2}^{(0,2)}, BT_2^{(\beta-1,2)}, R_1 = (\beta-1,2) \to (1,\delta) \to (1,\delta-1) \to (\beta-2,2), R_2 = (\beta,0) \to (\beta,1) \to (\beta-1,2) \to (\beta-1,0), R_3 = (\beta-1,0) \to (\beta-3,2) \to (0,2) \to (0,1) \to (1,0)\}$ of R. By Lemmas 3.1 and 3.3, there exists a proper triangulation of R.

Note that both R_1 and R_2 are equivalent to Z_2. So unless R_3 is equivalent to Z_2 or Z_4, by Lemmas 3.1 and 3.3, there exists a second proper triangulation of R.

Suppose R_3 is equivalent to Z_2; then $\beta = 3$ and so R is equivalent to X_3. In this case by inspection there does not exist a second distinct proper triangulation of R.

Suppose R_3 is equivalent to Z_4; then $\beta = 4$. Then $R = (4,0) \to (4,1) \to (1,4) \to (1,3) \to (0,4) \to (0,1) \to (1,0)$ and $\{FT_2^{(2,0)}, FT_2^{(1,2)}, FT_1^{(0,1)}, FT_1^{(0,2)}, FT_1^{(0,3)}, FT_1^{(1,0)}, FT_1^{(3,1)}, BT_2^{(2,2)}, BT_1^{(1,1)}, BT_1^{(1,3)}, BT_1^{(3,2)}, BT_1^{(4,1)}\}$ is a second distinct proper triangulation of R.

Subcase C: $\gamma < \delta$ and $\beta = \gamma + 1$.

Since $\delta + \gamma - \beta \leq \alpha$, it follows, from the conditions for this subcase, that $\delta - \alpha \leq 1$. However, from the conditions for this lemma, $\alpha < \delta$, so $\delta = \alpha + 1$.

Now, from the conditions for this subcase, $0 < \delta - \gamma$ and $\beta - 1 = \gamma$, so $0 < (\alpha+1) - (\beta-1)$; thus $\beta - 2 < \alpha$. From the conditions for this lemma, $\alpha < \beta$, so it follows that $\alpha = \beta - 1$.

Thus, $R = X_1$ and by inspection there does not exist a proper triangulation of R.

Subcase D: $\delta = \gamma$.

In this case the region R is the union of the region $R_1 = (\gamma,0) \to (0,\gamma) \to (0,\alpha) \to (\alpha,0)$ and the region $R_2 = (\beta,0) \to (\beta,2\gamma-\beta) \to (\gamma,\gamma) \to (\gamma,0)$. Furthermore, $R_1 \cap R_2 = \{(0,\gamma)\}$. By Lemma 3.1, there exists a proper t riangulation of R and if either one of R_1 or R_2 is not equivalent to Z_2, then there exists a second distinct proper triangulation. If R_1 and R_2 are both equivalent to Z_2, then R is equivalent to X_5 and by inspection has only one possible proper triangulation. □

Now that we have established Lemma 3.4 we can use it in conjunction with Lemmas 3.1 and 3.2 to establish when a convex region with six sides has a proper triangulation and when it has at least two distinct proper triangulations.

Lemma 3.5. *Let R be a convex region with six corners (sides). Whenever R is not equivalent to Z_6 then R has a proper triangulation; moreover, whenever R is not equivalent to Z_5 or Z_6 then R has a second distinct proper triangulation.*

Proof: Recall that if a proper triangulation for some region, R, exists, then a proper triangulation exists for all regions equivalent to R.

The region R is equivalent to $(\beta,0) \to (\beta,\gamma-\delta) \to (\beta-\delta,\gamma) \to (0,\gamma) \to (0,\alpha) \to (\alpha,0)$, where $0<\alpha,\delta<\beta,\gamma$.

Two cases are considered; Case A where $\gamma-\delta \le \alpha+\delta-\beta$ and Case B where $\gamma-\delta > \alpha+\delta-\beta$.

Case B has several additional subcases which are summarized in the following table.

$B1$	The condition $\alpha=\beta-\delta=\gamma-\delta=1$ does not hold		
$B2$	$\alpha=\beta-\delta=\gamma-\delta=1$	$B2.1$	The condition $\delta=\beta-\alpha=\gamma-\alpha=1$ does not hold
		$B2.2$	$\delta=\beta-\alpha=\gamma-\alpha=1$

Subcase A: $\gamma-\delta \le \alpha+\delta-\beta$.

Consider the tessellation $\{FT_{\beta+\gamma-\alpha-\delta}^{(\beta-\delta,\alpha+\delta-\beta)}, R_1=(\alpha,0)\to(\beta,0)\to(\beta,\gamma-\delta)\to(2\beta+\gamma-\alpha-2\delta,\alpha+\delta-\beta)\to(\beta-\delta,\alpha+\delta-\beta), R_2=(\beta-\delta,\alpha+\delta-\beta)\to(\beta-\delta,\gamma)\to(0,\gamma)\to(0,\alpha)\}$ of R. By Lemmas 3.1 and 3.3 there exists a proper triangulation of R. Moreover if neither R_1 nor R_2 is equivalent to an element of $\{Z_0,Z_2,Z_4\}$, then there exists a second distinct proper triangulation of R.

As $\alpha,\delta<\beta,\gamma$, both R_1 and R_2 are not equivalent to Z_0. Let R_1 and R_2 both be equivalent to Z_2. Then $1=\beta-\delta=\gamma-\delta=\alpha+\delta-\beta$, thus, $\beta=\gamma=\delta+1$ and $\alpha=2$.

From the condition for this case $\gamma+\beta-\alpha\le 2\delta$. Recall that $0<\alpha<\beta,\gamma$, so $1\le\beta-\alpha$ and $2\le\gamma$. Hence, $3\le 2\delta$, thus, $2\le\delta$. Consider the tessellation $\{FT_{\beta-2}^{(2,1)}, BT_2^{(2,2)}, R_3=(2,0)\to(\beta,0)\to(\beta,1)\to(2,1), R_4=(0,2)\to(2,2)\to(2,\gamma-1)\to(1,\gamma)\to(0,\gamma)\}$ of R. By Lemmas 3.1 and 3.3 there exists a second distinct proper triangulation of R.

As R_2 has four corners it is not equivalent to Z_4. Let R_1 be equivalent to Z_2 and R_2 be equivalent to Z_4 then $(\alpha,\beta,\gamma,\delta)=(3,4,4,3)$ and, $\{FT_2^{(2,1)}, FT_1^{(3,0)}, FT_1^{(0,3)}, FT_1^{(1,3)}, BT_2^{(2,3)}, BT_1^{(3,1)}, BT_1^{(4,1)}, BT_1^{(1,4)}\}$ is a second distinct proper triangulation of R.

Subcase $B1$: $\gamma-\delta>\alpha+\delta-\beta$ and the condition $\alpha=\beta-\delta=\gamma-\delta=1$ does not hold.

Consider the tessellation $\{FT_\delta^{(\beta-\delta,\gamma-\delta)}, R_1=(\alpha,0)\to(\beta,0)\to(\beta,\gamma-\delta)\to(\beta-\delta,\gamma-\delta)\to(\beta-\delta,\gamma)\to(0,\gamma)\to(0,\alpha)\}$ of R. As the condition $\alpha=\beta-\delta=\gamma-\delta=1$ does not hold, R_1 is not equivalent to X_1. By Lemma 3.4, there exists a proper triangulation of R and if $R_1\notin\mathcal{X}$, there exists a second distinct proper triangulation of R.

Let $R_1\in\mathcal{X}$. As $\gamma-\delta>\alpha+\delta-\beta$, R_1 is not equivalent to X_5. Recall that $\alpha<\gamma,\beta$, so, R_1 is not equivalent to X_2. Thus, R_1 is equivalent to either X_3 or X_4. If R_1 is equivalent to X_4, then, by inspection, there exists a second distinct proper triangulation of R. If R_1 is equivalent to X_3, then R is equivalent to Z_5 and by inspection there exists precisely one proper triangulation of R.

Subcase $B2.1$**:** $\gamma-\delta>\alpha+\delta-\beta$**,** $\alpha=\beta-\delta=\gamma-\delta=1$ **and the condition** $\delta=\beta-\alpha=\gamma-\alpha=1$ **does not hold.**

Consider the linear transformation $R\lambda+(\gamma,\beta)$ where $\lambda=\left[\begin{smallmatrix}0 & -1\\ -1 & 0\end{smallmatrix}\right]\in G$. This transformation interchanges β with γ and α with δ. Thus, in this subcase the region R is equivalent to a region in Subcase $B1$.

Subcase $B2.2$**:** $\gamma-\delta>\alpha+\delta-\beta$**,** $\alpha=\beta-\delta=\gamma-\delta=\delta=\beta-\alpha=\gamma-\alpha=1$**.**

In this subcase $(\alpha,\beta,\gamma,\delta)=(1,2,2,1)$, hence, R is equivalent to Z_6 and, by inspection, R has no possible proper triangulation. □

We can now state the first major result of this paper.

Theorem 3.6. *Let R be any convex region. Then if R is not equivalent to Z_6, it has a proper triangulation. Moreover, whenever $R\notin\mathcal{Z}$ then R has a second distinct proper triangulation.*

Proof: This follows immediately from Lemmas 3.1, 3.3 and 3.5. □

4. Distinct proper triangulations containing a fixed triangle

We now move on to the question of establishing when it is possible to find two distinct proper triangulations of a region when some triangle is forced to occur in both proper triangulations. We answer this question completely for three-sided regions, and also for non-rectangular four-sided regions. We deal with the four-sided regions first; the two choices for the direction of the internal triangle yield two theorems.

Theorem 4.1. *Let $1\le\alpha\le\beta$, $0<\chi\le\alpha$, $0<\gamma\le\beta-\chi$, $0\le\delta\le\alpha-\chi$ and $\alpha\le\gamma+\delta$; then there exists a proper triangulation of the region $R=(\beta,0)\to(\beta,\alpha)\to(0,\alpha)\to(\alpha,0)$ which contains $FT_\chi^{(\gamma,\delta)}$ if and only if $(\alpha,\beta,\gamma,\delta,\chi)\neq(2,3,1,1,1)$.*

If $\alpha=1$ or $(\alpha,\beta)=(2,2)$ or $(\alpha,\beta,\gamma\delta,\chi)\in\{(2,3,2,1,1),(2,3,2,0,1),(2,4,2,1,1)\}$, then there exists precisely one proper triangulations of R containing $FT_\chi^{(\gamma,\delta)}$.

Otherwise there exists a second distinct proper triangulations of R containing $FT_\chi^{(\gamma,\delta)}$.

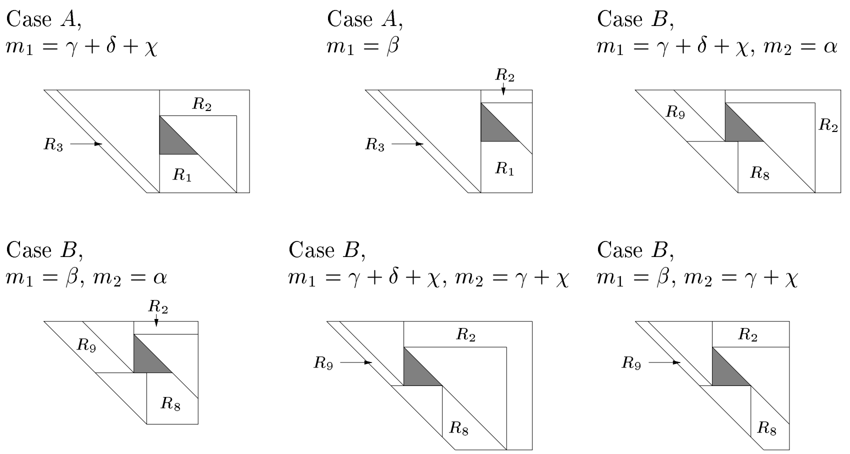
Proof: If $(\alpha,\beta,\gamma,\delta,\chi)=(2,3,1,1,1)$, then, by inspection, there does not exist a proper triangulation of R containing $FT_\chi^{(\gamma,\delta)}$.

If $\alpha=1$ or $(\alpha,\beta)=(2,2)$ or $(\alpha,\beta,\gamma\delta,\chi)\in\{(2,3,2,1,1),(2,3,2,0,1),(2,4,2,1,1)\}$, then, by inspection, there exists precisely one proper triangulation of R containing $FT_\chi^{(\gamma,\delta)}$.

By inspection, the other cases where $\alpha+\beta\le 6$ have two distinct proper triangulations of R containing $FT_\chi^{(\gamma,\delta)}$.

Henceforth, assume that $1 < \alpha$ and $6 < \alpha + \beta$. Two subcases are considered: Case A where $\alpha \leq \gamma$ and Case B where $\alpha > \gamma$ (see Figure 5 for illustrations of these cases and their subcases).

FIGURE 5. Tessellations for cases of Lemma 4.1.



Let $m_1 = \min\{\gamma + \delta + \chi, \beta\}$ and $m_2 = \min\{\gamma + \chi, \alpha\}$.

Case A**:** $\alpha \leq \gamma$**.**

Note that for this case $m_2 = \alpha$.

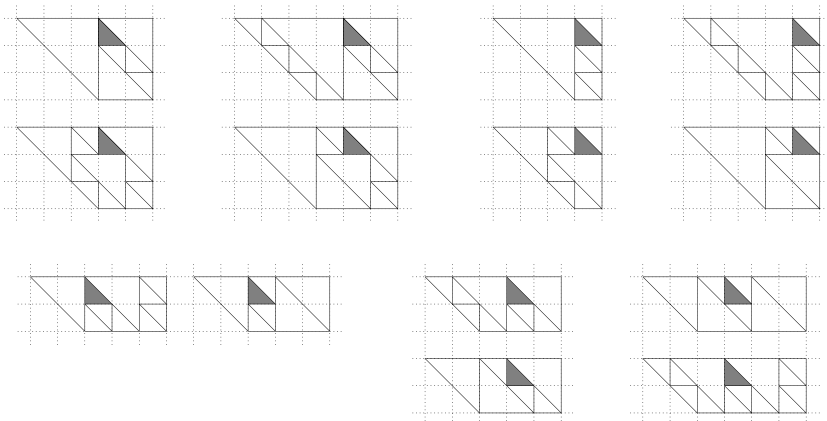
Consider the tessellation $\{BT_{\alpha}^{(\gamma,\alpha)}, BT_{m_1-\gamma}^{(m_1,\delta+\chi)}, FT_{\chi}^{(\gamma,\delta)}, R_1 = (\beta, 0) \to (m_1, \gamma + \delta + \chi - m_1) \to (\gamma + \chi, \delta) \to (\gamma, \delta) \to (\gamma, 0), R_2 = (\beta, 0) \to (\beta, \alpha) \to (\gamma, \alpha) \to (\gamma, \delta + \chi) \to (m_1, \delta + \chi) \to (m_1, 0), R_3 = (\gamma, 0) \to (\gamma - \alpha, \alpha) \to (0, \alpha) \to (\alpha, 0)\}$ of R. By Lemmas 3.1, 3.2 and 3.3, there exists a proper triangulation of R containing $FT_{\chi}^{(\gamma,\delta)}$, and if $\{R_1, R_2, R_3\} \not\subset \mathcal{Z}$, then there exists a second distinct proper triangulation of R containing $FT_{\chi}^{(\gamma,\delta)}$. So, assume that $\{R_1, R_2, R_3\} \subset \mathcal{Z}$.

Consider the tessellation $\{BT_{\delta+\chi}^{(\gamma,\delta+\chi)}, BT_{\alpha+m_1-\gamma-\delta-\chi}^{(m_1,\alpha)}, FT_{\chi}^{(\gamma,\delta)}, R_1, R_4 = (\beta, 0) \to (\beta, \alpha) \to (m_1, \alpha) \to (m_1, 0), R_5 = (\gamma, \delta + \chi) \to (\gamma + \delta + \chi - \alpha, \alpha) \to (0, \alpha) \to (\alpha, 0) \to (\gamma, 0) \to (\gamma - \delta - \chi, \delta + \chi)\}$ of R. By Lemmas 3.1, 3.2 and 3.3, unless $\delta + \chi = \alpha$, there exists a second distinct proper triangulation of R containing $FT_{\chi}^{(\gamma,\delta)}$.

If $\delta + \chi = \alpha$ and $\chi \neq 1$, consider the tessellation $\{BT_{\alpha-1}^{(\gamma,\alpha-1)}, BT_{m_1-\gamma}^{(m_1,\alpha)}, BT_1^{(\gamma,\alpha)}, FT_{\chi}^{(\gamma,\delta)}, R_1, R_4, R_6 = (\gamma, \alpha - 1) \to (\gamma - 1, \alpha) \to (0, \alpha) \to (\alpha, 0) \to (\gamma, 0) \to (\gamma - \alpha + 1, \alpha - 1)\}$ of R. By Lemmas 3.1, 3.2 and 3.3, there exists a second distinct proper triangulation of R containing $FT_{\chi}^{(\gamma,\delta)}$.

If $\delta + \chi = \alpha$, $\chi = 1$ and $\gamma + \chi < \beta$, recall that $\{R_1, R_2, R_3\} \subset \mathcal{Z}$, hence, $\alpha \leq 3$ and $\beta \leq 6$. For $6 < \alpha + \beta \leq 9$ a pair of proper triangulations of R both containing $FT_{\chi}^{(\gamma,\delta)} = FT_1^{(\gamma,\delta)}$ is shown in Figure 6.

Figure 6. Proper triangulations for Subcase A of Lemma 4.1.



If $\delta+\chi=\alpha$, $\chi=1$ and $\gamma+\chi=\beta$, consider the tessellation $\{BT_{\alpha-1}^{(\beta,\delta)}, BT_1^{(\gamma,\alpha)}, BT_1^{(\beta,\alpha)}, FT_\chi^{(\gamma,\delta)}, R_7=(\beta,0)\rightarrow(\beta-\alpha+1,\delta)\rightarrow(\gamma,\delta)\rightarrow(\gamma-1,\alpha)\rightarrow(0,\alpha)\rightarrow(\alpha,0)\}$ of R. By Lemma 3.2, there exists a second distinct proper triangulation of R.

Case B: $\alpha>\gamma$.

Consider the tessellation $\{BT_{m_2+\delta-\alpha}^{(m_2,\delta)}, BT_{\alpha-\delta}^{(\gamma,\alpha)}, BT_{m_1-\gamma}^{(m_1,\delta+\chi)}, FT_\chi^{(\gamma,\delta)}, R_2, R_8=(\beta,0)\rightarrow(m_1,\gamma+\delta+\chi-m_1)\rightarrow(\gamma+\chi,\delta)\rightarrow(m_2,\delta)\rightarrow(m_2,\alpha-m_2)\rightarrow(\alpha,0), R_9=(\gamma,\delta)\rightarrow(\gamma+\delta-\alpha,\alpha)\rightarrow(0,\alpha)\rightarrow(\alpha-\delta,\delta)\}$ of R. By Lemmas 3.1, 3.2 and 3.3, there exists a proper triangulation of R containing $FT_\chi^{(\gamma,\delta)}$, and if $\{R_2,R_8,R_9\}\not\subset\mathcal{Z}$, then there exists a second distinct proper triangulation of R containing $FT_\chi^{(\gamma,\delta)}$.

So, assume that $\{R_2,R_8,R_9\}\subset\mathcal{Z}$.

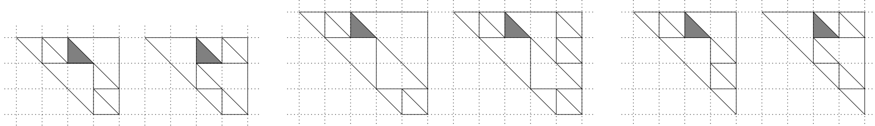
Recall the above assumption that $6<\alpha+\beta$. Consider the tessellation $\{BT_{m_2+\delta-\alpha}^{(m_2,\delta)}, BT_\chi^{(\gamma,\delta+\chi)}, BT_{\alpha+m_1-\gamma-\delta-\chi}^{(m_1,\alpha)}, FT_\chi^{(\gamma,\delta)}, R_4, R_8, R_{10}=(\gamma,\delta+\chi)\rightarrow(\gamma+\delta+\chi-\alpha,\alpha)\rightarrow(0,\alpha)\rightarrow(\alpha-\delta,\delta)\rightarrow(\gamma,\delta)\rightarrow(\gamma-\chi,\delta+\chi)\}$ of R. By Lemmas 3.1, 3.2 and 3.3, unless $\delta+\chi=\alpha$, there exists a second distinct proper triangulation of R containing $FT_\chi^{(\gamma,\delta)}$.

If $\delta+\chi=\alpha$ and $\chi\neq 1$, consider the tessellation $\{BT_{m_2+\delta-\alpha}^{(m_2,\delta)}, BT_{\chi-1}^{(\gamma,\alpha-1)}, BT_{m_1-\gamma}^{(m_1,\alpha)}, BT_1^{(\gamma,\alpha)}, FT_\chi^{(\gamma,\delta)}, R_4, R_8, R_{11}=(\gamma,\alpha-1)\rightarrow(\gamma-1,\alpha)\rightarrow(0,\alpha)\rightarrow(\alpha-\delta,\delta)\rightarrow(\gamma,\delta)\rightarrow(\gamma-\chi+1,\alpha-1)\}$ of R. By Lemmas 3.1, 3.2 and 3.3, there exists a second distinct proper triangulation of R containing $FT_\chi^{(\gamma,\delta)}$.

Otherwise $\delta+\chi=\alpha$ and $\chi=1$. Since $\alpha>\gamma$ for this subcase, $m_2=\gamma+1$. Observe that R_8 is equivalent to one of Z_0, Z_1, Z_2 or Z_4.

If R_8 is equivalent to Z_4, then $\alpha = \beta - 1 = 4$ and a pair of proper triangulations of R both containing $FT_\chi^{(\gamma,\delta)} = FT_1^{(\gamma,\delta)}$ is shown in Figure 7.

FIGURE 7. Proper triangulations for Subcase B of Lemma 4.1.



Otherwise R_8 is equivalent to Z_0, Z_1 or Z_2. Thus either $\gamma = 1$ or $\beta - \gamma \leq 2$.

When R_8 is equivalent to Z_0 and $\gamma = 1$ then $\beta \leq 2$. But from the conditions of this lemma, $\alpha \leq \beta$ and from above $6 < \alpha + \beta$, creating a contradiction.

When R_8 is equivalent to Z_1 or Z_2 and $\gamma = 1$ consider the tessellation $\{BT_\delta^{(\alpha,\delta)}$, $BT_1^{(1,\alpha)}, BT_1^{(2,\alpha)}, FT_\chi^{(\gamma,\delta)} = FT_1^{(1,\alpha-1)}, R_{11} = (\beta,0) \to (\beta,\alpha) \to (2,\alpha) \to (2,\alpha-1) \to (\alpha,\delta) \to (\alpha,0)\}$ of R. As R_8 is equivalent to Z_1 or Z_2 and $R_2 \in \mathcal{Z}$, $\beta \leq \alpha + 2$. So, if $\alpha \leq 2$, then $\alpha + \beta \leq 6$ contradicting our assumption that $6 < \alpha + \beta$. Thus, by Lemmas 3.1 and 3.2, we have a proper triangulation of R containing $FT_\chi^{(\gamma,\delta)}$ which is distinct to the one above.

Otherwise $1 < \gamma$ and $\beta - \gamma \leq 2$. Note that $\alpha = \delta + \chi < \delta + \gamma$. Suppose that $\gamma - 1 \geq \beta - \gamma$. (Equivalently, as $\chi = 1$, $\gamma + \delta - \beta \geq \alpha - \gamma$.) Then consider the tessellation $\{BT_\gamma^{(\gamma,\alpha)}, BT_{\beta-\gamma}^{(\beta,\alpha)}, FT_{\beta-\gamma}^{(\gamma,\gamma+\delta-\beta)}, FT_\chi^{(\gamma,\delta)} = FT_1^{(\gamma,\alpha-1)}, R_{13} = (\beta,0) \to (\beta,\gamma+\delta-\beta) \to (\gamma,\gamma+\delta-\beta) \to (\gamma,\alpha-\gamma) \to (\alpha,0), R_{14} = (\beta,\gamma+\delta-\beta) \to (\beta,\alpha+\gamma-\beta) \to (\gamma+1,\delta) \to (\gamma,\delta)\}$ of R. By Lemmas 3.1 and 3.3, there exists a second distinct proper triangulation of R containing $FT_\chi^{(\gamma,\delta)}$.

Otherwise $\gamma - 1 < \beta - \gamma$. Then $2 = \gamma < \alpha \leq \beta \leq 4$. A pair of proper triangulations of R both containing $FT_\chi^{(\gamma,\delta)} = FT_1^{(\gamma,\delta)}$ is shown in Figure 7. □

Theorem 4.2. *Let* $0 < \chi \leq \delta \leq \alpha \leq \beta$; $\gamma \leq \beta$; $\alpha \leq \gamma + \delta - \chi$ *and* $R = (\beta,0) \to (\beta,\alpha) \to (0,\alpha) \to (\alpha,0)$.

If $1 = \beta - \gamma = \beta - \alpha = \delta - \chi$, *then there does not exist a proper triangulation of* R *containing* $BT_\chi^{(\gamma,\delta)}$.

Otherwise, if $1 = \alpha$ *or* $(\alpha,\beta,\gamma,\delta,\chi) = (2,3,3,2,1)$ *or* $\alpha-\delta, \beta-\gamma, \gamma+\delta-\alpha-\chi \leq 1$, *then there exists precisely one proper triangulation of* R *containing* $BT_\chi^{(\gamma,\delta)}$.

In cases other than the above, there exists at least two distinct proper triangulations of R *both containing* $BT_\chi^{(\gamma,\delta)}$.

PROOF: If $1 = \beta - \gamma = \beta - \alpha = \delta - \chi$, then, by inspection, there does not exist a proper triangulation of R containing $BT_\chi^{(\gamma,\delta)}$.

Similarly, if $1 = \alpha$ or $(\alpha,\beta,\gamma,\delta,\chi) = (2,3,3,2,1)$ or $\alpha-\delta, \beta-\gamma, \gamma+\delta-\alpha-\chi \leq 1$, then, by inspection, there exists precisely one proper triangulation of R containing $BT_\chi^{(\gamma,\delta)}$.

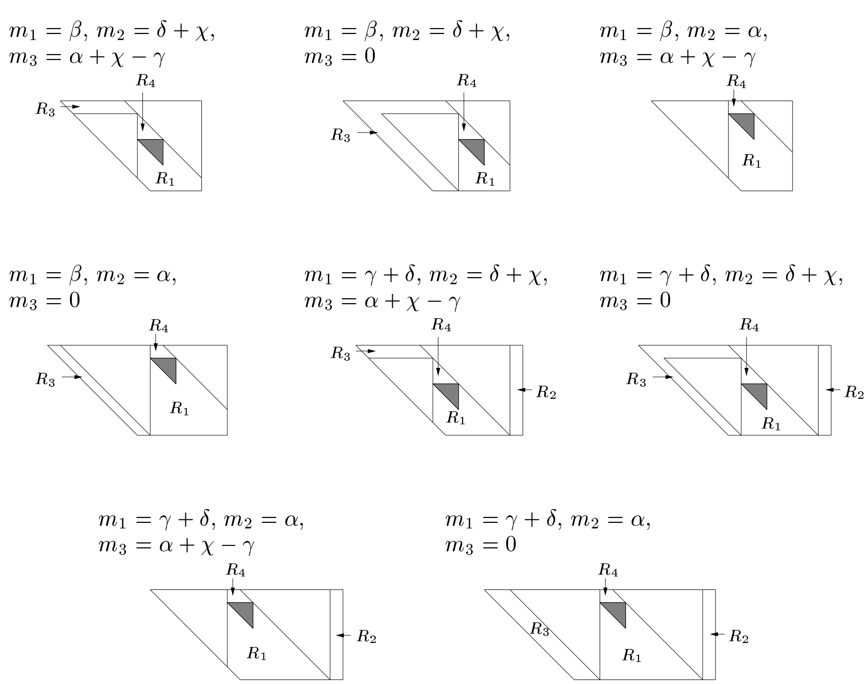
By inspection, the other cases where $\alpha + \beta \leq 6$ have two distinct proper triangulations of R containing $BT_\chi^{(\gamma,\delta)}$.

Henceforth, assume that $1 < \alpha$, $6 < \alpha + \beta$ and either $1 < \alpha - \delta$ or $1 < \beta - \gamma$ or $1 < \gamma + \delta - \alpha - \chi$.

Let $m_1 = \min\{\gamma + \delta, \beta\}$, $m_2 = \min\{\delta + \chi, \alpha\}$ and $m_3 = \max\{\alpha + \chi - \gamma, 0\}$.

Consider the tessellation $\{BT_{\alpha+m_1-\gamma-\delta}^{(m_1,\alpha)}, BT_{m_2-m_3}^{(\gamma-\chi,m_2)}, BT_\chi^{(\gamma,\delta)}, R_1 = (m_1, 0) \to (m_1, \gamma+\delta-m_1) \to (\gamma,\delta) \to (\gamma, \delta-\chi) \to (\gamma-\chi,\delta) \to (\gamma-\chi, m_3) \to (\alpha, 0), R_2 = (\beta, 0) \to (\beta,\alpha) \to (m_1,\alpha) \to (m_1, 0), R_3 = (\gamma-\chi, m_3) \to (\gamma+m_3-m_2-\chi, m_2) \to (\gamma-\chi, m_2) \to (\gamma+\delta-\alpha,\alpha) \to (0,\alpha) \to (\alpha, 0), R_4 = (\gamma,\delta) \to (\gamma+\delta-m_2, m_2) \to (\gamma-\chi, m_2) \to (\gamma-\chi,\delta)\}$ of R (see Figure 8 for an illustration of these cases). Since the condition $1 = \beta - \gamma = \beta - \alpha = \delta - \chi$ does not hold, the region R_1 is not equivalent to the region X_1. By Lemmas 3.1, 3.2, 3.3 and 3.4, there exists a proper triangulation of R containing $BT_\chi^{(\gamma,\delta)}$. Furthermore if $\{R_1, R_2, R_3, R_4\} \not\subset \mathcal{X} \cup \mathcal{Z}$, then there exists a second distinct proper triangulation.

FIGURE 8. Tessellations for cases of Lemma 4.2.



Otherwise $\{R_1, R_2, R_3, R_4\} \subset \mathcal{X} \cup \mathcal{Z}$ and we wish to establish the existence of a second distinct proper triangulation when $1 < \alpha$. For $6 < \alpha + \beta \leq 12$ there are a small number of cases; these are dealt with individually in [11]. Henceforth we assume that $12 < \alpha + \beta$. Three cases are considered: Case A where $\chi = 1$, Case B where $\chi = 2$ and Case C where $3 \le \chi$.

Case A**:** $\chi = 1$.

Suppose first that $\gamma \neq \beta$ and $\chi = 1 < \gamma + \delta - \alpha$.

From Figure 8, R_3 is equivalent to Z_0 or Z_1 or Z_3. Thus $\alpha - \delta \le 2$ and $\gamma - \alpha - \chi \le 1$. Since R_2 is equivalent to Z_0 or Z_1, $\beta - m_1 \le 1$. Since $R_1 \in \mathcal{Z} \cup \mathcal{X}$ and $\chi = 1 < \gamma + \delta - \alpha$, the region R_1 is equivalent to either Z_3 or X_2 or X_3 or X_5.

If R_1 is equivalent to Z_3, then $\alpha = \beta$ and $\delta = 3$. Thus, $\alpha = \beta \le \delta + 2 = 5$, so $\alpha + \beta \le 10$, a contradiction.

If R_1 is equivalent to X_2, then $\delta = 2$ and either $m_1 - \gamma = 1$, in which case $m_1 = \beta$ and $m_1 - \alpha \le 3$, or $m_1 - \gamma = 2$, in which case $\beta - m_1 \le 1$ and $m_1 - \alpha = 2$. Thus, $\alpha \le \delta + 2 = 4$ and $\beta - \alpha \le 3$, so, $\beta \le 7$. Hence $\alpha + \beta \le 11$, a contradiction.

If R_1 is equivalent to X_3, then $\delta = 3$, $\alpha + \chi - \gamma = 1$ and $\beta + \chi - \gamma = 3$. Thus, $\alpha \le \delta + 2 = 5$ and $\beta = 2 + \alpha \le 7$. So $\alpha + \beta \le 12$, a contradiction.

If R_1 is equivalent to X_5, then $\delta = 1$, $\beta - m_1 \le 1$, $0 \le \gamma - \alpha - \chi \le 1$ and $m_1 - \gamma = 1$. Thus, $\alpha \le \delta + 2 = 3$ and $\beta \le \alpha + 4 \le 7$. So, $\alpha + \beta \le 10$, a contradiction.

Secondly, suppose that $\gamma \neq \beta$ and $\gamma + \delta - \alpha = \chi = 1$. Hence, R_1 is equivalent to Z_1 or Z_2.

If $3 \le \gamma$, then consider the tessellation $\{BT_2^{(2,\alpha)}, BT_{m_1-2}^{(m_1,\alpha-1)}, BT_\chi^{(\gamma,\delta)} = BT_1^{(\gamma,\alpha-\gamma+1)}, R_1, R_5 = (\gamma,\delta) \to (2,\alpha-1) \to (2,\alpha-2) \to (\gamma-1,\delta), R_6 = (\beta,0) \to (\beta,\alpha) \to (2,\alpha) \to (2,\alpha-1) \to (m_1,\alpha-1) \to (m_1,0)\}$ of R. By Lemmas 3.1 and 3.2, there exists a second distinct proper triangulation of R containing $BT_\chi^{(\gamma,\delta)}$.

If $\gamma \le \alpha - 2$, consider the tessellation $\{BT_2^{(\alpha,2)}, BT_{\alpha-2}^{(\alpha-1,\alpha)}, BT_\chi^{(\gamma,\delta)} = BT_1^{(\gamma,\alpha-\gamma+1)}, R_7 = (\alpha-1,2) \to (\gamma,\delta) \to (\gamma,\delta-1) \to (\alpha-2,2), R_8 = (\gamma,\delta) \to (1,\alpha) \to (0,\alpha) \to (\gamma-1,\delta), R_9 = (\beta,0) \to (\beta,\alpha) \to (\alpha-1,\alpha) \to (\alpha-1,2) \to (\alpha,2) \to (\alpha,0)\}$ of R. By Lemmas 3.1 and 3.2, there exists a second distinct proper triangulation of R containing $BT_\chi^{(\gamma,\delta)}$.

Otherwise $\alpha - 2 < \gamma < 3$, so $\alpha \le 3$. As R_2 is equivalent to Z_0 or Z_1 and $\gamma + \delta - \alpha = \chi = 1$ it follows that $\beta \le \alpha + 2$. Thus, $\alpha + \beta \le 8$, a contradiction.

Thirdly, suppose that $\gamma = \beta$ and $\gamma + \delta - \alpha = \chi = 1$; then $\delta - 1 = \alpha - \beta$. From the conditions for this lemma $0 \le \delta - 1$ and $\alpha - \beta \le 0$. Thus $\alpha = \beta$ and $\delta = 1$. Note that $6 < \alpha + \beta$, so, $4 \le \alpha = \beta$. Consider the tessellation $\{BT_{\alpha-1}^{(\alpha-1,\alpha)}, BT_\chi^{(\gamma,\delta)} = BT_1^{(\alpha,1)}, R_{10} = (\alpha,1) \to (\alpha,\alpha) \to (\alpha-1,\alpha) \to (\alpha-1,1)\}$ of R. By Lemma 3.1, there exists a second distinct proper triangulation of R containing $BT_\chi^{(\gamma,\delta)}$.

Finally assume $\gamma = \beta$ and $\chi = 1 < \gamma + \delta - \alpha$. Recall that R_3 is equivalent to Z_0, Z_1 or Z_3, so, $\alpha - \delta \le 2$ and $\gamma - \alpha - \chi \le 1$. Hence, as $\gamma = \beta$ and $\chi = 1$, it follows that $\beta - 2 \le \alpha$. If $\alpha \le 2$, then $\beta \le 4$ and thus $\alpha + \beta \le 6$; therefore, $3 \le \alpha$.

If $\beta - 2 < \alpha + \chi - \delta$, then $\beta - 2 < 2 + \chi$, hence, $\beta < 5$, thus, $6 < \alpha + \beta \leq 8$, a contradiction.

Hence, assume $\alpha + \chi - \delta \leq \beta - 2$.

If $\alpha - 2 - (\alpha - (\beta - 2)) < 0$, then $\beta < 4$, so, $\alpha + \beta \leq 6$, a contradiction. Hence, $\alpha - 2 - (\alpha - (\beta - 2)) \geq 0$.

Let $m_4 = \min\{\alpha - 2, \delta - 1\}$. Consider the tessellation $\{FT_2^{(\beta-2,m_4)}, BT_{\beta-2}^{(\beta-2,\alpha)}, BT_\chi^{(\gamma,\delta)} = BT_1^{(\beta,\delta)}, R_{11} = (\beta,0) \to (\beta,m_4) \to (\beta-2,m_4) \to (\beta-2,\alpha-\beta+2) \to (\alpha,0), R_{12} = (\beta,m_4) \to (\beta,\delta-1) \to (\beta-1,\delta) \to (\beta,\delta) \to (\beta,\alpha) \to (\beta-2,\alpha) \to (\beta-2,m_4+2)\}$ of R. By Lemmas 3.1 and 3.3 there exists a second distinct proper triangulation of R containing $BT_\chi^{(\gamma,\delta)}$.

Case B: $\chi = 2$.

As R_3 is equivalent to Z_0, Z_1 or Z_3 and R_4 is equivalent to Z_0 or Z_2, $\alpha - \delta \leq 1$ (so, $m_2 = \alpha$) and $\gamma - \alpha - \chi \leq 1$. Since R_1 is equivalent to Z_0, Z_1, Z_2, Z_3, Z_4, X_4 or X_5, $m_1 = \beta$, $\beta - \alpha \leq 2$, $m_1 + \chi - \gamma \leq 4$ and $\delta \leq 4$. If $\delta = 4$, then R_1 is equivalent to Z_3, so, $\alpha = \beta$ and $\alpha - \delta, \beta - \gamma, \gamma + \delta - \alpha - \chi \leq 1$ contradicting our assumption that either $1 < \alpha - \delta$ or $1 < \beta - \gamma$ or $1 < \gamma + \delta - \alpha - \chi$. So, $\delta \leq 3$. Thus, $\alpha \leq 4$ and $\beta \leq 2 + \alpha$, so, $\alpha + \beta \leq 10$, a contradiction.

Case C: $3 \leq \chi$.

As $R_i \in \mathcal{X} \cup \mathcal{Z}$ for all $1 \leq i \leq 4$ it follows that $\beta - \gamma, \alpha - \delta, \gamma + \delta - \alpha - \chi \leq 1$, a contradiction. ☐

Letting $\alpha = \beta$ in Theorem 4.1 and 4.2 describes precisely when at least one or two proper triangulations exist for a three-sided region containing any fixed triangle.

Note that, this result is described in terms of finding proper triangulations of a backward triangle. However, the equivalences of regions (and indeed proper triangulations) discussed earlier mean that a similar result holds for forward triangles.

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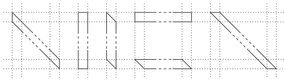
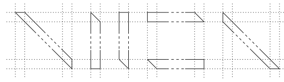
Appendix

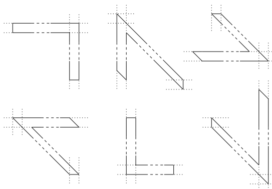
Regions equivalent to Z_1



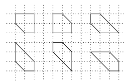
Regions equivalent to Z_2



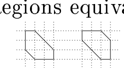
Regions equivalent to Z_3



Regions equivalent to Z_4

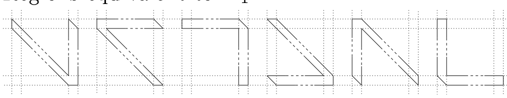


Regions equivalent to Z_5

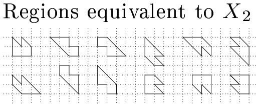


A Region equivalent to Z_6

Regions equivalent to X_1



Regions equivalent to X_2



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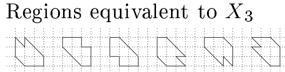
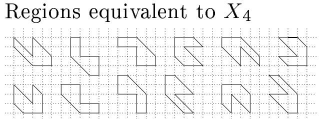
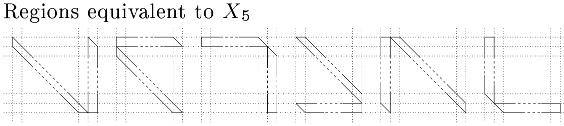


Regions equivalent to X_4



Regions equivalent to X_5



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