Do finite Bruck loops behave like groups?

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Abstract. This note contains Sylow's theorem, Lagrange's theorem and Hall's theorem for finite Bruck loops. Moreover, we explore the subloop structure of finite Bruck loops.

Keywords: finite loops, finite Bruck loops, finite Bol loops, finite A_r-loops, classical theorems for finite loops

Classification: 20N05, 20B05

1. Introduction

Let $(X, \circ)$ be a finite loop; that is a finite set together with a binary operation $\circ$ on X, such that there exists an element $1 \in X$ with $1 \circ x = x \circ 1 = x$ for all $x \in X$ and such that the left and right translations

$$\lambda_x : X \to X,\ y \mapsto x \circ y, \quad \rho_y : X \to X,\ x \mapsto x \circ y$$

are bijections. Loops can be thought of as groups without the associativity law.

Though loops are a generalisation of groups, general loops can be very wild due to the missing associativity. For instance left and right inverses may not be identical, the powers of elements may not be definable in the usual way.

Bol [Bo37] introduced the so called (*right*) *Bol identity*:

$$((x \circ y) \circ z) \circ y = x \circ ((y \circ z) \circ y) \ \text{ for all } \ x, y, z \in X$$

and a loop is called a (right) *Bol loop* if it satisfies this identity. A consequence of the identity is that the subloop generated by one element is a (cyclic) group. Therefore powers and inverses of elements are well defined.

Nagy [N] showed that general Bol loops behave still very differently from groups. While groups of odd order are soluble due to work of Feit and Thompson, there are simple Bol loops of odd non-prime order [N].

Bol loops which satisfy the *automorphic inverse property* AIP:

$$(x \circ y)^{-1} = x^{-1} \circ y^{-1} \ \text{ for all } \ x, y \in X$$

are called *Bruck loops*. Groups satisfy (AIP) if and only if they are abelian. Thus Bruck loops generalise abelian groups.

There are general similarities between loops and groups: In every loop X the order of every normal subloop T, i.e. T is the kernel of some homomorphism π

from X to another loop, divides the order of X, see [AA43, Theorem 4]. Thus Lagrange‘s theorem holds for normal subloops. Moreover, the theorem on homomorphisms $X/T \cong \pi(X)$ holds, see [AA43, Theorem 4]. See also the isomorphism theorems for loops in [AA43], [S44] and the Jordan-Hölder Theorem in [Ba45].

There are certainly huge differences between Bruck loops and groups. In this note we will see that some basic classical group theoretical theorems can be extended to the theory of finite Bruck loops. We will focus on the similarities between groups and Bruck loops by extending Lagrange's theorem and to some extent Sylow's theorem and Hall's theorem to finite Bruck loops. Thus Bruck loops are more general than abelian groups — they are neither associative nor abelian nor soluble in general — but behave nevertheless nicely. Note also the work of Gagola [Ga11] on similarities between certain Moufang loops and groups as well as the paper by Jedlička, Kinyon and Vojtěchovský showing Cauchy's theorem, Lagrange's theorem and the odd order theorem for commutative automorphic loops [JKV11]. Here, we study the subloop structure of the finite Bruck loops as well.

This paper is organized as follows. After fixing the notation in the next section, we recall the general structure theorem of finite Bruck loops. In the fourth section we discuss Sylow's theorem and in the fifth the subloop structure. In the last two sections we present Lagrange's and Hall's theorems. Some parts of the paper already have been proven in [G64], [G68] or in [BS11] and some are new.

2. Notation and definitions

Here we follow the notation of Aschbacher [A05] and [AKP06]. Given a loop X, let $G := \langle \rho_x : x \in X \rangle \leq \mathrm{Sym}(X)$, which is usually called *right multiplication group* and by Aschbacher *enveloping group of* X. The set $K := \{\rho_x : x \in X\}$ is a transversal to $H := \mathrm{Stab}_G(1)$, the *right inner mapping group*, in G, and (G, H, K) is called the *Baer envelope of* X.

The following easily verified properties hold.

(1) $1 \in K$ and K is a right transversal to all conjugates of H in G.
(2) H is core free.
(3) $G = \langle K \rangle$.

Baer [Ba39] observed that whenever (G, H, K) is a triple with G a group, $H \leq G$ and $K \subseteq G$ satisfying condition (1), then we get a loop on K by setting $x \circ y = z$, $x, y \in K$ whenever z is the element in K such that $Hxy = Hz$. This loop is called the *loop related to* (G, H, K).

The triple (G, H, K) with G a group, $H \leq G$ and $K \subseteq G$ is called a *loop folder*, *faithful loop folder* or *loop envelope* if (1), (1) and (2) or (1) and (3) hold, respectively. In general there are many different loop folders for a given loop.

If X is a Bol loop and (G, H, K) the Baer envelope of X, then K is a *twisted subgroup*, that is $1 \in K$ and whenever $x, y \in K$, then x^{-1} and xyx is in K. If, moreover, X is a Bruck loop, then H acts on K by conjugation, [A05, 4.1].

Subloops, homomorphisms, normal subloops, factor loops and simple loops are defined as usual in universal algebra: A *subloop* is a nonempty subset which is closed under loop multiplication and is itself a loop with that operation.

Homomorphisms are maps between loops which commute with loop multiplication. The map defines an equivalence relation on the loop, such that the product of equivalence classes is again an equivalence class. *Normal subloops* are preimages of 1 under a homomorphism and therefore subloops. A normal subloop defines a partition of the loop into blocks (cosets), such that the set of products of elements from two blocks is again a block. Such a construction gives factor loops as homomorphic images with the block containing 1 as the kernel. *Simple loops* have only the full loop and the trivial loop as normal subloops.

For instance if (G, H, K) is a loop folder defining a loop X and G_0 a normal subgroup of G which contains H, then $(G_0, H, G_0 \cap K)$ is a loop folder for a normal subloop X_0 of X.

A loop X is *soluble* if there exists a series $1 = X_0 \leq \cdots \leq X_n = X$ of subloops with X_i normal in X_{i+1} and X_{i+1}/X_i an abelian group.

3. The general structure theorem

In this and the following sections X always is a finite Bruck loop and (G, H, K) always denotes the Baer envelope of X.

In this section we recall the fundamental theorem on Bruck loops proved in [BS11]. We need some more notation. As usual a 2-element is an element whose order is a power of 2. Moreover, for G a group $O(G)$ is the biggest normal subgroup of G of odd order, $O_2(G)$ the biggest normal subgroup of G whose order is a power of 2 and $O^{2'}(G)$ is the smallest normal subgroup of G such that $G/O^{2'}(G)$ is a group of odd order, see for instance [A86].

For X a loop, $O(X)$ is the largest normal subloop of X of odd order and $O^{2'}(X)$ is the subloop of X which is generated by all the 2-elements of X, see [AKP06].

Theorem 1 ([BS11, Theorem 1]). *Let X be a finite Bruck loop. Then the following holds.*

(a) $X = O(X) \times O^{2'}(X)$.

(b) *$O^{2'}(X)$ is the set of 2-elements of X.*

(c) *Let G be the enveloping group of X. Then $G = O(G) \times O^{2'}(G)$.*

(d) *A loop envelope (G, H, K) of $O^{2'}(X)$ where H acts on K and K is a twisted subgroup consisting of 2-power elements satisfies the following.*

  (1) *$\overline{G} := G/O_2(G) \cong D_1 \times D_2 \times \cdots \times D_e$ with $D_i \cong \mathrm{PGL}_2(q_i)$, $q_i \geq 5$ a Fermat prime or $q_i = 9$ and e a non-negative integer.*

  (2) *$D_i \cap \overline{H}$ is a Borel subgroup in D_i.*

  (3) *$F^*(G) = O_2(G)$.*

  (4) *$\overline{K}$ is the set of involutions in $\overline{G} \setminus \overline{G}'$.*

Remark to Theorem 1. In particular, in a finite Bruck loop the set of 2-elements forms a subloop.

The theorem implies that if X is a Bruck loop of 2-power exponent, then its order is very restricted. Recall that for n a natural number and p a prime n_p is the p-part of n, i.e. $n = n_p x$, p does not divide x and n_p is a power of p.

Lemma 3.1 ([BS11, Corollary 3.6])**.** *Let X be a Bruck loop of 2-power exponent. Then*

$$|X| = 2^a \prod_{i=1}^{e} (q_i + 1)$$

for some $e \in \mathbb{N} \cup \{0\}$, where $q_i = 9$ or $q_i \geq 5$ a Fermat prime. Moreover, $|X|_2 = 2^{a+e}$. If (G, H, K) is the Baer envelope of X, then $2^a = |O_2(G) : O_2(G) \cap H|$.

Remark. If X is a Bruck loop of 2-power exponent which is not soluble, then $e \geq 1$. If $e = 1$ and $q_1 = 5$, then $a \geq 4$, see [BS10, Theorem 3].

4. Sylow's theorem

In order to be able to formulate Sylow's or later Hall's theorem we need further notation: Let π be a set of primes. A natural number n is a *π-number* if $n = 1$ or n is the product of powers of primes in π. Assume that X is a loop such that every element of the loop generates a group. We say that X is a *π-loop* if the order of X is a π-number. Notice that this definition is different from the one given in [G64]. For Bruck loops of odd order these two concepts coincide (see [G64, p. 394, Corollary 2]), but not for loops of even order (see the Aschbacher loop in [BS10]).

In order to distinguish the two concepts we propose to use the following notations: A *local π-loop* is a loop such that the orders of the elements are all π-numbers — for instance a loop of 2-power exponent is a local 2-loop — and a *global π-loop* is a loop such that the order of the loop is a π-number. In general we will simply say π-loop instead of global π-loop.

As already mentioned there are local 2-loops which are not global 2-loops: the Aschbacher loop is of order 96, so not a 2-loop, but every element in that loop is of order 2, see [BS10]. We say that a subloop Y of X is a *Sylow p-subloop* of X, if $|Y| = |X|_p$.

In this section we discuss for finite Bruck loops X the following *Sylow-properties*.

- (S1) If p divides $|X|$, then X contains a Sylow p-subloop.
- (S2) The Sylow p-subloops are all conjugate under the action of the right inner mapping group H.
- (S3) Every p-subloop is contained in a Sylow p-subloop.
- (S4) The number of Sylow p-subloops of X is not divisible by p.

In Bruck loops of even order Sylow's theorem holds for the prime $p = 2$, see [BS11]:

Theorem 2 ([BS11] Sylow's Theorem)**.** *Let X be a Bruck loop of even order. Then the Sylow-properties* (S1)–(S4) *hold for the prime $p = 2$.*

Proof: We still need to prove (S4), which was not shown in [BS11]. By Theorem 1 $X = Y \times Z$ where Y is the largest normal subloop of X of odd order and Z the subloop of X of 2-elements of X.

If T is a Sylow 2-subloop, then, as Y does not contain elements of even order [G64], T projects trivially onto Y and is therefore contained in Z. According to [BS11], Proof of Corollary 4.10, there is a Sylow 2-subgroup P of the enveloping group of Z, such that $(P, P \cap H, P \cap K)$ is a loop folder for T. We may identify T with $P \cap K$. By [BS11, Corollary 4.10], all the Sylow 2-subloops of X are conjugate under the action of H. Thus the number of the Sylow 2-subloops of X is $|H : N_H(P \cap K)|$. As $P \cap H$ normalises $P \cap K$ and as $P \cap H$ is a Sylow 2-subgroup of H, see [BS11, Lemma 4.8(2)], this number is odd. □

Remark. We obtain the loop folders to the Sylow 2-subloops as follows: Take a Sylow 2-subgroup Q of H. Then $P := N_G(Q)O_2(G)$ is in $\mathrm{Syl}_2(G)$. Moreover, $P = (P \cap H)(P \cap K)$ and $(P, P \cap H, P \cap K)$ is a loop folder for a Sylow 2-subloop of X, see [BS11, Lemma 4.7]. In the next section we will describe how to get the Sylow 2-subloops directly.

Notice:

Lemma 4.1. *Let X be a Bruck loop of 2-power exponent. Then the only subloop of X of odd order is the trivial subloop $\{1\}$.*

Proof: Burn showed the elementwise Lagrange Theorem for finite Bol loops, i.e. that the order of every element divides the order of the loop [Bu78]. Let Y be a subloop of X of odd order. As Y is a Bol loop as well, every element of Y has odd order. Since X is of 2-power exponent, we get $Y = \{1\}$. □

If X is a Bruck loop of odd order, then Sylow's theorem holds in full generality [G64]. This then yields the following:

Theorem 3. *Let X be a finite Bruck loop and let p be a prime. Then the Sylow-properties* (S1)–(S4) *hold for p if and only if $p = 2$ or if p is odd and p does not divide the number of 2-elements of X.*

In particular if $p = 2$ or if p is an odd prime different from 5 and not dividing $q + 1$ for q a Fermat prime, then (S1)–(S4) *hold.*

Proof: If X is odd, then by Corollary 3 to Theorem 9 of [G64], X satisfies the Sylow-properties.

Now let X be a Bruck loop of even order. Then $X = Y \times Z$, where Y is the largest normal subloop of X of odd order and Z the subloop of X consisting of all the 2-elements of X. If $p = 2$ or p divides $|Y|$, but not $|Z|$, then (S1)–(S4) hold by Theorem 2 and the first paragraph of this proof, respectively.

Thus assume that p is an odd prime which divides $|Z|$ and assume that there is a Sylow p-subloop T of X. Then according to Lemma 4.1 T projects trivially onto Z. Therefore, T is contained in Y which is not possible, as Y satisfies (S3), but by assumption $|Y|_p < |T| = |X|_p$. □

5. More on subloops

The following result shows that knowing the subloops of the odd order Bruck loops as well as those of the Bruck loops of 2-power exponent already gives the full knowledge of all the subloops.

Proposition 5.1. *Let T be a subloop of the finite Bruck loop $X = O(X) \times O^{2'}(X)$. Then $T = (T \cap O(X)) \times (T \cap O^{2'}(X))$.*

Proof: Let $x = y \circ z$ be an element in T with $y \in O(X)$ and $z \in O^{2'}(X)$. According to [AKP06, Theorem 2], $y \circ z = z \circ y$ and $\rho_y \rho_z = \rho_{y \circ z} = \rho_z \rho_y$. This implies that $(y \circ z)^2 = y^2 \circ z^2$:

$$(y \circ z)^2 = (y \circ z) \circ (y \circ z) = (y\rho_z)\rho_{y \circ z} = (y\rho_z)\rho_y \rho_z = (y\rho_y)\rho_z^2 = y^2 \rho_{z^2} = y^2 \circ z^2.$$

Thus

$$(y \circ z)^{2^i} = y^{2^i} \circ z^{2^i}$$

for every i in $\mathbb{N}$. As y is of odd order and the order of z is a 2-power, it follows that y and z are in $\langle x \rangle = \langle y \circ z \rangle$.

Let π_o and π_e be the projections of X onto $O(X)$ and $O^{2'}(X)$, respectively. Then $T = \pi_o(T) \times \pi_e(T)$ which is the assertion. □

The subloops of the odd order Bruck loops can be extracted from Glauberman's papers [G64], [G68]. So assume that X is a finite Bruck loop of 2-power exponent. Then by Lemma 3.1

$$|X| = 2^a \prod_{i=1}^{e} (q_i + 1)$$

for some $e \in \mathbb{N} \cup \{0\}$, where $q_i = 9$ or $q_i \geq 5$ a Fermat prime. If $e = 0$, then $|X| = 2^a$ and X is soluble, see [BS11, Lemma 3.5] or [N98, Corollary 3.3]. In this case we will not study the subloops as there are too many.

So assume that X is a non-soluble Bruck loop of 2-power exponent, i.e. that $O(X) = 1$ and $e \geq 1$. Then, $G/O_2(G) \cong D_1 \times D_2 \times \cdots \times D_e$ with $D_i \cong \mathrm{PGL}_2(q_i)$ for $q_i \geq 5$ a Fermat prime or $q_i = 9$, see Theorem 1.

We distinguish between the soluble and the non-soluble subloops of X.

5.1 Non-soluble subloops. Let $I := \{1, 2, \ldots, e\}$ and let G_J be the full preimage of $\prod_{j \in J} D_j$ in G for $J \subseteq I$.

Lemma 5.2. *For every $J \subseteq I$ we have $(G_J, G_J \cap H, G_J \cap K)$ is a loop folder for a subloop X_J of X.*

Proof: According to [BS11, Lemma 4.4], $G_J = (G_J \cap H)(G_J \cap K)$ which yields according to [BSS11, Lemma 2.1] the assertion. □

Lemma 5.3. *Let $e = 1$ and let Y be a non-soluble subloop of the simple loop $X = X_1$. Then $Y = X$.*

Proof: Suppose $Y < X$. Let $(S, S\cap H, S\cap K)$ be the subloop folder of (G, H, K) related to Y. Then $M := \langle S \cap K\rangle$ is as described in Theorem 1. In particular $M/O_2(M)$ is isomorphic to a direct product of groups isomorphic to $L_2(q_i)$ with $q_i = 9$ or $q_i \geq 5$ a Fermat prime. Then $G/O_2(G)$ contains subgroups isomorphic to $L_2(q_i)$. This implies $G/O_2(G) \cong PGL_2(9)$, $M/O_2(M) \cong PGL_2(5)$ and $O_2(M) \leq O_2(G)$. Then $HO_2(G)/O_2(G) \cong 3^2 : 8$, but $(M \cap H)O_2(G)/O_2(G) \cong (M \cap H)O_2(M)/O_2(M) \cong 5 : 4$, which is not possible as the second is a subgroup of the first group. Thus $Y = X$. □

5.2 Soluble subloops. Let T be a soluble subloop of X. Then by [BS11, Lemma 3.5] the order of T is a power of 2. Hence T is contained in a Sylow 2-subloop of X. Next we describe these subloops T. Let X_J be the subloop defined in the previous paragraph.

Lemma 5.4 ([BS11, Lemma 4.3]). *There is a subloop Z of X of order 2^a with subfolder $(O_2(G), O_2(G) \cap H, O_2(G) \cap K)$.*

Proof: According to [BS11, Lemma 4.3], $(O_2(G), O_2(G) \cap H, O_2(G) \cap K)$ is a folder for a subloop Z of X and by [BS11, Corollary 3.6], $|Z| = |O_2(G) \cap K| = 2^a$. □

As G_i is the full preimage of D_i in G, it follows that Z is a subloop of X_i for $1 \leq i \leq e$. Let $x \in X_i \setminus Z$ and set

$$M_x := Z \cup x \circ Z.$$

Lemma 5.5. *The following hold.*

(a) *M_x is a subloop of X_i of order 2^{a+1}.*
(b) *If $x, y \in X_i \setminus Z$, then $M_x = M_y$ if and only if $y \in M_x$.*
(c) *M_x is non-associative.*
(d) *The loops M_x are the Sylow 2-subloops of X_i.*
(e) *Z is a normal subloop of M_x.*

Proof: Let $k \in K$ such that $k = \rho_x$. Moreover, let Q be a Sylow 2-subgroup of $H_i := H \cap G_i$ such that $[k, Q] \leq QO_2(G)$. Then $P := O_2(G)Q\langle k\rangle$ is in $\mathrm{Syl}_2(G_i)$ and $P \cap H_i = Q$ is in $\mathrm{Syl}_2(H_i)$. Therefore, $P \cap O_2(G_i)H_i \in \mathrm{Syl}_2(H_i)$, which implies that $P = (P\cap H)(P\cap K)$, see [BS11, Lemma 4.7]. It follows that $(P, P\cap H_i, P\cap K)$ is a subfolder for a subloop T_i of X_i by [BSS11, Lemma 2.1], which is a Sylow 2-loop.

Then Z is a normal subloop of T_i, as $|T_i| = 2|Z|$ and the map π_i from T_i onto $\mathbb{Z}_2$ sending every element from Z to 0 and every element in $T_i \setminus Z$ to 1, is a homomorphism with kernel Z. Thus by Theorem 2 of [S44] $T_i = M_x$, which yields (a), (b), (d) and (e).

It remains to show (c). Assume that M_x is associative. This is equivalent to the property that $\rho_x\rho_y = \rho_{x\circ y}$ for all $x, y \in M_x$. Thus $P \cap K$ is a group. As M_x is a Bruck loop, this group is then elementary abelian of order 2^{a+1}. Thus

$(K \cap P)O_2(G)/O_2(G) \cong (K \cap P)/(O_2(G) \cap K)$ is of order 2 in contradiction to $\langle K \cap P \rangle O_2(G)/O_2(G) \cong D_{2((q-1)/2)}$. Thus M_x is non-associative. □

Corollary 5.6. *Let Y_i be a soluble subloop of X_i. Then Y_i is contained in M_x for some $x \in X_i \setminus Z$.*

Proof: The assertion follows from [BSS11, Corollary 3.11] and Theorem 2 and Lemma 5.5(d) of this note. □

Moreover, notice:

Lemma 5.7. *The subloop Z is a normal subloop of every Sylow 2-subloop T of X.*

Proof: Extend the homomorphisms π_i for $1 \le i \le e$ to the homomorphism π from T onto $(\mathbb{Z}_2)^e$. Then we obtain the assertion, as $Z = \ker \pi$. □

Observe

Lemma 5.8. *Let $x, y \in X_i \setminus Z$ with $x \circ y \notin Z$. Then $\langle x, y \rangle$ is not a soluble loop.*

Proof: Assume that $Y := \langle x, y \rangle$ is a soluble loop. Then by [BSS11, Corollary 3.11] $\langle \rho_x, \rho_y \rangle$ is a 2-group and Y a 2-loop. Then by Corollary 5.6 $Y \le M_t$ for some $t \in X_i \setminus Z$. Thus $x = t \circ z_1$ and $y = t \circ z_2$ for some $z_1, z_2 \in Z$. As Z is normal in M_t, see Lemma 5.5(e), it follows that $x \circ y \in Z$ in contradiction with our assumption. □

This lemma together with Lemma 5.3 immediately imply the following:

Corollary 5.9. *If $e = 1$ and if $x, y \in X_i \setminus Z$ with $x \circ y \notin Z$ then $\langle x, y \rangle = X$.*

The following is a helpful observation to write down the Sylow 2-subloops.

Lemma 5.10. *Let $x_i \in X_i$ for $1 \le i \le e$. Then the expression $x_1 \circ x_2 \cdots \circ x_e \circ Z$ is independent of the order of the x_i's and independent of brackets.*

Proof: There is a Sylow 2-subloop T of X which contains $x_1, \ldots, x_e$. By Lemma 5.7 Z is a normal subloop of T and T/Z is an elementary abelian group. This yields the assertion. □

Let

$$x_i \in X_i \setminus Z, \quad \text{for} \quad 1 \le i \le e, \quad \text{and set} \quad x := (x_1, \ldots, x_e).$$

By Lemma 5.10 for $J \subseteq I$ the following set is well defined:

$$x^J \circ Z := \prod_{j \in J} x_j \circ Z,$$

where the product is that one in the loop X. If $J = \emptyset$, then $x^J \circ Z = Z$. Set

$$T_x := \bigcup_{J \subseteq I} x^J \circ Z.$$

Corollary 5.11. *The subloops T_x with $x = (x_1, \dots, x_e)$ and x_i in $X_i \setminus Z$, for $1 \le i \le e$, are the Sylow 2-subloops of X.*

PROOF: By construction and by Lemma 5.7, T_x is a loop. Further,

$$|T_x| = 2^{|I|}|Z| = 2^e 2^a = 2^{e+a}.$$

So Lemma 3.1 yields $|T_x| = |X|_2$, the assertion. □

6. Lagrange's theorem

Here we summarize the results of [G64] and [BS11] on Lagrange's theorem.

Theorem 4 ([G64], [BS11] Lagrange's Theorem)**.** *Let X be a finite Bruck loop and $Y \le X$ a subloop. Then $|Y|$ divides $|X|$.*

PROOF: It is a direct consequence of Theorem 1, Corollary 4, p. 395, in [G64] and of Theorem 3 in [BS11]. □

7. Hall's theorem

We say that a subloop Y of X is a *Hall π-subloop*, if $|Y|_\pi = |X|_\pi$. The Theorem of Hall holds as well:

Theorem 5 ([G68], [BS11] Hall's Theorem)**.** *Let X be a finite Bruck loop and let Π be the set of primes dividing the order of X. Then X is soluble if and only if there is a Hall π-subloop in X for every subset π of Π.*

PROOF: This is Theorem 12 of [G68] if X is of odd order and Theorem 4 of [BS11] in the general case. □

There is even a stronger version of that theorem:

Theorem 6 ([BS11, Theorem 5])**.** *Let X be a finite Bruck loop. Then X is soluble if and only if there is a Sylow subloop in X for every prime dividing $|X|$.*

8. Open questions

Here we just want to mention two of the open questions concerning Bruck loops:

(1.) In the known simple Bruck loop of order 96 the subloop related to $O_2(G)$ is a group in fact. Does this hold in general?

(2.) Is there a simple Bruck loop with enveloping group G such that $G/O_2(G) \cong L_2(9) : 2$ or $L_2(q) : 2$ with $q > 5$ a Fermat prime? Let X be such a loop. Then X contains a quotient Z which is an M-loop, for the definition see [AKP06, p. 3062]. Then $|Z| = 2^a(q+1)$ by Lemma 3.1, and by Theorem 3 of [AKP06] $|Z| = n_1 2^n(2^{n-1}+1)$ where $n_1 = |kO_2(G) \cap K|$ with $k \in K \setminus O_2(G)$ and $q = 2^n + 1$. Hence $2^a = n_1 2^{n-1}$ and $2^a \ge 2^{n-1}$, which implies $a \ge 2$. In fact, in such an example a would have to be much bigger than 2.

Acknowledgment. I like to thank the referee for his helpful comments and for improving the style of the paper.

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(*Received* November 15, 2011, *revised* March 8, 2012)