Solution of distributive-like quasigroup functional equations

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Abstract. We are investigating quasigroup functional equation classification up to parastrophic equivalence [Sokhatsky F.M., *On classification of functional equations on quasigroups*, Ukrainian Math. J. **56** (2004), no. 4, 1259–1266 (in Ukrainian)]. If functional equations are parastrophically equivalent, then their functional variables can be renamed in such a way that the obtained equations are equivalent, i.e., their solution sets are equal. There exist five classes of generalized distributive-like quasigroup functional equations up to parastrophic equivalence [Sokhatsky F.M., *On classification of distributive-like functional equations*, Book of Abstracts of the 8^{th} International Algebraic Conference in Ukraine, July 5–12 (2011), Lugansk, Ukraine, p. 79].

In the article, we find the solution sets of four generalized distributive-like quasigroup functional equations of different classes. In consequence, we solve one of the equations on topological quasigroup operations, defined on arbitrary topological space as well as on the space of real numbers with the natural topology.

The fifth class contains the generalized left distributivity functional equation. V.D. Belousov [*Some remarks on the functional equation of generalized distributivity*, Aequationes Math. **1** (1968), no. 1–2, 54–65] described only a subset of its solution set. The set of all solutions still remains an open problem in the quasigroup theory and in the functional equation theory.

Keywords: quasigroup, functional equation, distributive quasigroup, distributive-like functional equation, quasigroup solution, solution set, quasigroup identity, parastrophic equivalence

Classification: 20N05, 05B15

Introduction

We continue investigation of the problem of quasigroup functional equations classification up to parastrophic equivalence [11]. This problem was considered in many articles, in particular, in [7], [9], [10].

In [12] it was stated that every generalized distributive-like quasigroup functional equation is parastrophically equivalent to at least one of the equations (2)–(6). The notion '*distributive-like*' means that the equation has three different individual variables and the number of their appearances is equal to 2, 2, 3. To classify generalized distributive-like quasigroup functional equations up to parastrophic equivalence, we have to find their solution sets.

In this paper, we solve the equations (3)–(6). Earlier the results were stated in [8]. As a consequence, one can get a set of all solutions of (3)–(6) over arbitrary set of functions which are closed under composition. To give an example, we solve the equation (3) over the set of all topological quasigroup functions, defined on a topological space (Corollary 5). In particular, we solve (3) when the space coincides with the topological space of real numbers with the natural topology (Corollary 6).

The set of all solutions of the quasigroup functional equation (2) is still unsolved and it is a well-known problem in the quasigroup theory and in the functional equation theory. V.D.Belousov [5] described its partial solution.

1. Preliminaries

An operation f, defined on a *carrier set* Q, is said to be *left-invertible* (*right-invertible*) if every of its right (left) translation is a permutation of Q. In other words, the equation $f(x;a) = b$ (respectively, $f(a;y) = b$) has a unique solution for all a, $b \in Q$ and it is denoted by $f^{\ell}(b;a)$ (respectively, by $f^{r}(a;b)$). It is easy to see that f^{ℓ} and f^{r} are binary operations on Q which are called *left* and *right divisions* of f. A quasigroup operation f, its divisions, divisions of the divisions, ... are called *parastrophs* of f. It is easy to verify that every quasigroup operation has at most six different parastrophs. Left- and right-invertible operation is called *invertible* or *quasigroup* operation. A groupoid $(Q;f)$ is called a *quasigroup* if f is invertible.

So, the equalities

$$F(F^{\ell}(x;y),y) = x, \quad F^{\ell}(F(x;y),y) = x, \qquad (1)$$
$$F(x;F^{r}(x;y)) = y, \quad F^{r}(x;F(x;y)) = y$$

are *superidentities* on Δ, i.e., they hold for all $x, y \in Q$ and for all values of F in the set Δ of all invertible functions, defined on Q.

Let W, V be terms and $[W]$ denote the set of all individual variables appearing in W. Let

$$\{x_1, \dots, x_n\} := [W] \cup [V];$$

then the formula

$$(\forall x_1)\dots(\forall x_n) \quad W = V$$

is called a *functional equation*. As usual, the universal quantifiers are omitted. A sequence of operations, defined on a set Q, is called a *solution on Q of a functional equation* if the sequence reduces the equation to an identity [1]. If all components of a solution are invertible, then it is called a *quasigroup solution*. The set of all solutions on Q will be called *solution set* of the equation. A functional equation is called:

- a *generalized functional equation* if all its functional variables are pairwise different;

- a *binary functional equation* if all functional variables are binary, i.e., they are assumed to take their values in a set of binary operations;
- a *quasigroup functional equation* if its functional variables are assumed to take their values in a set of quasigroup operations (i.e., invertible functions);
- a *distributive-like functional equation* if it is binary and has three individual variables with appearances 2, 2, 3.

Here we consider binary quasigroup functional equations having neither individual nor functional constants.

Two functional equations are said to be *parastrophically equivalent* [11] if one can be obtained from the other in a finite number of the following steps:

1) application (1);
2) changing the sides of the equation;
3) renaming of individual variables;
4) renaming of functional variables.

In [12], it was stated that every generalized distributive-like quasigroup functional equation is parastrophically equivalent to at least one of the equations

$$F_1(x; F_2(y; z)) = F_3(F_4(x; y); F_5(x; z)), \tag{2}$$

$$F_1(y; F_2(x; z)) = F_3(F_4(y; F_5(x; z)); x), \tag{3}$$

$$F_1(F_2(x; y); y) = F_3(x; F_4(F_5(x; z); z)), \tag{4}$$

$$F_1(x; F_2(x; z)) = F_3(F_4(F_5(x; y); y); z), \tag{5}$$

$$F_1(y; F_2(x; z)) = F_3(y; F_4(x; F_5(x; z))). \tag{6}$$

If an operation is denoted by f_i and an element is denoted by e, then we agree to denote the corresponding left and right translations by L_i and R_i respectively, i.e.,

$$L_i x := f_i(e; x), \qquad R_i x := f_i(x; e), \qquad i = 1, 2, 3, \ldots \tag{7}$$

Operations f, g are called *orthogonal* ($f \perp g$) if the system

$$\begin{cases} f(x; y) = a, \\ g(x; y) = b \end{cases}$$

has a unique solution for all $a, b \in Q$.

Let $g(x; y) := \gamma^{-1} f(\alpha x; \beta y)$ for some permutations α, β, γ of Q; then g is called an *isotope* of f and is denoted by $g := f(\alpha, \beta, \gamma)$; the triplet (α, β, γ) is called *isotopism* between g and f; the corresponding relation on the set of all binary operations, defined on Q, is called an *isotopy*.

Any isotope of a parastroph of f is called an *isostroph* of f. It is easy to verify that an arbitrary isostroph is a parastroph of an isotope of f and any isostroph of an invertible operation is invertible as well.

The well-known functional equation of generalized associativity

$$F_1(F_2(x;y);z) = F_3(x;F_4(y;z)) \tag{8}$$

was solved by V.D. Belousov in [3] but its proof was published in [2]. Here we need some specification of Belousov's solution and so, we are giving proof of the corresponding theorem.

Note that if $(Q;\cdot)$ is an arbitrary semigroup, α, β, γ are arbitrary transformations and δ, ν are permutations of Q, then it is easy to see that a quadruple $(g_1;g_2;g_3;g_4)$ of functions, defined by

$$\begin{aligned} g_1(t;z) &= \delta t\cdot\gamma z, \qquad & g_2(x;y) &= \delta^{-1}(\alpha x\cdot\beta y),\\ g_3(x;u) &= \alpha x\cdot\nu u, \qquad & g_4(y;z) &= \nu^{-1}(\beta y\cdot\gamma z), \end{aligned} \tag{9}$$

is a solution of (8). Here we consider *quasigroup solutions* only.

Theorem 1. *Let $(Q;\cdot)$ be a group and α, β, γ, δ, ν be permutations on Q. Then a quadruple $(g_1;g_2;g_3;g_4)$, defined by (9), is a quasigroup solution of (8) on Q.*

Vice versa, if a quadruple $(g_1;g_2;g_3;g_4)$ of operations is a quasigroup solution of (8), then for any element $e\in Q$ there exists a unique sequence $(\cdot;\alpha;\beta;\gamma;\delta;\nu)$ of invertible operations, defined on Q, such that $(Q;\cdot)$ is a group with the neutral element e, $\delta e=\nu e=e$ and the equalities (9) are true. In this case, the operations $(\cdot)$, α, β, γ, δ, ν can be defined by

$$\begin{gathered} \delta x = g_1(x;g_1^r(e;e)),\ \alpha x = g_3(x;e),\ \nu x = g_3(g_3^\ell(e;e);x)\\ \gamma x = g_1(e;x),\ x\cdot y = g_1(\delta^{-1}(x);\gamma^{-1}(y)),\ \beta x = \delta g_2(g_3^\ell(e;e);x). \end{gathered} \tag{10}$$

Proof: Let a quadruple of operations $(g_1;g_2;g_3;g_4)$ be defined by (9). All operations are invertible, since all of them are isotopic to $(\cdot)$.

Vice versa, let $(g_1;g_2;g_3;g_4)$ be a quasigroup solution of (8) on Q. This means that the equality

$$g_1(g_2(x;y);z) = g_3(x;g_4(y;z)) \tag{11}$$

is an identity on Q and each of the operations g_1, g_2, g_3, g_4 is invertible. Let e be an arbitrary element of Q. We define $(\cdot)$ and α, β, γ, δ, ν by (10). Operations $(\cdot)$, α, β, γ, δ, ν are invertible, since g_1, g_2, g_3, g_4 are invertible.

Note that $\delta e=\nu e=e$ and the element e is neutral for the operation $(\cdot)$. Really,

$$\delta e \overset{(10)}{=} g_1(e;g_1^r(e;e)) \overset{(1)}{=} e, \qquad \nu e \overset{(10)}{=} g_3(g_3^\ell(e;e);e) \overset{(1)}{=} e.$$

Taking into consideration (1), the equalities (10) imply $\gamma^{-1}x = g_1^r(e;x)$ and $\delta^{-1}(x) = g_1^\ell(x; g_1^r(e;e))$. That is why

$$x \cdot e \overset{(10)}{=} g_1(g_1^\ell(x; g_1^r(e;e)); g_1^r(e;e)) \overset{(1)}{=} x.$$

Since $\delta e = e$, we have $\delta^{-1}e = e$ and

$$e \cdot x = g_1(\delta^{-1}e; \gamma^{-1}x) = g_1(e; g_1^r(e;x)) \overset{(1)}{=} x.$$

Thus, $(Q;\cdot)$ is a loop and e is its neutral element.

The equalities (10) imply the first identity of (9). We put the obtained expression for g_1 in (11):

$$\delta g_2(x;y) \cdot \gamma z = g_3(x; g_4(y;z)). \tag{12}$$

Combining $x := g_3^\ell(e;e)$, (10), (12), we get $\beta y \cdot \gamma z = \nu g_4(y;z)$. Consequently, the forth equality of (9) is true. Substituting the obtained value for g_4 in (12), we have

$$\delta g_2(x;y) \cdot \gamma z = g_3(x; \nu^{-1}(\beta y \cdot \gamma z)). \tag{13}$$

We put $z = \gamma^{-1}e$:

$$\delta g_2(x;y) = g_3(x; \nu^{-1}\beta y). \tag{14}$$

Integrating (14) in (13), we obtain

$$g_3(x; \nu^{-1}\beta y) \cdot \gamma z = g_3(x; \nu^{-1}(\beta y \cdot \gamma z)).$$

We replace γz with z and we put $y = \beta^{-1}e$: $\alpha x \cdot z = g_3(x; \nu^{-1}z)$, i.e., the third equality of (9) is true. Putting the expression for g_3 in (14), we get the second equality of (9).

Combining (9) and (11), we obtain associativity of $(\cdot)$. So, the existence of (9) is established.

To prove the uniqueness, we assume that an operation sequence $(\circ; \alpha_1, \beta_1, \gamma_1, \delta_1, \nu_1)$ satisfies the conditions of the theorem, i.e., $(Q;\circ)$ is a group, e is its neutral element, $\delta_1 e = \nu_1 e = e$ and

$$\begin{aligned} g_1(t;z) &= \delta_1 t \circ \gamma_1 z, & g_2(x;y) &= \delta^{-1}(\alpha_1 x \circ \beta_1 y), \\ g_3(x;u) &= \alpha_1 x \circ \nu_1 u, & g_4(y;z) &= \nu_1^{-1}(\beta_1 y \circ \gamma_1 z). \end{aligned} \tag{15}$$

Comparing (9) and (15) for g_1, we obtain the identity:

$$\delta t \cdot \gamma z = \delta_1 t \circ \gamma_1 z.$$

If $t = e$, then $\gamma = \gamma_1$. Replacing z with $\gamma^{-1}e$, we come to $\delta = \delta_1$. Thus, $(\cdot) = (\circ)$. Equating two expressions for g_3 of (9) and (15), we get the identity

$\alpha x\cdot \nu u = \alpha_1 x\cdot \nu_1 u$. If $u=e$, then $\alpha=\alpha_1$. If $x=\alpha^{-1}e$, then $\nu=\nu_1$. Equating two expressions for g_2, we obtain $\beta=\beta_1$. □

Recall that the left multiplication $\underset{\ell}{\oplus}$ and the right multiplication $\underset{r}{\oplus}$ of binary operations are defined by

$$(g \underset{\ell}{\oplus} h)(x;y) := g(h(x;y);y), \qquad (g \underset{r}{\oplus} h)(x;y) := g(x;h(x;y)).$$

Lemma 2 ([4]). *Let g, h be invertible operations; then the following assertions are true:*

$$g \underset{\ell}{\oplus} h \text{ is invertible } \Leftrightarrow\ g\perp h^{\ell}, \qquad g \underset{r}{\oplus} h \text{ is invertible } \Leftrightarrow\ g\perp h^{r}.$$

Recall that an invertible operation f, defined on Q, is called topological in a topological space $(Q;T)$ if f, f^{ℓ}, f^{r} are continuous.

Lemma 3. *Let $(Q;T)$ be an arbitrary topological space, f be a topological quasigroup operation in $(Q;T)$, g be defined on Q, and g have a neutral element. If f, g are isotopic and at least one component of the isotopism is a homeomorphism of $(Q;T)$, then g is a topological quasigroup in $(Q;T)$ and all components of the isotopism are homeomorphisms of $(Q;T)$.*

Proof: Let (α,β,γ) denote isotopism between operations f and g, i.e.,

$$f(x;y)=\gamma^{-1}g(\alpha x;\beta y) \tag{16}$$

for all $x,y\in Q$. So, g is invertible. Let $a:=\alpha^{-1}e$ and $b:=\beta^{-1}e$, where e denotes the neutral element of g. Put $x=a$ and $y=b$ in (16):

$$L_a^f=\gamma^{-1}\beta \quad\text{and}\quad R_b^f=\gamma^{-1}\alpha.$$

Since f is a topological quasigroup operation, all its translations and their inverses are homeomorphisms. So, the lemma follows from the above equalities. □

2. Solution of distributive-like functional equations

Functional equation (2) is well known as a *generalized left distributivity quasigroup functional equation.* Its solutions set is unknown. Here the solution of (3)–(6) and some corollaries are given. The other functional equation which is parastrophically equivalent to (3), has been solved in [7].

Theorem 4. *Let $(Q;\cdot)$ be a group; g be a quasigroup and $g^{\ell}\perp(\cdot)$; α, β, γ, δ, μ be permutations of Q; then the quintuple $(f_1,\dots,f_5)$ of operations, defined on a set Q by*

$$\begin{gathered} f_1(x;y)=\alpha x\cdot\delta y; \qquad f_2(x;z)=\delta^{-1}\big(g(z;\gamma x)\cdot\gamma x\big);\\ f_3(x;y)=\beta x\cdot\gamma y; \quad f_4(x;y)=\beta^{-1}(\alpha x\cdot\mu y);\\ f_5(x;z)=\mu^{-1}g(z;\gamma x), \end{gathered} \tag{17}$$

is a quasigroup solution of (3).

Conversely, if a quintuple $(f_1, \dots, f_5)$ *is a quasigroup solution of* (3)*, then for an arbitrary element* $e \in Q$ *there exists a unique sequence* $(\cdot, g, \alpha, \beta, \gamma, \delta, \mu)$ *of quasigroup operations such that* $(Q; \cdot)$ *is a group with neutral element* e*,* $\alpha e = \beta e = \delta e = e$*,* $g^\ell \perp (\cdot)$*,* (17) *is valid and*

$$\begin{gathered}\alpha x = f_1(x; e), \quad \beta x = f_3(x; f_3^r(e; e)), \quad \gamma x = f_3(e; x), \\ \delta y = f_1(e; y) \quad \mu x = f_3(f_4(e; x); f_3^r(e; e)), \\ x \cdot y = f_3(\beta^{-1} x; \gamma^{-1} y), \quad g(z; x) = \mu f_5(\gamma^{-1} x; z).\end{gathered} \tag{18}$$

Proof: Let $(Q; \cdot)$ be a group, α, β, γ, δ, μ be arbitrary permutations of Q, g be an arbitrary binary quasigroup operation, $g^\ell \perp (\cdot)$, and operations $f_1, \dots, f_5$ be defined by (17). Because the operations f_1, f_3, f_4, f_5 are isostrophs of an invertible operation, each of them is invertible. According to Lemma 2, orthogonality $g^\ell \perp (\cdot)$ and invertibility of g imply invertibility of f_2. Now we prove the identity

$$f_1(y; f_2(x; z)) = f_3(f_4(y; f_5(x; z)); x). \tag{19}$$

For this purpose we calculate its left and right parts:

$$f_1(y; f_2(x; z)) = \alpha y \cdot \delta f_2(x; z) = \alpha y \cdot g(z; \gamma x) \cdot \gamma x,$$

$$\begin{gathered}f_3(f_4(y; f_5(x; z)); x) = \beta f_4(y; f_5(x; z)) \cdot \gamma x = \\ = \alpha y \cdot \mu f_5(x; z) \cdot \gamma x = \alpha y \cdot g(z; \gamma x) \cdot \gamma x.\end{gathered}$$

These right parts are identically equal that is why the left parts are identically equal too. This means that $(f_1, \dots, f_5)$ is a quasigroup solution of (3).

Conversely, let $(f_1, \dots, f_5)$ be a quasigroup solution of (3). This means that the identity (19) is true.

Let e be an arbitrary element of Q. Combining (7) and (19) with $y = e$, we obtain $f_2(x; z) = L_1^{-1} f_3(L_4 f_5(x; z); x)$. We put the expression in (19):

$$f_1(y; L_1^{-1} f_3(L_4 f_5(x; z); x)) = f_3(f_4(y; f_5(x; z)); x).$$

The variable $t := L_4 f_5(x; z)$ together with z takes all values in Q, therefore,

$$f_3(f_4(y; L_4^{-1} t); x) = f_1(y; L_1^{-1} f_3(t; x)). \tag{20}$$

for all $x, y, t \in Q$. We introduce the following notation:

$$g_1 := f_3, \qquad g_2(y; t) := f_4(y; L_4^{-1} t), \qquad g_3(y; u) := f_1(y; L_1^{-1} u). \tag{21}$$

(20) means that the quadruple $(g_1; g_2; g_3; g_1)$ of operations is a solution of the generalized functional equation of associativity (8). Theorem 1 implies the equalities (9) and (10) with $g_4 = g_1$. (21) and (7) imply that e is a left neutral element

for both g_2 and g_3:

$$g_2(e;x) \overset{(21)}{=} f_4(e;L_4^{-1}x) \overset{(7)}{=} L_4L_4^{-1}x = x,$$
$$g_3(e;x) \overset{(21)}{=} f_1(e;L_1^{-1}x) \overset{(7)}{=} L_1L_1^{-1}x = x.$$

Therefore, $g_3^\ell(e;e) = e$ and from (10) we have $\nu = \varepsilon$, $\alpha e = e$ and $\beta = \delta$:

$$\beta x = \delta g_2(g_3^\ell(e;e);x) = \delta g_2(e;x) = \delta x.$$

Combining (21) and (9), we obtain:

$$f_1(x;L_1^{-1}u) \overset{(21)}{=} g_3(x;u) \overset{(9)}{=} \alpha x \cdot u, \qquad f_3(t;z) \overset{(21)}{=} g_1(t;z) \overset{(9)}{=} \beta t \cdot \gamma z,$$

$$f_4(x;L_4^{-1}y) \overset{(21)}{=} g_2(x;y) \overset{(9)}{=} \beta^{-1}(\alpha x \cdot \beta y).$$

Denoting $\delta := L_1$ and $\mu := \beta L_4$, we get the equalities for f_1, f_3, f_4 of (17) and dependence (18) for α, β, γ, δ, $(\cdot)$, μ. Now we return to (19):

$$\alpha y \cdot \delta f_2(x;z) = \alpha y \cdot \mu f_5(x;z) \cdot \gamma x.$$

We reduce the equality by αy and get $\delta f_2(x;z) = \mu f_5(x;z) \cdot \gamma x$. Define

$$g(z;x) := \mu f_5(\gamma^{-1}x;z).$$

Therefore, $\delta f_2(\gamma^{-1}x;z) = g(z;x) \cdot x$. Since the operation f_2 is invertible, by Lemma 2, this equality implies orthogonality $g^\ell \perp (\cdot)$. Thus, we obtain the expressions (17) for f_2 and f_5. The proof of uniqueness is the same as in Theorem 1. □

Corollary 5. *Let $(Q;T)$ be an arbitrary topological space and a quintuple $(f_1;\dots;f_5)$ of operations be defined on a set Q by* (17)*, where $(Q;\cdot)$ is a topological group, $(Q;g)$ is a topological quasigroup, $g^\ell \perp (\cdot)$, α, β, γ, δ, μ are homeomorphisms of $(Q;T)$. Then $(f_1;\dots;f_5)$ is a topological quasigroup solution of the functional equation* (3)*.*

Conversely, if a quintuple $(f_1;\dots;f_5)$ of topological quasigroup operations is a solution of (3)*, then for an arbitrary element $e \in Q$ there exists a single sequence $(\cdot;g;\alpha;\beta;\gamma;\delta;\mu)$ of operations such that $(Q;\cdot)$ is a topological group and e is its neutral element, g is a topological quasigroup operation and $g^\ell \perp (\cdot)$, α, β, γ, δ, μ are homeomorphisms, $\alpha e = \beta e = \delta e = e$ and* (17) *are fulfilled. In this case the sequence $(\cdot;g;\alpha;\beta;\gamma;\delta;\mu)$ is defined by* (18)*.*

Proof: Let operations $f_1,\dots,f_5$ be defined by (17); then they are topological, since each of them is a composition of topological operations. According to Theorem 4, the quintuple $(f_1,\dots,f_5)$ of operations is a quasigroup solution of the functional equation (3).

Conversely, let $f_1,\dots,f_5$ be topological quasigroup operations in the topological space $(Q;T)$ and a quintuple $(f_1;\dots;f_5)$ be a solution of (3). Theorem 4 implies (17) and (18). The equalities (18) imply that the operations α, β, γ, δ, μ, $(\cdot)$, g are topological in $(Q;T)$. $\square$

Corollary 6. *Let* $\mathbf{R}$ *be the topological space of the real number with the natural topology and binary operations* $f_1,\dots,f_5$ *be defined on* $\mathbf{R}$. *Then a quintuple* $(f_1;\dots;f_5)$ *is a topological quasigroup solution of the functional equation* (3) *if and only if there exist homeomorphisms* α, β, γ, μ, δ, φ *of the space and a quasigroup topological operation* g *such that* g^ℓ *is orthogonal to the additive operation* $(+)$ *of the field* $\mathbf{R}$ *and*

$$\begin{gathered} f_1(x;y)=\varphi(\alpha x+\delta y), \quad f_2(x;z)=\delta^{-1}(g(z;\gamma x)+\gamma x),\\ f_3(x;y)=\varphi(\beta y+\gamma x), \quad f_4(x;y)=\beta^{-1}(\alpha x+\mu y),\\ f_5(x;y)=\mu^{-1}g(y;\gamma x). \end{gathered} \tag{22}$$

Proof: Let $(f_1,\dots,f_5)$ be arbitrary topological quasigroup solution of (3) on $\mathbf{R}$. According to Corollary 5, there exists a topological group $(\mathbf{R};\cdot)$, topological quasigroup $(\mathbf{R};g)$ and homeomorphisms α, β, γ, δ, μ of $\mathbf{R}$ such that the equalities (17) are valid. It is well known [6] that the topological groups $(\mathbf{R};\cdot)$ and $(\mathbf{R};+)$ are topologically isomorphic, i.e., there exists a homeomorphism φ of $\mathbf{R}$ such that

$$x\cdot y=\varphi(\varphi^{-1}(x)+\varphi^{-1}(y)), \tag{23}$$

where $(+)$ denotes addition of the real numbers. Using this relationship, the equalities (17) can be written as follows:

$$\begin{gathered} f_1(x;y)=\varphi(\varphi^{-1}\alpha x+\varphi^{-1}\delta y),\\ f_2(x;z)=\delta^{-1}\varphi(\varphi^{-1}g(z;\gamma x)+\varphi^{-1}\gamma x),\\ f_3(x;y)=\varphi(\varphi^{-1}\beta y+\varphi^{-1}\gamma x),\\ f_4(x;y)=\beta^{-1}\varphi(\varphi^{-1}\alpha x+\varphi^{-1}\mu y). \end{gathered} \tag{24}$$

Now we make another notation:

$$\begin{gathered} \alpha_0:=\varphi^{-1}\alpha, \quad \beta_0:=\varphi^{-1}\beta, \quad \gamma_0:=\varphi^{-1}\gamma,\\ \mu_0:=\varphi^{-1}\mu, \quad \delta_0:=\varphi^{-1}\delta, \quad g_0(x;y):=\varphi^{-1}g(x;\varphi y) \end{gathered} \tag{25}$$

and we obtain the following expressions for f_1, f_3 and f_4:

$$\begin{gathered} f_1(x;y)=\varphi(\alpha_0 x+\delta_0 y), \quad f_3(x;y)=\varphi(\beta_0 y+\gamma_0 x),\\ f_4(x;y)=\beta_0^{-1}(\alpha_0 x+\mu_0 y). \end{gathered}$$

For f_2 and f_5 we have:

$$f_2(x;z) \overset{(24)}{=} \delta^{-1}\varphi(\varphi^{-1}g(z;\gamma x) + \varphi^{-1}\gamma x) =$$
$$= \delta^{-1}\varphi(\varphi^{-1}g(z;\varphi(\varphi^{-1}\gamma x)) + \varphi^{-1}\gamma x) \overset{(25)}{=} \delta_0^{-1}(g_0(z;\gamma_0 x) + \gamma_0 x),$$
$$f_5(x;y) \overset{(17)}{=} \mu^{-1}g(y;\gamma x) = \mu^{-1}\varphi(\varphi^{-1}g(y;\varphi(\varphi^{-1}\gamma x))) \overset{(25)}{=}$$
$$\overset{(25)}{=} \mu_0^{-1}\varphi^{-1}g(y;\varphi\gamma_0 x) \overset{(25)}{=} \mu_0^{-1}g_0(y;\gamma_0 x).$$

Since $f_2(x;z) = \delta_0^{-1}(g_0(z;\gamma_0 x) + \gamma_0 x)$, it follows that

$$\delta_0 f_2(z;\gamma_0^{-1}x) = g_0(z;x) + x = \left((+) \underset{\ell}{\oplus} g_0\right)(x;y).$$

By Lemma 2, $(+) \perp g_0^\ell$.

The inverse statement immediately follows from Corollary 5. □

Corollary 7. *Let a set Q have a prime order p and binary operations $f_1, \dots, f_5$ be defined on Q. Then $(f_1; \dots; f_5)$ is a quasigroup solution of the functional equation* (3) *if and only if there exist bijections α, β, γ, μ, δ, φ between Q and $\mathbf{Z}_p$,*[1] *a quasigroup operation g such that $(+) \perp g^\ell$, and the equalities* (22) *are true.*

Proof: It is well known that all groups of the same prime order are pairwise isomorphic. In other words, there exists a bijection φ between Q and the set $\mathbf{Z}_p$ of residues modulo p such that the groups $(Q;\cdot)$ and $(\mathbf{Z}_p;+)$ are isomorphic, i.e., the equality (23) is true. Combining (23), (25) and Theorem 4, we obtain the corollary. □

Theorem 8. *Let $f_1, \dots, f_5$ be binary operations, defined on a set Q. Then $(f_1; \dots; f_5)$ is a quasigroup solution of the functional equation* (4) *if and only if f_1, f_3 and f_4 are quasigroup operations and there exist permutations α and θ of Q such that the identities*

(26) $$f_3(x;\theta x) = \alpha x, \qquad f_2(x;y) = f_1^\ell(\alpha x;y), \qquad f_5(x;y) = f_4^\ell(\theta x;y)$$

hold.

Proof: Let a quintuple $(f_1; \dots; f_5)$ of quasigroup operations be a solution of (4). This means that

(27) $$f_1(f_2(x;y);y) = f_3(x;f_4(f_5(x;z);z))$$

is an identity on Q. Let e be an element of Q. We define α and θ by $\alpha := R_1R_2$, $\theta := R_4R_5$.[2]

[1] $\mathbf{Z}_p$ denotes the field of residues modulo p and $(+)$ is the additive operation of the field.
[2] Notation is given in (7).

If $y = z = e$, then (27) implies the first identity of (26). It implies that $f_3^r(x;\alpha x) = \theta x$. Put $z = e$ in (27) and obtain $f_1(f_2(x;y);y) = f_3(x;\theta x) = \alpha x$. So, the second identity of (26) is true. Putting $y = e$ in (27), we have $f_3(x;f_4(f_5(x;z);z)) = \alpha x$. Applying the definition of right division for f_3, we get $f_4(f_5(x;z);z) = f_3^r(x;\alpha x) = \theta x$. Using the definition of left division for f_4, we obtain the third identity of (26).

Vice versa, let the relationships (26) be true and the operations f_1, f_3, f_4 be invertible. Then f_2 and f_5 are invertible, since each of them is an isostroph of an invertible operation. Moreover, we have

$$\begin{aligned} f_1(f_2(x;y);y) &= f_1(f_1^\ell(\alpha x;y);y) = \alpha x = f_3(x;\theta x) = \\ &= f_3(x;f_4(f_4^\ell(\theta x;z);z)) = f_3(x;f_4(f_5(x;z);z)), \end{aligned}$$

i.e., (27) is an identity. Thus, the quintuple $(f_1;\ldots;f_5)$ of operations is a quasigroup solution of the functional equation (4). □

Proposition 1. *For any solution $(f_1;\ldots;f_5)$ there exists only one pair $(\alpha;\theta)$ of permutations of Q such that the equalities (26) are valid.*

Really, let $(\alpha_1;\theta_2)$ and $(\alpha;\theta)$ be pairs of permutations of Q satisfying (26). Therefore,

$$\begin{aligned} f_2(x;y) &= f_1^\ell(\alpha_1 x;y), \quad f_5(x;y) = f_4^\ell(\theta_1 x;y), \\ f_2(x;y) &= f_1^\ell(\alpha_2 x;y), \quad f_5(x;y) = f_4^\ell(\theta_2 x;y). \end{aligned}$$

So, $f_1^\ell(\alpha_1 x;y) = f_1^\ell(\alpha_2 x;y)$ and $f_4^\ell(\theta_1 x;y) = f_4^\ell(\theta_2 x;y)$, therefore, $\alpha_1 = \alpha_2$, $\theta_1 = \theta_2$.

Theorem 9. *Let Q be a set and $f_1,\ldots,f_5$ be binary operations, defined on Q. Then $(f_1;\ldots;f_5)$ is a quasigroup solution of the functional equation (5) if and only if f_1, f_2 and f_4 are quasigroup operations, $f_1 \perp f_2^r$ and there exists a permutation α of Q such that the identities*

$$f_3(x;z) = f_1(\alpha^{-1}x;f_2(\alpha^{-1}x;z)); \quad f_5(x;y) = f_4^\ell(\alpha x;y) \tag{28}$$

hold.

Proof: Let a quintuple $(f_1;\ldots;f_5)$ satisfy the conditions of the theorem. The operation f_5 is invertible, since it is an isostroph of the invertible operation f_4. By Lemma 2, the invertibility of f_3 follows from $f_1 \perp f_2^r$. Using the definition of the left division for f_6, the relation (28) implies

$$f_1(x;f_2(x;z)) = f_3(\alpha x;z), \quad f_4(f_5(x;y);y) = \alpha x. \tag{29}$$

This means that the identity

$$f_1(x;f_2(x;z)) = f_3(f_4(f_5(x;y);y);z) \tag{30}$$

holds. Consequently, $(f_1,\ldots,f_5)$ is a quasigroup solution of the functional equation (5).

Vice versa, let a quintuple $(f_1; \dots; f_5)$ of quasigroup operations be a solution of (5), therefore, the identity (30) is valid. Let e be an element of Q and $\alpha := R_4 R_5$. Substituting $y = e$ in (30), we obtain the first equality of (29). Combining the obtained identity and (30), we get the second equality of (29). From these identities, we obtain (28). According to Lemma 2, the first equality of (29) and invertibility of f_1, f_2, f_3 follow $f_1 \perp f_2^r$. □

Proposition 2. *For any solution* $(f_1; \dots; f_5)$ *of* (5) *there exists exactly one permutation* α *such that* (28) *hold.*

Really, let α, α_1 be permutations of Q satisfying (28). Therefore,

$$f_5(x; y) = f_4^\ell(\alpha x; y), \qquad f_5(x; y) = f_4^\ell(\alpha_1 x; y).$$

Comparing the identities, we obtain $\alpha_1 = \alpha$.

Theorem 10. *Let* Q *be a set and* $f_1, \dots, f_5$ *be binary operations, defined on* Q*. Then* $(f_1; \dots; f_5)$ *is a quasigroup solution of the functional equation* (6) *if and only if the operations* f_2, f_3 *and* f_5 *are quasigroups,* $f_2 \perp f_5$ *and there exists a permutation* α *of* Q *such that the identities*

$$f_1(x; y) = f_3(x; \alpha y), \qquad f_4(x; y) = \alpha f_2(x; f_5^r(x; y)) \tag{31}$$

hold.

Proof: Let a quintuple $(f_1; \dots; f_5)$ of operations satisfy conditions of the theorem. The operation f_1 is invertible, since it is an isotope of an invertible operation. According to Lemma 2, $f_2 \perp f_5$ implies invertibility of f_4. Let us prove that the equality

$$f_1(y; f_2(x; z)) = f_3(y; f_4(x; f_5(x; z))) \tag{32}$$

is an identity. For this purpose we calculate its left and right sides:

$$f_1(y; f_2(x; z)) = f_3(y; \alpha f_2(x; z)),$$
$$f_3(y; f_4(x; f_5(x; z))) = f_3(y; \alpha f_2(x; f_5^r(x; f_5(x; z)))) = f_3(y; \alpha f_2(x; z)).$$

We obtain the same expression, so (32) is true. This means that $(f_1; \dots; f_5)$ is a quasigroup solution of (6).

Vice versa, let a quintuple $(f_1; \dots; f_5)$ of quasigroup operations be a solution of (6), i.e., the identity (32) is true, and let $e \in Q$. Combining (32), $y = e$ and $\alpha := L_3^{-1} L_1$, we get

$$f_4(x; f_5(x; z)) = \alpha f_2(x; z). \tag{33}$$

Using the equality, we make replacement in the right side of (32):

$$f_1(y; f_2(x; z)) = f_3(y; \alpha f_2(x; z)).$$

Replacing $f_2(x;z)$ with t, we obtain the first identity of (31). Defining the operation f_4 from the equality (33), we get the second identity of (31). According to Lemma 2, invertibility of $\alpha^{-1}f_4$ implies $f_2 \perp f_5$. $\square$

Proposition 3. *For any solution* $(f_1,\ldots,f_5)$ *of the functional equation* (6) *there exists exactly one permutation* α *such that the equalities* (31) *are valid.*

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