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Metrization of function spaces with the Fell topology

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Abstract. For a Tychonoff space X, let $\downarrow \mathrm{C}_F(X)$ be the family of hypographs of all continuous maps from X to $[0,1]$ endowed with the Fell topology. It is proved that X has a dense separable metrizable locally compact open subset if $\downarrow \mathrm{C}_F(X)$ is metrizable. Moreover, for a first-countable space X, $\downarrow \mathrm{C}_F(X)$ is metrizable if and only if X itself is a locally compact separable metrizable space. There exists a Tychonoff space X such that $\downarrow \mathrm{C}_F(X)$ is metrizable but X is not first-countable.

Keywords: space of continuous maps, Fell topology, hyperspace, metrizable, hypograph, separable, first-countable

Classification: 54C35, 54E45, 54B20

1. Introduction and main results

For a topological space X, let $\mathrm{C}(X)$ denote the set of all continuous maps from X to the unit closed interval $\mathbf{I} = [0,1]$ with the usual topology. Then we can endow $\mathrm{C}(X)$ with various topologies. For example, the topology of uniform convergence, the topology of pointwise convergence and the compact-open topology are well known. In [4]–[10], $\mathrm{C}(X)$ is endowed with other natural topologies inherited from the spaces $\mathrm{Cld}(X \times \mathbf{I})$ of nonempty closed sets in $X \times \mathbf{I}$.

For a space Y, let $\mathrm{Cld}(Y)$ be the set of all nonempty closed sets in Y. For an open set U in Y, let

$$U^- = \{A \in \mathrm{Cld}(Y) : A \cap U \neq \emptyset\} \quad \text{and} \quad U^+ = \{A \in \mathrm{Cld}(Y) : A \subset U\}.$$

The most well-known topology of $\mathrm{Cld}(Y)$, called the *Vietoris topology*, is generated by

$$\{U^-, U^+ : U \ \text{ is open in } Y\}.$$

In this paper, we consider the *Fell topology* of $\mathrm{Cld}(Y)$, which is generated by

$$\{U^-, (Y \setminus K)^+ : U \ \text{ is open and } K \text{ is compact in } Y\}.$$

The hyperspaces $\mathrm{Cld}(Y)$ with the above two topologies are denoted by $\mathrm{Cld}_V(Y)$ and $\mathrm{Cld}_F(Y)$, respectively. It is well-known that $\mathrm{Cld}_V(Y)$ (resp. $\mathrm{Cld}_F(Y)$) is metrizable if and only if Y is a compact (resp. locally compact and separable) metrizable space. Obviously, when Y is compact, the Fell topology of $\mathrm{Cld}(Y)$ is equal to the Vietoris topology.

For every $f \in \mathrm{C}(X)$, let

$$\downarrow f = \{(x,s) \in X \times \mathbf{I} : s \leq f(x)\} \in \mathrm{Cld}(X \times \mathbf{I}),$$

which is called the *hypograph* of f. By identifying each $f \in \mathrm{C}(X)$ with $\downarrow f \in \mathrm{Cld}_V(X \times \mathbf{I})$, we can regard $\mathrm{C}(X)$ as the subset

$$\downarrow \mathrm{C}(X) = \{\downarrow f : f \in \mathrm{C}(X)\} \subset \mathrm{Cld}(X \times \mathbf{I}).$$

Let $\downarrow \mathrm{C}_V(X)$ and $\downarrow \mathrm{C}_F(X)$ be the spaces with the topologies inherited from $\mathrm{Cld}_V(X \times \mathbf{I})$ and $\mathrm{Cld}_F(X \times \mathbf{I})$, respectively. These topologies are different from the three topologies mentioned previously (see [4, Corollary 1]). In [9, Theorem 1], it was proved that, for a Tychonoff space X, $\downarrow \mathrm{C}_V(X)$ is metrizable if and only if $\downarrow \mathrm{C}_V(X)$ is second-countable if and only if X is compact and metrizable. The following theorem is our main result.

Theorem 1. *For a Tychonoff space X, the following conditions are equivalent:*

(a) *$\downarrow \mathrm{C}_F(X)$ is separable metrizable;*
(b) *$\downarrow \mathrm{C}_F(X)$ is metrizable.*

In case X is first-countable, the above two conditions are equivalent to

(c) *X is a locally compact and separable metrizable space.*

We also prove the following theorem.

Theorem 2. *Let $\bigoplus_{s \in S} Y_s$ be the topological sum of Tychonoff spaces Y_s, $s \in S$, and $a_s \in Y_s$ a non-isolated point for every $s \in S$. Let, further, Y be the quotient space of $\bigoplus_{s \in S} Y_s$ with the set $\{a_s : s \in S\}$ identified to a point. Then $\downarrow \mathrm{C}_F(Y)$ is homeomorphic to a subspace of the product space $\prod_{s \in S} \downarrow \mathrm{C}_F(Y_s)$.*

Applying this theorem, we show the following.

Corollary 1. *There exists a Tychonoff space X such that $\downarrow \mathrm{C}_F(X)$ is separable metrizable but X is not first-countable.*

The above corollary shows that the first-countability of X is essential for the equivalence between (a) and (c) in Theorem 1. The following Theorem 3 tells us that, the non-compact case is very different from the compact one.

Theorem 3. *There exists a countable Tychonoff space X such that $\downarrow \mathrm{C}_F(X)$ is Hausdorff and second-countable but not regular.*

In [1, 5.1.2 Proposition], it was proved that, for a Tychonoff space X, the following conditions are equivalent: (a) $\mathrm{Cld}_F(X)$ is Hausdorff, (b) $\mathrm{Cld}_F(X)$ is regular, (c) $\mathrm{Cld}_F(X)$ is Tychonoff, and (d) X is locally compact. Theorem 3 shows that we cannot replace $\mathrm{Cld}_F(X)$ by $\downarrow \mathrm{C}_F(X)$ in [1, 5.1.2 Proposition].

The following Theorem 4 states that, even for a compact space X, the regularity and the first-countability of $\downarrow \mathrm{C}_F(X)$ do not imply the metrizability of it.

Theorem 4. *There exists a compact space X such that $\downarrow \mathrm{C}_F(X)$ is Tychonoff, separable and first-countable but not metrizable.*

Finally, we will give a necessary condition for the metrizability of ${\downarrow}\mathrm{C}_F(X)$.

Theorem 5. *For a Tychonoff space X, if ${\downarrow}\mathrm{C}_F(X)$ is metrizable, then there exists a dense, locally compact, open and separable metrizable subspace of X. But the converse is not true.*

2. Preparatory results

In the following, we always assume that X is a Tychonoff space and $p : X \times \mathbf{I} \to X$ is the projection. For $s \in \mathbf{I}$, we use $\underline{s}$ to denote the constant function from X to $\mathbf{I}$ which maps all elements to s. By $\mathbb{R}$ and $\mathbb{Q}$, we denote the sets of all real numbers and of all rational numbers, respectively. Let cl_Y and int_Y be the closure-operator and the interior-operator in a space Y. If $Y = X$, the subscript in the above operators will be omitted. And, for a closed set F in Y, let

$$F^* = (Y \setminus F)^+ = \{A \in \mathrm{Cld}(Y) : A \cap F = \emptyset\}.$$

By the definition, the topology of ${\downarrow}\mathrm{C}_F(X)$ is generated, as a base, by the following sets:

$$\bigcap_{i=1}^{n} G_i^- \cap K^* \cap {\downarrow}\mathrm{C}(X),$$

where $G_1, G_2, \cdots, G_n$ are open sets in $X \times (0,1]$ and K is a compact set in $X \times (0,1]$. In particular,

$$\left\{ \bigcap_{i=1}^{n} G_i^- \cap {\downarrow}\mathrm{C}(X) : G_1, \cdots, G_n \text{ are nonempty open in } X \times (0,1] \right\}$$
$$\text{and } \{K^* \cap {\downarrow}\mathrm{C}(X) : K \text{ is compact in } X \times (0,1]\}$$

are neighborhood bases at ${\downarrow}\underline{1}$ and ${\downarrow}\underline{0}$ in ${\downarrow}\mathrm{C}_F(X)$, respectively.

To prove our theorems, we need some lemmas. At first, we show the following lemma.

Lemma 1. *For a space X, the following hold:*

(1) *${\downarrow}\mathrm{C}_F(X)$ is T_1;*
(2) *${\downarrow}\mathrm{C}_F(X)$ is Hausdorff if and only if there exists a dense open subset U of X which is locally compact.*

Proof: (1): Let $f \neq g \in \mathrm{C}(X)$. We may assume that $f(x_0) < g(x_0)$ for some $x_0 \in X$. Then x_0 has an open neighborhood W such that $f(x) < a < g(x)$ for every $x \in W$, where $a = \frac{f(x_0)+g(x_0)}{2}$. Thus ${\downarrow}f \in (\{x_0\} \times [a,1])^* \not\ni {\downarrow}g$ and ${\downarrow}g \in (W \times (a,1])^- \not\ni {\downarrow}f$.

(2): The "if" part: Take $f, g \in \mathrm{C}(X)$, $x_0 \in W$ and $a \in \mathbf{I}$ as the same as in (1). Since f and g are continuous, we assume that $x_0 \in U$. Because U is locally compact, we have an open set V in X such that $x_0 \in V \subset \mathrm{cl}\, V \subset U \cap W$ and $\mathrm{cl}\, V$ is compact. Since $f(x) < a < g(x)$ for $x \in \mathrm{cl}\, V$, $(\mathrm{cl}\, V \times [a,1])^* \cap {\downarrow}\mathrm{C}(X)$ and $(V \times (a,1])^- \cap {\downarrow}\mathrm{C}(X)$ are disjoint neighborhoods of ${\downarrow}f$ and ${\downarrow}g$, respectively.

The "only if" part: We define an open set

$$U = \bigcup\{\operatorname{int} K : K \text{ is compact in } X\} \subset X.$$

Then U is locally compact. We show that U is dense in X. Assume that U is not dense in X. Then there exists a nonempty open set V in X such that the interior of every compact subset of V is empty. Because X is Tychonoff, we can choose $f \in \mathrm{C}(X)$ such that $f(X \setminus V) \subset \{1\}$ and $f(x_0) = 0$ for some $x_0 \in V$. Since $\downarrow\mathrm{C}_F(X)$ is Hausdorff, there exist disjoint open sets $\mathcal{U}$ and $\mathcal{V}$ in $\downarrow\mathrm{C}_F(X)$ such that $\downarrow\underline{1} \in \mathcal{U}$ and $\downarrow f \in \mathcal{V}$. Then we can find nonempty open sets $G_1, G_2, \cdots, G_n, \cdots, G_m \subset X \times (0,1]$ and a compact set $K \subset X \times (0,1]$ such that

$$\downarrow\underline{1} \in G_1^- \cap G_2^- \cap \cdots \cap G_n^- \cap \downarrow\mathrm{C}(X) \subset \mathcal{U} \text{ and}$$
$$\downarrow f \in G_{n+1}^- \cap \cdots \cap G_m^- \cap K^* \cap \downarrow\mathrm{C}(X) \subset \mathcal{V}.$$

Since $f(X \setminus V) \subset \{1\}$, it follows that $p(K) \subset V$, which implies that $\operatorname{int} p(K) = \emptyset$. For every $i \le m$, $p(G_i) \setminus p(K) \ne \emptyset$ since $p(G_i)$ is a nonempty open set in X. Take $x_i \in p(G_i) \setminus p(K)$. Because X is Tychonoff, we have $g \in \mathrm{C}(X)$ satisfying

$$g(x_i) = 1 \text{ for } i \le m \text{ and } g(p(K)) = \{0\}.$$

Then $\downarrow g \in \mathcal{U} \cap \mathcal{V}$, which contradicts that $\mathcal{U} \cap \mathcal{V} = \emptyset$. □

Lemma 2. *If $\downarrow\mathrm{C}_F(X)$ is first-countable, then there exist compact sets $C_1 \subset C_2 \subset \cdots$ in X such that every compact set in X is contained in some C_n. In particular, $X = \bigcup_{n=1}^{\infty} C_n$.*

Proof: Because $\downarrow\mathrm{C}_F(X)$ is first-countable, we can find compact sets $K_1 \subset K_2 \subset \cdots$ in $X \times (0,1]$ such that $\{K_n^* \cap \downarrow\mathrm{C}(X) : n = 1, 2, \dots\}$ is a neighborhood base of $\downarrow\underline{0}$ in $\downarrow\mathrm{C}_F(X)$. Then $C_n = p(K_n)$, $n = 1, 2, \dots$, are the desired compact sets in X. We verify that every compact set C in X is contained in some C_n. Otherwise, for every n, we can choose $x_n \in C \setminus C_n$ and define $f_n \in \mathrm{C}(X)$ such that $f_n(x_n) = 1$ and $f_n(C_n) = \{0\}$. Then $\downarrow f_n \in K_n^*$ for every n and hence $\downarrow f_n \to \downarrow\underline{0}$ in $\downarrow\mathrm{C}_F(X)$. But every $\downarrow f_n$ is not contained in the neighborhood $(C \times \{1\})^*$ of $\downarrow\underline{0}$, which is a contradiction. □

Lemma 3. *If X and $\downarrow\mathrm{C}_F(X)$ are first-countable, then X is locally compact.*

Proof: Suppose there exists $x_0 \in X$, which has no compact neighborhood. Because X is first-countable, x_0 has a countable open neighborhood base $\{U_n : n = 1, 2, \dots\}$, where $U_n \supset U_{n+1}$ for every n. Since $\downarrow\mathrm{C}_F(X)$ is also first-countable, we can find compact sets $K_1 \subset K_2 \subset \cdots$ in $X \times (0,1]$ such that $\{K_n^* \cap \downarrow\mathrm{C}(X) : n = 1, 2, \dots\}$ is a neighborhood base at $\downarrow\underline{0}$ in $\downarrow\mathrm{C}(X)$. By the assumption, $p(K_n) \not\supset U_n$ for every $n = 1, 2, \dots$, hence we can take $x_n \in U_n \setminus p(K_n)$. Then $x_n \to x_0$ in X. Since X is Tychonoff, we have $f_n \in \mathrm{C}(X)$ such that

$$f_n(x_n) = 1 \text{ and } f_n(p(K_n) \cup (X \setminus U_n)) = \{0\}.$$

Then $\downarrow f_n \in K_n^*$ and hence $\downarrow f_n \to \downarrow \underline{0}$. On the contrary,

$$(\{x_n : n = 0, 1, 2, \cdots\} \times \{1\})^* \cap \downarrow \mathrm{C}(X)$$

is a neighborhood of $\downarrow \underline{0}$ in $\downarrow \mathrm{C}_F(X)$ which does not contain any $\downarrow f_n$. □

When X is locally compact and non-compact, let $\alpha X = X \cup \{\infty\}$ be the one-point compactification of X. Using Lemmas 2 and 3, we may prove the following

Proposition 1. *If X and $\downarrow \mathrm{C}_F(X)$ are first-countable, then*

(1) *X is locally compact and αX is also first-countable;*
(2) *$\downarrow \mathrm{C}_F(\alpha X)$ is first-countable;*
(3) *$\downarrow \mathrm{C}_F(\alpha X)$ is second-countable if $\downarrow \mathrm{C}_F(X)$ is second-countable.*

Proof: The assertion (1) directly follows from Lemmas 2 and 3. To show (2) and (3), we only consider the case that X is not compact. Let $\{U_n : n = 1, 2, \ldots\}$ be a countable open neighborhood base at ∞ in αX, and let $\phi : \mathrm{C}(\alpha X) \to \mathrm{C}(X)$ be the restriction, that is,

$$\phi(f) = f|X \quad \text{for every} \quad f \in \mathrm{C}(\alpha X).$$

Then it is not hard to verify that $\downarrow \phi : \downarrow \mathrm{C}_F(\alpha X) \to \downarrow \mathrm{C}_F(X)$ is a continuous injection. Unfortunately, it is not an embedding. However, the following $\mathcal{S}$ is a subbase of $\downarrow \mathrm{C}_F(\alpha X)$:

$$\mathcal{S} = \{(\downarrow\phi)^{-1}(G) : G \in \mathcal{G}\} \\ \cup \{(\mathrm{cl}_{\alpha X} U_n \times [r, 1])^* \cap \downarrow \mathrm{C}(\alpha X) : r \in \mathbb{Q} \cap (0, 1], n = 1, 2, \ldots\},$$

where $\mathcal{G}$ is an open base for $\downarrow \mathrm{C}_F(X)$. Obviously, $\mathcal{S}$ is a subfamily of the topology of $\downarrow \mathrm{C}_F(\alpha X)$. For every open set V in $\alpha X \times \mathbf{I}$, $V \cap (X \times \mathbf{I})$ is open in $X \times \mathbf{I}$ and

$$V^- \cap \downarrow \mathrm{C}(\alpha X) = (\downarrow\phi)^{-1}((V \cap (X \times \mathbf{I}))^- \cap \downarrow \mathrm{C}(\alpha X)).$$

For every compact set K in $\alpha X \times (0, 1]$, if $K \cap (\{\infty\} \times \mathbf{I}) = \emptyset$, then K is also compact in $X \times \mathbf{I}$ and

$$K^* \cap \downarrow \mathrm{C}(\alpha X) = (\downarrow\phi)^{-1}(K^* \cap \downarrow \mathrm{C}(X)).$$

If $K \cap (\{\infty\} \times \mathbf{I}) \neq \emptyset$, then for every $\downarrow f \in K^* \cap \downarrow \mathrm{C}(\alpha X)$, using the Wallace's Theorem, there exist n and a rational number $r \in (0, 1]$ such that

$$(\mathrm{cl}_{\alpha X} U_n \times [r, 1]) \cap \downarrow f = \emptyset \quad \text{and}$$
$$K \cap (\mathrm{cl}_{\alpha X} U_n \times \mathbf{I}) \subset \mathrm{cl}_{\alpha X} U_n \times [r, 1].$$

Let

$$K_1 = (K \cap ((\alpha X \setminus U_n) \times \mathbf{I})) \cup (\mathrm{cl}_{\alpha X} U_n \times [r, 1]).$$

Then K_1 is compact in $\alpha X \times (0,1]$, $K_1 \supset K$ and $K_1 \cap \downarrow f = \emptyset$. Thus, $\downarrow f \in K_1^* \subset K^*$. Note that

$$K_1^* \cap \downarrow \mathrm{C}_F(\alpha X) = (\downarrow\phi)^{-1}((K \cap ((\alpha X \setminus U_n) \times \mathbf{I}))^*) \\ \cap (\mathrm{cl}(U_n) \times [r,1])^* \cap \downarrow \mathrm{C}_F(\alpha X),$$

that is, $K_1^* \cap \downarrow \mathrm{C}_F(\alpha X)$ is an intersection of two elements of $\mathcal{S}$.

As a conclusion, $\mathcal{S}$ is a subbase for $\downarrow \mathrm{C}_F(\alpha X)$. Therefore, $\downarrow \mathrm{C}_F(\alpha X)$ is first-countable. Moreover, $\downarrow \mathrm{C}_F(\alpha X)$ is second-countable if $\downarrow \mathrm{C}_F(X)$ is second-countable. Hence (2) and (3) hold. $\square$

Lemma 4. *We consider the following statements.*

(a) $\downarrow \mathrm{C}_F(X)$ *is first-countable.*

(b) $\downarrow \mathrm{C}_F(X)$ *has a countable neighborhood base at* $\downarrow \underline{1}$.

(c) *There exists a countable family* $\mathcal{U}$ *of nonempty open sets in* X *such that every nonempty open set in* X *includes an element of* $\mathcal{U}$, *that is,* $\mathcal{U}$ *is a countable* π*-base for* X.

(d) $\downarrow \mathrm{C}_F(X)$ *is separable.*

Then the implications (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d) *hold.*

Furthermore, when X *is compact, the implication* (c)$\Rightarrow$(a) *holds and hence* (a), (b) *and* (c) *are equivalent.*

Proof: The implication (a)$\Rightarrow$(b) is trivial.

(b)$\Rightarrow$(c): We may assume that

$$\{(G_1^n)^- \cap (G_2^n)^- \cap \cdots \cap (G_{k(n)}^n)^- \cap \downarrow \mathrm{C}(X) : n = 1, 2, \dots\}$$

is a countable neighborhood base at $\downarrow \underline{1}$ in $\downarrow \mathrm{C}_F(X)$. Let

$$\mathcal{U} = \{p(G_i^n) : i = 1, 2, \dots, k(n), n = 1, 2, \dots\}.$$

Then $\mathcal{U}$ is a countable family of nonempty open sets in X. We show that every nonempty open set U in X includes an element of $\mathcal{U}$. Take $f \in \mathrm{C}(X)$ such that $f(X \setminus U) \subset \{1\}$ and $f(x_0) = 0$ for some point $x_0 \in U$. Because $\downarrow \mathrm{C}_F(X)$ is T_1 by Lemma 1(1), $\downarrow f \notin \bigcap_{i=1}^{k(n)} (G_i^n)^-$ for some n, hence $\downarrow f \notin (G_i^n)^-$ for some $i \le k(n)$. Then $\downarrow f \cap G_i^n = \emptyset$. Since $f(X \setminus U) \subset \{1\}$, we have $U \supset p(G_i^n)$, as required.

(c)$\Rightarrow$(d): Let $\mathcal{U}$ be a countable π-base for X. For every $U \in \mathcal{U}$ and $r \in \mathbb{Q} \cap (0,1]$, we can take a continuous map $f_{(U,r)} : X \to [0,r]$ such that $f_{(U,r)}(X \setminus U) \subset \{0\}$ and $f_{(U,r)}(x) = r$ for some $x \in U$. Let

$$D = \{\max\{f_{(U,r)} : U \in \mathcal{F}, r \in F\} : \mathcal{F} \text{ and } F \text{ are} \\ \text{finite subsets of } \mathcal{U} \text{ and } \mathbb{Q} \cap (0,1], \text{ resp.}\}.$$

Then $\downarrow D = \{\downarrow f : f \in D\}$ is a countable subset of $\downarrow \mathrm{C}(X)$. It remains to verify that $\downarrow D$ is dense in $\downarrow \mathrm{C}_F(X)$. Let $f \in \mathrm{C}(X)$, K be compact in $X \times (0,1]$ and G_i,

$i \leq k$, open in $X \times (0,1]$, such that

$$\downarrow f \in G_1^- \cap G_2^- \cap \cdots \cap G_k^- \cap K^* \cap \downarrow \mathrm{C}(X).$$

We have $x_1, \cdots, x_k \in$ X such that $\{x_i\} \times [0, f(x_i)] \cap G_i \neq \emptyset$ for each $i \leq k$. Because $\{x_i\} \times [0, f(x_i)] \cap K = \emptyset$, we have an open neighborhood W_i of x_i in X and $s_i < t_i$ such that $W_i \times (s_i, t_i) \subset G_i$ and $W_i \times [0, t_i] \cap K = \emptyset$. Thus, by (c), choose $r_i \in \mathbb{Q} \cap (s_i, t_i)$ and $U_i \in \mathcal{U}$ such that $U_i \subset W_i$. Then $\downarrow f_{(U_i, r_i)} \in G_i^- \cap K^*$ and hence

$$\downarrow \max\{f_{(U_i, r_i)} : i \leq k\} \in \downarrow D \cap G_1^- \cap G_2^- \cap \cdots \cap G_k^- \cap K^*.$$

Now, we show (c)$\Rightarrow$(a) under the assumption that X is compact. Let $\mathcal{U}$ be a countable π-base of X. Then, $X \times \mathbf{I}$ has the following countable π-base:

$$\mathcal{G} = \{U \times (s,t) : U \in \mathcal{U}, s < t \in \mathbb{Q} \cap (0,1)\}.$$

For every $f \in \mathrm{C}(X)$ and $n = 1, 2, \ldots$, let

$$\mathcal{G}(f) = \{G \in \mathcal{G} : \downarrow f \in G^-\}, \quad K_n(f) = \{(x,t) \in X \times \mathbf{I} : t \geq f(x) + n^{-1}\}.$$

For every open set H in $X \times (0,1]$ with $H^- \ni \downarrow f$, there exists $x_0 \in X$ such that $\{x_0\} \times [0, f(x_0)] \cap H \neq \emptyset$. Since $f(x_0) > 0$, we can find an open neighborhood V of x_0 in X and $s < t \in \mathbb{Q} \times (0,1)$ such that $s < f(x_0)$, $V \times (s,t) \subset H$ and $s < f(x)$ for every $x \in V$. Since $\mathcal{U}$ is a π-base for X, V contains some $U \in \mathcal{U}$. Then we have $G = U \times (s,t) \in \mathcal{G}$ and $\downarrow f \in G^- \subset H^-$. Moreover, for every compact set K in $X \times \mathbf{I}$ with $K^* \ni \downarrow f$, by the compactness of X, there exists n such that $K_n(f) \supset K$ and hence $\downarrow f \in K_n(f)^* \subset K^*$. Therefore,

$$\{G_1^- \cap \cdots \cap G_k^- \cap K_n(f)^* \cap \downarrow \mathrm{C}(X) : G_i \in \mathcal{G}(f) \ \text{ for } \ i \leq k, k, n = 1, 2, \ldots\}$$

is a countable neighborhood base at $\downarrow f$ in $\downarrow \mathrm{C}_F(X)$. □

As a consequence of Lemma 4, we have the equivalence between (a) and (b) in Theorem 1, that is,

Proposition 2. *The space $\downarrow \mathrm{C}_F(X)$ is metrizable if and only if it is separable metrizable.* □

We need the following two lemmas which were proved in [8], [9], respectively.

Lemma 5. *If V is open in X such that $\mathrm{cl}\, V$ is compact, then the restriction $\phi : \downarrow \mathrm{C}_F(X) \to \downarrow \mathrm{C}_F(\mathrm{cl}\, V)$ defined by $\phi(\downarrow f) = \downarrow f | \mathrm{cl}\, V$ is a continuous open surjection.* □

Lemma 6. *If X is compact and $\downarrow \mathrm{C}_F(X) = \downarrow \mathrm{C}_V(X)$ is second-countable, then X is metrizable.* □

3. Proofs of main results

In this section, we show our main results.

Proof of Theorem 1: The equivalence between (a) and (b) is Proposition 2. If X is first-countable, then X is locally compact by Proposition 1(1). Using Proposition 1(3), the condition (b) implies that $\downarrow\mathrm{C}(\alpha X)$ is second-countable. It follows from Lemma 6 that αX is metrizable. Hence the condition (c) holds. That is, the implication (b)$\Rightarrow$(c) holds under the assumption that X is first-countable. The condition (c) implies that $\mathrm{Cld}_F(X \times \mathbf{I})$ is metrizable ([1, 5.1.5 Theorem]), hence so is $\downarrow\mathrm{C}_F(X)$, i.e., (b) holds. Therefore, the implication (c)$\Rightarrow$(b) holds. $\square$

Proof of Theorem 2: We may think that every Y_s is a subspace of Y. Define $\phi : \mathrm{C}(Y) \to \prod_{s\in S} \mathrm{C}(Y_s)$ by

$$\phi(f) = (f|Y_s)_{s\in S} \quad \text{for each} \quad f \in \mathrm{C}(Y).$$

Evidently, ϕ is an injection and its image is

$$\phi(\mathrm{C}(Y)) = \left\{ g \in \prod_{s\in S} \mathrm{C}(Y_s) : g(s)(a_s) = g(s')(a_{s'}) \text{ for } s, s' \in S \right\}.$$

Now we show that $\downarrow\phi :\downarrow\mathrm{C}_F(Y) \to \prod_{s\in S} \downarrow\mathrm{C}_F(Y_s)$ is an embedding. Let $p_s : \prod_{s\in S} \downarrow\mathrm{C}_F(Y_s) \to \downarrow\mathrm{C}_F(Y_s)$ be the projection.

To show the continuity of $\downarrow\phi$, it is sufficient to verify that $p_s\circ\downarrow\phi$ is continuous for every $s \in S$. For every open set G in $Y_s \times (0,1]$, $G \setminus (\{a_s\} \times \mathbf{I})$ is open in $Y \times (0,1]$. Since a_s is a non-isolated point in Y_s,

$$(p_s\circ\downarrow\phi)^{-1}(G^-\cap\downarrow\mathrm{C}(Y_s)) = (G \setminus (\{a_s\} \times \mathbf{I}))^-\cap\downarrow\mathrm{C}(Y).$$

For each compact set K in $Y_s \times (0,1]$,

$$(p_s\circ\downarrow\phi)^{-1}(K^*\cap\downarrow\mathrm{C}(Y_s)) = K^*\cap\downarrow\mathrm{C}(Y).$$

Hence, $p_s\circ\downarrow\phi :\downarrow\mathrm{C}_F(Y) \to\downarrow\mathrm{C}_F(Y_s)$ is continuous for every $s \in S$.

Moreover, for every open set H in $Y \times (0,1]$, if $\downarrow f \in H^-\downarrow\mathrm{C}_F(Y)$, then there exists $s \in S$ such that $\downarrow f|Y_s \in (H \cap (Y_s \times \mathbf{I}))^-$. Hence

$$\downarrow\phi(H^-\cap\downarrow\mathrm{C}_F(Y)) = \bigcup_{s\in S}\left((H \cap (Y_s \times \mathbf{I}))^- \times \prod_{t\in S\setminus\{s\}} \downarrow\mathrm{C}(Y_t)\right)\cap\downarrow\phi(\downarrow(\mathrm{C}(Y))).$$

It shows that $\downarrow\phi(H^-\cap\downarrow\mathrm{C}_F(Y))$ is open in $\downarrow\phi(\downarrow(\mathrm{C}_F(Y)))$. For every compact set K in $Y \times (0,1]$, there exists a finite subset S_0 of S such that $K \subset \bigcup_{s\in S_0} Y_s \times (0,1]$. Then $K \cap Y_s \times (0,1]$ is compact for every $s \in S_0$ and

$$\downarrow\phi(K^*\cap\downarrow\mathrm{C}(Y)) = \left(\prod_{s\in S_0} (K \cap Y_s \times (0,1])^* \times \prod_{s\in S\setminus S_0} \downarrow\mathrm{C}(Y_s)\right)\cap\downarrow\phi(\downarrow\mathrm{C}(Y)).$$

It follows that $\downarrow\phi(K^*\cap\downarrow\mathrm{C}(Y))$ is open in $\downarrow\phi(\downarrow(\mathrm{C}_F(Y)))$. Since ϕ is one-to-one, we have that $\downarrow\phi$ maps every open set in $\downarrow\mathrm{C}_F(Y)$ to an open set in $\downarrow\phi(\downarrow(\mathrm{C}_F(Y)))$.

Therefore, $\downarrow\phi:\downarrow\mathrm{C}_F(Y)\to\prod_{s\in S}\downarrow\mathrm{C}_F(Y_s)$ is an embedding. □

Remark 1. Even for a set S of two points, if a_s is an isolated point in Y_s for some s, the map $\downarrow\phi$ defined in the above proof needs not be continuous. For example, let $Y_1=\{1\}\times(\{0\}\cup[1,2]), Y_2=\{2\}\times\mathbf{I}$ as subspaces of $\mathbb{R}^2$. If we think that $a_1=(1,0), a_2=(2,0)$, then $p_1\circ\downarrow\phi:\downarrow\mathrm{C}(Y)\to\downarrow\mathrm{C}(Y_1)$ is not continuous. In fact, choose $f_n\in\mathrm{C}(Y)$ such that $f_n(2,0)=f_n(1,0)=0$ and $f_n(x)=1$ for every $x\in Y\setminus(\{2\}\times[0,n^{-1}])$. Then $\downarrow f_n\to\downarrow\underline{1}$ but $(p_1\circ\downarrow\phi)(\downarrow f_n)\not\to(p_1\circ\downarrow\phi)(\downarrow\underline{1})$.

Proof of Corollary 1: Let $\{Y_n:n=1,2,\dots\}$ be a family of pairwise disjoint locally compact separable metrizable spaces Y_n with a non-isolated point a_n. Then, by Theorems 1 and 2, the space Y defined in Theorem 2 is as required. □

Proof of Theorem 3: Let $\beta\omega$ be the Čech-Stone compactification of the discrete space ω of non-negative integers and $q\in\beta\omega\setminus\omega$. Then the subspace $X=\omega\cup\{q\}$ of $\beta\omega$ satisfies the conditions in Theorem 3. By Lemma 1(2), $\downarrow\mathrm{C}_F(X)$ is Hausdorff.

Before showing that $\downarrow\mathrm{C}_F(X)$ is second-countable but not regular, we verify that every compact subset of X is finite. In fact, let C be an infinite compact subset of X. Then $q\in C$. Write $C=A\cup B\cup\{q\}$ such that A and B are disjoint infinite subsets of ω. Define a continuous map $f:\omega\to\{0,1\}$ as $f^{-1}(0)=A$. Then there exists a continuous extension $\overline{f}:X\to\{0,1\}$ since X is a subspace of $\beta\omega$. If $\overline{f}(q)=0$, then B is closed in X and hence is compact. But it is impossible since B is infinite discrete. If $\overline{f}(q)=1$, then A is closed in X and hence is compact. It is also impossible since A is also infinite discrete.

Now, we define a product space $Y=\prod_{x\in X}\mathbf{I}_x$, where $\mathbf{I}_x$ is a copy of the unit interval $[0,1]$ with the usual topology for $x\in\omega$ and $\mathbf{I}_q$ is $[0,1]$ with the topology generated by $\{[0,r):r\in[0,1]\cap\mathbb{Q}\}\cup\{[0,1]\}$. Then Y is second-countable. We may regard $\downarrow\mathrm{C}(X)\subset Y$ by identifying $\downarrow f$ with $(f(x))_{x\in X}$ for every $f\in\mathrm{C}(X)$. To show that $\downarrow\mathrm{C}_F(X)$ is second-countable, it suffices to verify that $\downarrow\mathrm{C}_F(X)$ is the subspace of the space Y. It is easy to see that for each $x\in X$, the map $p_x:\downarrow\mathrm{C}_F(Y)\to\mathbf{I}_x$ defined by $p_x(\downarrow f)=f(x)$ is continuous. Hence the subspace topology is coarser than the Fell topology on $\downarrow\mathrm{C}(X)$. Conversely, take a compact set $K\subset X\times(0,1]$ and $f\in\mathrm{C}(X)$. Then $p(K)$ is compact in X. Then $p(K)$ is a finite set in X and $\downarrow f\cap K=\emptyset$ if and only if $f(x)<m(x)=\min\{s:(x,s)\in K\}$ for every $x\in p(X)$. Hence we can identify

$$K^*\cap\downarrow\mathrm{C}(X)=\left(\prod_{x\in p(K)}[0,m_x)\times\prod_{x\in X\setminus p(K)}\mathbf{I}_x\right)\cap\downarrow\mathrm{C}(X)$$

is open in the subspace topology of Y. For every open set G in $X\times(0,1]$ and

$f \in \mathrm{C}(X)$, ${\downarrow}f \cap G \neq \emptyset$ if and only if ${\downarrow}f \cap G \setminus (\{q\} \times \mathbf{I}) \neq \emptyset$ if and only if $f(n) > s_n$ for some $n \in p(G) \cap \omega$, where $s_n = \inf\{s : (n,s) \in G\}$. Hence

$$G^- \cap {\downarrow}\mathrm{C}(X) = \left(\bigcup_{n \in p(G)\cap\omega} p_n^{-1}(s_n, 1] \right) \cap {\downarrow}\mathrm{C}(X),$$

where $p_n : Y \to \mathbf{I}_n$ is the projection, is open in the subspace topology of Y. Therefore, ${\downarrow}\mathrm{C}_F(X)$ is the subspace of Y.

To show that ${\downarrow}\mathrm{C}_F(X)$ is not regular, we consider an open neighborhood $\mathcal{U} = (\{q\} \times [\frac{1}{2}, 1])^* \cap {\downarrow}\mathrm{C}(X)$ of ${\downarrow}0$. For every compact set K in $X \times (0,1]$, $p(K)$ is finite. Define $f \in \mathrm{C}(X)$ such that $f^{-1}(0) = p(K) \cap \omega$ and $f^{-1}(1) = X \setminus (p(K) \cap \omega)$. Then ${\downarrow}f \in \mathrm{cl}_{{\downarrow}\mathrm{C}_F(X)}(K^* \cap {\downarrow}\mathrm{C}_F(X)) \setminus \mathcal{U}$. In fact, every neighborhood of ${\downarrow}f$ in ${\downarrow}\mathrm{C}_F(Y)$ contains the following neighborhood of ${\downarrow}f$:

$$\mathcal{G} = G_1^- \cap \cdots \cap G_k^- \cap G^- \cap C^* \cap {\downarrow}\mathrm{C}_F(X),$$

where $G_i = \{n_i\} \times (s_i, t_i)$ for $1 \le i \le k$ and $G = (A \cup \{q\}) \times (s,t)$ are open and C is compact in $X \times (0,1]$. Then A is an infinite subset of ω and hence we may choose $n_0 \in A \setminus p(K \cup C)$. Now, define $g \in \mathrm{C}(X)$ as

$$g(x) = \begin{cases} 0 & \text{if } \ x \in A \cup \{q\} \setminus \{n_i : 0 \le i \le k\}; \\ 1 & \text{if } \ x = n_0; \\ f(x) & \text{otherwise.} \end{cases}$$

Then it is easy to verify that ${\downarrow}g \in \mathcal{G} \cap K^*$. This shows that ${\downarrow}f \in \mathrm{cl}_{{\downarrow}\mathrm{C}_F(X)}(K^* \cap {\downarrow}\mathrm{C}_F(X))$. Because $f(q) = 1$, we have ${\downarrow}f \notin \mathcal{U}$. Hence, $\mathrm{cl}_{{\downarrow}\mathrm{C}_F(X)}(K^* \cap {\downarrow}\mathrm{C}(X)) \not\subset \mathcal{U}$ for any compact K in $X \times (0,1]$. Note that the family of all of such $K^* \cap {\downarrow}\mathrm{C}_F(X)$ is a neighborhood base at ${\downarrow}\underline{0}$ in ${\downarrow}\mathrm{C}_F(X)$. Therefore, ${\downarrow}\mathrm{C}_F(X)$ is not regular. $\square$

Proof of Theorem 4: Choose a compact Hausdorff non-metrizable space X satisfying (c) in Lemma 4, for example, $\beta\omega$ or Helly space (see [2, Problem 5.M]). Then, by Lemma 4, ${\downarrow}\mathrm{C}_F(X)$ is separable and first-countable. By [3] (cf. [1, 5.1.2 Proposition]), $\mathrm{Cld}_F(X \times \mathbf{I}) = \mathrm{Cld}_V(X \times \mathbf{I})$ is Tychonoff and hence so is ${\downarrow}\mathrm{C}_F(X)$. Since X is compact and non-metrizable, ${\downarrow}\mathrm{C}_F(X)$ is not second-countable because of Lemma 6. According to Proposition 2, if ${\downarrow}\mathrm{C}_F(X)$ is metrizable, then ${\downarrow}\mathrm{C}_F(X)$ is separable metrizable, hence second-countable. Therefore, ${\downarrow}\mathrm{C}_F(X)$ is not metrizable. $\square$

Proof of Theorem 5: Assume that ${\downarrow}\mathrm{C}_F(X)$ is metrizable, which means that ${\downarrow}\mathrm{C}_F(X)$ is separable metrizable by Proposition 2. Then ${\downarrow}\mathrm{C}_F(X)$ is second-countable. By Lemma 1(2), there exists a dense open set U in X such that U is locally compact. To complete the proof, it remains to verify that U is separable metrizable. By Lemma 2, there exists a countable family $\mathcal{C} = \{C_1, C_2, \cdots\}$ of compact sets in X such that every compact set in X is contained in some C_n. For each n, let $U_n = \mathrm{int}(U \cap C_n)$. Then, $\mathrm{cl}\, U_n$ is compact because $\mathrm{cl}\, U_n \subset C_n$. By

Lemma 5, there exists a continuous open surjection from $\downarrow\mathrm{C}_F(X)$ onto $\downarrow\mathrm{C}_F(\mathrm{cl}\,U_n)$. Therefore, $\downarrow\mathrm{C}_F(\mathrm{cl}\,U_n)$ is second-countable, hence $\mathrm{cl}\,U_n$ is compact and metrizable by Lemma 6. Thus every U_n is also separable metrizable, hence it is second-countable. Moreover, for every $x \in U$, there exists an open set V such that $x \in V$, $\mathrm{cl}\,V$ is compact and $\mathrm{cl}\,V \subset U$. Hence there exists n such that $\mathrm{cl}\,V \subset C_n$. Then, $x \in V \subset \mathrm{int}(U \cap C_n) = U_n$. It follows that $U = \bigcup_{n=1}^{\infty} U_n$. Therefore, U is second-countable, hence it is separable metrizable.

As mentioned in proof of Theorem 4, $\beta\omega$ is a compact space and $\downarrow\mathrm{C}_F(\beta\omega)$ is not metrizable but ω is a dense, locally compact, open and separable metrizable subspace of $\beta\omega$. Namely, the converse is not true. □

Remark 2. The referee pointed out that McCoy and Ntantu [11] obtained analogous results in 1992. For example, Theorem 4.12 in [11] is similar to our Theorem 1. Our Theorem 3 for $\downarrow\mathrm{C}_F(X,\mathbf{I})$ is true for $\uparrow\mathrm{C}_F(X,\mathbb{R})$ using Theorems 3.5, 3.7, 4.11 and Example 3.3 in [11], where $\uparrow\mathrm{C}_F(X,\mathbb{R})$ is the subspace of $\mathrm{Cld}_F(X\times\mathbb{R})$ consisting of the epigraphs

$$\uparrow f = \{(x,s) \in X \times \mathbb{R} : f(x) \leq s\} \in \mathrm{Cld}(X \times \mathbb{R}),$$

of all $f \in \mathrm{C}(X,\mathbb{R})$. However our arguments are quite different from their arguments in [11].

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