Abelian differential modes are quasi-affine

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Abstract. We study a class of strongly solvable modes, called differential modes. We characterize abelian algebras in this class and prove that all of them are quasi-affine, i.e., they are subreducts of modules over commutative rings.

Keywords: differential modes, abelian algebras, quasi-affine algebras, subreducts of modules

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1. Introduction

Modes are idempotent algebras where every pair of operations commutes, or, in other terms, idempotent algebras where all operations are homomorphisms from the respective direct power (see Section 3 for a formal definition). One of the major open problems in the theory of modes is, to find an abstract characterization of modes that are *subreducts of a module over a commutative ring* (see the monograph [11] or the survey paper [9]). The problem has been addressed in several papers, the complete list of references can be found in a recent contribution [15], and many results are summarized in [11]. *Abelianess* is an obvious necessary condition and it seems plausible to conjecture that it is also sufficient. We confirm the conjecture for differential modes. (We consider abelian algebras in the abstract sense of universal algebra; they are called diagonally normal in [11]).

Our result also has some appeal to the "*abelian implies quasi-affine*" problem [17]. A mode is quasi-affine if and only if it is a subreduct of a module over a commutative ring (see Section 2 for explanation). All quasi-affine algebras are abelian, but not the other way around. One of the major projects in universal algebra is to determine abstract conditions that make abelian algebras quasi-affine. There was a significant progress over the years, from the initial results of H.P. Gumm [1] and J.D.H. Smith [12] proving the implication for congruence permutable varieties, to the strongest result so far, for varieties satisfying a non-trivial idempotent Mal'tsev condition [4] by K. Kearnes and Á. Szendrei. The full story is covered by the survey paper [17], or in a shorter way by the introductory notes of the most recent contribution [16].

The present paper settles the implication for the class of *differential modes* [5], consisting of modes with a single n-ary operation that possess a congruence such

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that all its blocks and the factor are left projection algebras (instead of left, we could have chosen any position to be the distinguished one). The main idea of the proof is, to syntactically verify the axioms of quasi-affine algebras recently found by M. Stronkowski and the author in [16].

Despite the fact the class we study is rather small, I find the result interesting for two reasons. First, all previous theorems on embedding modes into modules assumed some sort of cancellativity. But there are no cancellative differential modes (we can only have cancellativity in one coordinate). Second, the results of K. Kearnes [2], [3] indicate that modes come in three substantially different families. For *finite* modes, the families are: strongly solvable modes, affine modes and semilattice modes. Affine modes are trivially quasi-affine, and non-trivial semilattice modes are never abelian, so the interesting case is the strongly solvable one. It is natural to start with differential modes: they posses a strongly solvable chain of length 2. The class was investigated in a recent of papers [5], [7], [14] (and much earlier in the binary case, see [11]), providing tools and insight for our work.

The paper is organized as follows. In Section 2, we recall the folklore fact that quasi-affine modes are subreducts of modules over commutative rings. Section 3 contains an introduction to Szendrei modes and the observation that abelian modes are Szendrei modes. In the next section, we introduce a framework for Szendrei differential modes, to be used in Section 5 to characterize abelian differential modes, and in Section 6 to prove our main result. The final section contains remarks on differential modes that are reducts of modules.

2. Quasi-affine modes

An algebra $\mathbf{A}$ is called a *reduct* of an algebra $\mathbf{B}$, if they have the same universe and the operations of $\mathbf{A}$ can be expressed as term operations of $\mathbf{B}$. Subreduct means a subalgebra of a reduct. Two similar types of representation appear in literature:

- *Quasi-linear algebras* are subreducts of modules; it means their operations can be expressed as module terms
$$r_1x_1 + \cdots + r_nx_n.$$
- *Quasi-affine algebras* are subreducts of modules with additional constants for every element of the universe; it means their operations can be expressed as module polynomials
$$r_1x_1 + \cdots + r_nx_n + c,$$
with a constant c.

It has been shown recently [16] that, for algebras without nullary operations, the two notions coincide. It means that every quasi-affine algebra with no constants admits a quasi-linear representation. This is not an easy proof. However, it is very easy to prove it for idempotent algebras, and indeed this fact had been very well known before.

Proposition 2.1. *Every quasi-affine algebra containing an idempotent element e is a subreduct of a module such that $e = 0$.*

PROOF: Assume $\mathbf{A} = (A, F^A)$ admits a quasi-affine representation in a module $\mathbf{M}$ over a ring $\mathbf{R}$. It means, $A \subseteq M$ and for every basic operation $f^A \in F^A$,

$$f^A(a_1, \dots, a_n) = r_{f1}a_1 + \dots + r_{fn}a_n + c_f$$

for some $r_{f1}, \dots, r_{fn} \in R$ and $c_f \in M$. Consider the set $B = \{a - e : a \in A\}$ and a collection F^B of operations

$$f^B(a_1, \dots, a_n) = r_{f1}a_1 + \dots + r_{fn}a_n.$$

The mapping $\varphi : A \to B$, $a \mapsto a - e$, is bijective, maps e onto 0 and it is an isomorphism of $(A, F^A) \simeq (B, F^B)$, since

$$\begin{aligned} f^B(\varphi(a_1), \dots, \varphi(a_n)) &= f^B(a_1 - e, \dots, a_n - e) \\ &= r_{f1}(a_1 - e) + \dots + r_{fn}(a_n - e) \\ &= (r_{f1}a_1 + \dots + r_{fn}a_n + c_f) - (r_{f1}e + \dots + r_{fn}e + c_f) \\ &= f^A(a_1, \dots, a_n) - f^A(e, \dots, e) \\ &= f^A(a_1, \dots, a_n) - e = \varphi(f^A(a_1, \dots, a_n)) \end{aligned}$$

for every operation f and every tuple $a_1, \dots, a_n \in A$. □

Another folklore result says that, for modes, we can always assume the ring is commutative.

Proposition 2.2. *Every quasi-affine mode is a subreduct of a module over a commutative ring.*

PROOF: According to Proposition 2.1, we can assume that the mode $\mathbf{A} = (A, F)$ is a subreduct of a module $\mathbf{M}$ over a ring $\mathbf{R}$ such that $0 \in A$. Let

$$f(a_1, \dots, a_n) = r_{f1}a_1 + \dots + r_{fn}a_n$$

be the linear representation of the basic operations. We can assume that the module $\mathbf{M}$ is generated by the set A, that the ring $\mathbf{R}$ acts faithfully on $\mathbf{M}$ and that $\mathbf{R}$ is generated by the set $G = \{r_{f1}, \dots, r_{fn} : f \in F\}$ of all coefficients that appear in the linear representation. Let G^* denote the set of all products of elements from G. We start with a proof that

$$stu \cdot a = tsu \cdot a$$

for every $s, t \in G$, $u \in G^*$ and every $a \in A$. Assume $s = r_{fi}$, $t = r_{gj}$ and $u = u_1 \cdots u_p$, where $u_1 = r_{h_1k_1}, \dots, u_p = r_{h_pk_p}$. The fact that an m-ary operation

f and an n-ary operation g commute is expressed by the identity

$$f(g(x_{11},\dots,x_{1n}),\dots,g(x_{m1},\dots,x_{mn}))\\ = g(f(x_{11},\dots,x_{m1}),\dots,f(x_{1n},\dots,x_{mn})).$$

Replace x_{ij} with a term w_1 constructed in the following way: $w_{p+1} = y$, and $w_q = h_q(x,\dots,x,w_{q+1},x,\dots,x)$ for every $q = p,\dots,1$, where w_{q+1} sits at the k_q-th coordinate. Now, evaluate y with a and all other variables with zero. It results in the desired identity.

An easy induction show that in fact $stu \cdot a = tsu \cdot a$ for every $s,t,u \in G^*$ and every $a \in A$. The next step is to prove that

$$stu \cdot a = tsu \cdot a$$

for every $s,t,u \in R$ and every $a \in A$. Since every element of a ring is a sum of products of generators, we can write $s = \sum s_i$, $t = \sum t_i$, $u = \sum u_i$, where all $s_i,t_i,u_i \in G^*$. Now, $stu \cdot a = (\sum_{i,j,k} s_it_ju_k)\cdot a = \sum_{i,j,k}(s_it_ju_k \cdot a)$, and we can use the previous fact.

Finally, we show that

$$st \cdot m = ts \cdot m$$

for every $s,t \in R$ and every $m \in M$. Write $m = \sum r_i \cdot a_i$ for $r_i \in R$ and $a_i \in A$. Then $st \cdot m = st \cdot (\sum r_i \cdot a_i) = \sum (str_i \cdot a_i)$, and we can use the previous fact.

Consequently, since $\mathbf{R}$ acts faithfully, we have $st = ts$ for every $s,t \in R$. $\square$

The approach from the proof can be used for an arbitrary idempotent variety. For subvarieties of modes, one obtains the concept of the *affinization ring* of a variety, studied thoroughly in [11].

Example 2.3 ([10]). Consider the variety of binary differential modes. It is defined (relatively to modes) by the identity

$$x * (y * z) = x * y.$$

Let $\mathbf{A} = (A,*)$ be a quasi-affine binary differential mode and $a * b = (1-r)a + rb$ its linear representation in a module $\mathbf{M}$ over a commutative ring $\mathbf{R}$. We can assume that $0 \in A$, that $\mathbf{M}$ is generated by A, that $\mathbf{R}$ acts faithfully on $\mathbf{M}$ and that $\mathbf{R}$ is generated by $\{1,r\}$. Hence, $\mathbf{R}$ is a quotient of the polynomial ring $\mathbb{Z}[x]$. The identity $x*(y*z) = x*y$ translates into the equality

$$(1-r)a + r(1-r)b + r^2c = (1-r)a + rb$$

for every $a,b,c \in A$. Setting $a = b = 0$, we obtain $r^2c = 0$ for every $c \in A$, hence also for every $c \in M$, and so $r^2 = 0$. In this case, the equality is always satisfied. Consequently, for every quasi-affine binary differential mode, we can always take a module over the ring $\mathbf{R} = \mathbb{Z}[x]/(x^2)$.

3. Szendrei and abelian modes

The property that two operations f, g commute can be expressed by the so called *entropic law*:

$$f(g(x_{11},\dots,x_{1n}),\dots,g(x_{m1},\dots,x_{mn}))=g(f(x_{11},\dots,x_{m1}),\dots,f(x_{1n},\dots,x_{mn})).$$

If $f = g$, we can write it as

$$f(f(x_{11},\dots,x_{1n}),\dots,f(x_{n1},\dots,x_{nn})) = f(f(x_{\tilde{11}},\dots,x_{\tilde{1n}}),\dots,f(x_{\tilde{n1}},\dots,x_{\tilde{nn}}))$$

where $\tilde{ij} = ji$ for every i, j. However, subreducts of modules over commutative rings satisfy a more restrictive set of conditions:

$$f(f(x_{11},\dots,x_{1n}),\dots,f(x_{n1},\dots,x_{nn})) = f(f(x_{\bar{11}},\dots,x_{\bar{1n}}),\dots,f(x_{\bar{n1}},\dots,x_{\bar{nn}}))$$

whenever $\bar{\ }$ is an involution on indices that flips a single pair of indices ij, ji and leaves the other pairs still. The conditions will be called *Szendrei identities*, and modes satisfying Szendrei identities for every basic operation will be called *Szendrei modes*. Note that binary modes are always Szendrei modes, because the binary Szendrei identity is actually the entropic law. The importance of this concept is given by the following theorem [13], [15]: A mode satisfies Szendrei identities if and only if it is a subreduct of a semimodule over a commutative semiring.

An algebra $\mathbf{A}$ is called *abelian* if the diagonal is a block of a congruence of the square $\mathbf{A} \times \mathbf{A}$. Equivalently, if the quasi-identity

$$t(x, u_1,\dots,u_k) = t(x, v_1,\dots,v_k) \rightarrow t(y, u_1,\dots,u_k) = t(y, v_1,\dots,v_k)$$

is satisfied in $\mathbf{A}$ for every term t. Modules are obviously abelian, and so is every quasi-affine algebra.

The following observation is an obvious consequence of the conjecture that abelian modes are quasi-affine, and supports my belief in the conjecture.

Proposition 3.1. *Abelian modes are Szendrei modes.*

Proof: Let $t = f(f(x_{11},\dots,x_{1n}),\dots,f(x_{n1},\dots,x_{nn}))$. Then

$$t(x_{11},\dots,x_{nn}) = t(x_{\tilde{11}},\dots,x_{\tilde{nn}})$$

is the entropic law for an n-ary operation f. Fix $i < j$ and replace all variables, except x_{ij} and x_{ji}, by x. We obtain

$$t(x,\dots,x,x_{ij},x,\dots,x,x_{ji},x,\dots,x) = t(x,\dots,x,x_{ji},x,\dots,x,x_{ij},x,\dots,x).$$

Using (n^2-2)-times abelianess, we can replace the first occurrence of x by x_{11}, the second occurrence by x_{12}, and so on. The result is a Szendrei identity that flips $ij \leftrightarrow ji$. $\square$

(A more general result can be found in [6]: abelian entropic algebras with a single n-ary operation satisfy all equations true in every algebra $(\mathbb{R}, f)$, where f is an n-ary linear form.)

Remark 3.2. Neither abelian modes, nor quasi-affine modes, form a variety. Here is an example from [9]. The mode $(\mathbb{Z}_4, \circ)$, where $a \circ b = -a + 2b$, is a reduct of the $\mathbb{Z}$-module $\mathbb{Z}_4$, but its factor over the congruence $0|13|2$ is not abelian, because $[1] \circ [1] = [1] \circ [2]$, but $[0] \circ [1] \neq [0] \circ [2]$.

4. Szendrei differential modes

We describe a framework for Szendrei differential modes, useful for our arguments in the next two sections. It is similar to the one developed in [7], but we use a different notation. To avoid any confusion, we start from the very beginning. We recall from [5] that (left, n-ary) differential modes are axiomatized by the following identities:

$$f(x, x, \dots, x) = x \tag{I}$$

$$f(f(x, y_2, \dots, y_n), z_2, \dots, z_n) = f(f(x, z_2, \dots, z_n), y_2, \dots, y_n) \tag{E}$$

$$f(x, f(y_{21}, \dots, y_{2n}), \dots, f(y_{n1}, \dots, y_{nn})) = f(x, y_{21}, \dots, y_{n1}) \tag{R}$$

Using (R), every term is equivalent to a term in the *reduced form*

$$f(\dots(f(f(x, y_{12}, \dots, y_{1n}), y_{22}, \dots, y_{2n})\dots), y_{m2}, \dots, y_{mn}).$$

It is easy to check (or find in [7]) that the Szendrei identities are equivalent, in differential modes, to a single identity

$$f(x, y_2, \dots, y_n) = f(\dots(f(f(x, y_2, x, \dots, x), x, y_3, x, \dots, x)\dots), x, \dots, x, y_n).$$

We see that the action of every argument $y_2, \dots, y_n$ is in a sense independent of the action of the other ones, so instead of an algebra with a single n-ary operation, it is more convenient to consider a term equivalent algebra with $n - 1$ binary operations, defined by

$$x *_i y = f(\underbrace{x, \dots, x}_{i-1}, y, \underbrace{x, \dots, x}_{n-i}).$$

Using (R), the operation f can be recovered by

$$f(x, y_2, \dots, y_n) = (\dots((x *_2 y_2) *_3 y_3)\dots) *_n y_n.$$

We just proved the following statement.

Proposition 4.1. *Let (A, f) be a Szendrei differential mode. Then it is term equivalent to the algebra $(A, *_2, \dots, *_n)$.*

It is easy to check that the original set of identities (I) (E) (R) is equivalent to the following set, to be denoted in the same way.

(I) $$x *_i x = x \quad \text{for every } i$$

(E) $$(x *_i y) *_j z = (x *_j z) *_i y \quad \text{for every } i, j$$

(R) $$x *_i (y *_j z) = x *_i y \quad \text{for every } i, j$$

It follows that every term is equivalent to a term in the *reduced form*

(†) $$(((x *_{i_1} y_1) *_{i_2} y_2) \dots) *_{i_m} y_m,$$

for some m, some $i_1, \dots, i_m \in \{2, \dots, n\}$ and some choice of variables $x, y_1, \dots, y_m$. Using (I) and (E), we can find an equivalent term where x occurs only at the leftmost place; such expression is unique up to a permutation of the right actions (the proof is easy and can be found in [5]).

We shall use the following short notation for terms in the reduced form. Let $W(A)$ be the set of words over the alphabet $\{2, \dots, n\} \times A$. The term (†) will be denoted shortly $x\bar{w}$, where $\bar{w} = (i_1, y_1)(i_2, y_2) \dots (i_m, y_m)$ is a word from $W(A)$. We will always use overlined letters for words. Concatenation of words will be denoted by juxtaposition, so $x\bar{u}\bar{v}$ means the term $x\bar{w}$, where $\bar{w} = \bar{u}\bar{v}$. According to (E),

$$x\bar{u}\bar{v} = x\bar{v}\bar{u}$$

for every $\bar{u}, \bar{v} \in W(A)$, hence we are going to commute subwords freely without any explicit notice. According to (R),

$$(x\bar{u}) *_i (y\bar{v}) = x\bar{u}(i, y).$$

We shall need the following two technical notions. We say that two words $\bar{u} = (i_1, y_1) \dots (i_k, y_k)$, $\bar{v} = (j_1, z_1) \dots (j_l, z_l)$ are *similar*, and write $\bar{u} \sim \bar{v}$, if $k = l$ and there is a permutation π of the indices such that $i_m = j_{\pi(m)}$ for every $m = 1, \dots, k$. We say that they are *equivalent*, and write $\bar{u} \approx \bar{v}$, if there is a permutation π such that $i_m = j_{\pi(m)}$ and $y_m = z_{\pi(m)}$ for every $m = 1, \dots, k$, it means if the two words are equal up to a permutation of letters.

5. Abelian differential modes

Proposition 5.1. *A Szendrei differential mode* $\mathbf{A}$ *is abelian if and only if the following two conditions are satisfied:*

- (A1) *every operation* $*_i$ *is right cancellative;*
- (A2) *for every* $a, b \in A$ *and words* $\bar{c}, \bar{d} \in W(A)$ *with* $\bar{c} \sim \bar{d}$*, if* $a\bar{c} = a\bar{d}$*, then* $b\bar{c} = b\bar{d}$*.*

Proof: ($\Rightarrow$) (A1) Let $* = *_i$ and assume $a * c = b * c$. Using abelianess, $a * b = b * b = b$. Consequently, $b = a * b = a * (a * b) = a * a = a$, using (R) in the third step. (A2) is a special case of abelianess for $t(x, y_1, \dots, y_m) = x\bar{w}$, where $\bar{w} = (i_1, y_1) \dots (i_m, y_m)$ such that $\bar{w} \sim \bar{c} \sim \bar{d}$.

($\Leftarrow$) Let t be a term, we can assume it is in the reduced form $t(x_1,\dots,x_m) = x_k\bar{w}$ for a word $\bar{w} \in W(\{x_1,\dots,x_m\})$. Using (E) and (I), we can also assume that x_k does not appear in $\bar{w}$. We want to prove that $t(a,c_2,\dots,c_m) = t(a,d_2,\dots,d_m)$ implies $t(b,c_2,\dots,c_m) = t(b,d_2,\dots,d_m)$, for every $a,b,c_i,d_i \in A$.

For $k=1$, this is exactly condition (A2). For $k \neq 1$, using (E), we can further assume that $\bar{w} = \bar{u}\bar{v}$ such that $\bar{u}$ does not contain the variable x_1 and $\bar{v} \in W(\{x_1\})$. Now, if $t(a,c_2,\dots,c_m) = t(a,d_2,\dots,d_m)$, cancel from the right using (A1), and obtain $s(c_2,\dots,c_m) = s(d_2,\dots,d_m)$, where $s = x_k\bar{u}$. Then, multiply back from the right, and obtain $t(b,c_2,\dots,c_m) = t(b,d_2,\dots,d_m)$. $\square$

Conditions (A1) and (A2) are independent for general differential modes, but (A2) implies (A1) for the finite ones. To show independence, we present two binary examples. (Binary modes are always Szendrei modes.)

Example 5.2. The following table shows a non-abelian binary differential mode satisfying (A1) and failing (A2): $2*2 = 2*1$, but $0*2 \neq 0*1$.

	0	1	2
0	0	0	1
1	1	1	0
2	2	2	2

In the terminology of [14], this is the smallest cocyclic subdirectly irreducible binary differential mode and, in fact, all cocyclic SI's satisfy (A1) and fail (A2).

Example 5.3. The following construction shows a non-abelian binary differential mode satisfying (A2) and failing (A1). Let $X = \mathbb{N} \cup \{\circ,\bullet\}$ and let $f(x) = x+1$ for $x \in \mathbb{N}$ and $f(\circ) = f(\bullet) = 1$. Put $A = X \times \{0,1\}$ and define a binary operation by $(x,a)*(y,a) = (x,a)$ for both $a = 0,1$, and by $(x,a)*(y,b) = (f(x),a)$ for $a \neq b$. The operation is obviously not right cancellative, but we omit a rather technical proof that this is a differential mode satisfying (A2).

There is no such finite example, as asserted by the following proposition. An algebra is called *locally finite*, if every finitely generated subalgebra is finite.

Proposition 5.4. *A locally finite Szendrei differential mode is abelian if and only if condition* (A2) *holds.*

Proof: Let $\mathbf{A}$ be a locally finite Szendrei differential mode satisfying (A2). We prove that every operation $*_i$ is right cancellative. Let λ be a congruence of $\mathbf{A}$ such that all blocks of λ and the factor $\mathbf{A}/\lambda$ are left projection algebras (see [5]). Let R_a denote the right translation by a, it means $R_a(x) = x *_i a$. Since $\mathbf{A}/\lambda$ is a left projection algebra, we have $R_a(x)\ \lambda\ x$ for every a,x.

Now, fix $a \in A$ and a λ block B, and consider the subalgebra $\langle a,b\rangle$, generated by a and any element of B. Since $\mathbf{A}$ is locally finite, the subalgebra $\langle a,b\rangle$ is finite, so there is k and $x \in \langle a,b\rangle \cap B$ such that $R_a^k(x) = x$. Write it as $R_a^k(x) = R_x^k(x)$ and use (A2) to obtain that $R_a^k(y) = R_x^k(y) = y$ for every $y \in B$, the latter equality following from the fact that the blocks of λ are left projection algebras.

Consequently, the restriction of every right translation R_a on every block of λ is a bijection. But $\mathbf{A}/\lambda$ is a left projection algebra, hence R_a is a bijection on A. $\square$

Our final example shows that there indeed are (finite) abelian differential modes.

Example 5.5 ([11]). Let $(\mathbb{Z}_{k^2}, *)$ with $a * b = (1-k)a + kb$. This is a binary differential mode, it is a reduct of a module, hence abelian. All right translations are permutations of order k: we have $R_a^n(x) = (1-nk)x + nka$, and so $R_a^n(x) = x$ iff $nk = 0$.

Remark 5.6. Neither abelian differential modes, nor quasi-affine differential modes, form a variety. Example 5.2 is a factor of $(\mathbb{Z}_4, *)$ over the congruence $0|1|23$.

6. Quasi-affine representation of differential modes

Throughout the section, we implicitly use Proposition 4.1 and consider Szendrei differential modes as algebras $(A, *_2, \ldots, *_n)$. In particular, all terms are in the language of $*_2, \ldots, *_n$. The notions of being abelian, or quasi-affine, are invariant with respect to term equivalence.

Theorem 6.1. *A differential mode is abelian if and only if it is quasi-affine.*

Our proof is based on a syntactic verification of an axiomatization of quasi-affine algebras, found by M. Stronkowski and the author in [16]. First, we need to explain the axiomatization. A *multiset* is a generalization of a set in which members are allowed to appear more than once.

If T is a multiset of terms, let $B(T)$ denote the multiset of branches of terms from T. *A branch* of a term is defined for every occurrence of a variable, as the variable together with its address. Formally, if $t = x$, a variable, the only branch of t is x; and if $t = f(s_1, \ldots, s_n)$ for a basic operation f, then b is a branch of t if and only if $b = (f, i)b'$, where b' is a branch of s_i. (See [16] for an alternative description using free semimodules.)

Theorem 6.2 ([16]). *An algebra is quasi-affine if and only if it satisfies all quasi-identities*

$$t_1 = s_1 \ \& \ \ldots \ \& \ t_n = s_n \ \rightarrow \ t_0 = s_0$$

such that $B(\{t_0, t_1, \ldots, t_n\}) = B(\{s_0, s_1, \ldots, s_n\})$.

In Szendrei differential modes, it is convenient to assume terms are in the reduced form. The lemma states what equality of branch multisets means.

Lemma 6.3. *Let* $t_0, \ldots, t_n$, $s_0, \ldots, s_n$ *be terms such that*

$$B(\{t_0, \ldots, t_n\}) = B(\{s_0, \ldots, s_n\}).$$

Then there exist equivalent reduced forms $x_0\bar{u}_0, \ldots, x_n\bar{u}_n$, $y_0\bar{v}_0, \ldots, y_n\bar{v}_n$ *such that the following two conditions are satisfied:*

(B1) $\bar{u}_0\bar{u}_1 \dots \bar{u}_n \approx \bar{v}_0\bar{v}_1 \dots \bar{v}_n$;
(B2) *there is a permutation* π *of the indices such that* $x_i = y_{\pi(i)}$ *and* $u_i \sim v_{\pi(i)}$.

Proof: Consider the identity

$$(\mathrm{R}^+) \qquad x *_{i_0} ((((y *_{i_1} z_1) *_{i_2} z_2) \dots) *_{i_n} z_n) = x *_{i_0} y.$$

It is an obvious consequence of (R), for every choice of $i_0, \dots, i_n$. An occurrence of a variable in a term will be called *good*, if its address contains at most one right turn, i.e., at most one letter of the form $(*_i, 2)$. An application of the identity (R^+) only removes bad occurrences of variables, and good occurrences remain good. The other way around, every bad occurrence can be removed by an application of (R^+).

For a multiset T of terms, we define multisets $B_1(T), B_2(T)$ in the following way. For every $t \in T$ and every good occurrence of a variable x in t with precisely one right turn $(*_i, 2)$, we put one copy of the letter (i, x) into $B_1(T)$. For every $t \in T$ and every good occurrence of a variable x in t with no right turns, we put one copy of the corresponding branch into $B_2(T)$. Both multisets $B_1(T), B_2(T)$ are invariant with respect to an application of (R^+) to any of the terms in T.

Now, start with two multisets of terms $T = \{t_0, \dots, t_n\}$, $S = \{s_0, \dots, s_n\}$ such that $B(T) = B(S)$. Hence also $B_1(T) = B_1(S)$ and $B_2(T) = B_2(S)$. Using (R^+) sufficiently many times, we obtain multisets $T' = \{t'_0 \dots, t'_n\}$, $S' = \{s'_0, \dots, s'_n\}$ containing equivalent reduced forms. According to the previous paragraph, $B_i(T') = B_i(T) = B_i(S) = B_i(S')$ for both $i = 1, 2$. Condition (Bi) obviously follows from equality of the multisets B_i. □

Corollary 6.4. *A Szendrei differential mode is quasi-affine if it satisfies all quasi-identities*

$$t_1 = s_1 \ \& \ \dots \ \& \ t_n = s_n \ \rightarrow \ t_0 = s_0$$

such that the terms are in the reduced form and satisfy conditions (B1) *and* (B2) *of Lemma* 6.3.

Proof of Theorem 6.1: Every abelian mode is a Szendrei mode (Proposition 3.1), so we need to verify that abelianess, i.e. conditions (A1), (A2) of Proposition 5.1, implies every quasi-identity described in Corollary 6.4. Assume $t_1 = s_1, \dots, t_n = s_n$ holds, we prove $t_0 = s_0$. We will use the assumptions and notation introduced in Lemma 6.3 and start with an analysis of the permutation π from (B2).

Claim. *Let C be a cycle of length k in the permutation π such that $0 \notin C$. Let $c \in C$, and $\bar{w}_1 \sim \bar{w}_2$ be two similar words. Denote $\bar{u} = \bar{u}_c\bar{u}_{\pi(c)} \dots \bar{u}_{\pi^{k-1}(c)}$ and $\bar{v} = \bar{v}_c\bar{v}_{\pi(c)} \dots \bar{v}_{\pi^{k-1}(c)}$. If $a\bar{u}\bar{w}_1 = a\bar{v}\bar{w}_2$ for every a, then $a\bar{w}_1 = a\bar{w}_2$ for every a.*

Proof: Starting with the premise for $a = x_{\pi^{k-1}(c)}$, we obtain

$$\begin{aligned} x_{\pi^{k-1}(c)}\bar{u}\bar{w}_1 &= x_{\pi^{k-1}(c)}\bar{v}\bar{w}_2 \\ &= x_{\pi^{k-1}(c)}\bar{v}_c\bar{v}_{\pi(c)}\dots\bar{v}_{\pi^{k-1}(c)}\bar{w}_2 \\ &= y_c\bar{v}_c\bar{v}_{\pi(c)}\dots\bar{v}_{\pi^{k-1}(c)}\bar{w}_2 \\ &= x_c\bar{u}_c\bar{v}_{\pi(c)}\dots\bar{v}_{\pi^{k-1}(c)}\bar{w}_2, \end{aligned}$$

using (B2) and $t_c = s_c$ in the last two steps. Repeating the procedure performed on the last two lines k times, we obtain

$$\begin{aligned} x_{\pi^{k-1}(c)}\bar{u}\bar{w}_1 &= x_c\bar{u}_c\bar{v}_{\pi(c)}\bar{v}_{\pi^2(c)}\dots\bar{v}_{\pi^{k-1}(c)}\bar{w}_2 \\ &= y_{\pi(c)}\bar{u}_c\bar{v}_{\pi(c)}\bar{v}_{\pi^2(c)}\dots\bar{v}_{\pi^{k-1}(c)}\bar{w}_2 \\ &= x_{\pi(c)}\bar{u}_c\bar{u}_{\pi(c)}\bar{v}_{\pi^2(c)}\dots\bar{v}_{\pi^{k-1}(c)}\bar{w}_2 \\ &= \dots \\ &= x_{\pi^{k-1}(c)}\bar{u}_c\bar{u}_{\pi(c)}\bar{u}_{\pi^2(c)}\dots\bar{u}_{\pi^{k-1}(c)}\bar{w}_2 \\ &= x_{\pi^{k-1}(c)}\bar{u}\bar{w}_2. \end{aligned}$$

Now, use cancellation (A1) and obtain

$$x_{\pi^{k-1}(c)}\bar{w}_1 = x_{\pi^{k-1}(c)}\bar{w}_2.$$

Finally, use (A2) with the assumption that $\bar{w}_1 \sim \bar{w}_2$ and obtain

$$a\bar{w}_1 = a\bar{w}_2$$

for every a. □

Let $\pi = C_0C_1\dots C_l$ be the cycle decomposition of π such that 0 is contained in C_0. According to (B1), we have

$$a\bar{u}_0\bar{u}_1\dots\bar{u}_n = a\bar{v}_0\bar{v}_1\dots\bar{v}_n$$

for every a. By (B2), $u_i \sim v_{\pi(i)}$, so we can apply the claim on the previous identity l-times, for every cycle $C_1,\dots,C_l$. The result is that, for every a,

$$a\bar{u}_0\bar{u}_{\pi(0)}\dots\bar{u}_{\pi^{k-1}(0)} = a\bar{v}_0\bar{v}_{\pi(0)}\dots\bar{v}_{\pi^{k-1}(0)},$$

where k is the length of C_0. Now, start with $a = x_0$ and do exactly $k-1$ steps as in the proof of the claim. The result is

$$x_0\bar{u}_0\bar{u}_{\pi(0)}\dots\bar{u}_{\pi^{k-1}(0)} = x_{\pi^{k-1}(0)}\bar{v}_0\bar{u}_{\pi(0)}\dots\bar{u}_{\pi^{k-1}(0)}.$$

By cancellation (A1),

$$x_0\bar{u}_0 = x_{\pi^{k-1}(0)}\bar{v}_0 = y_0\bar{v}_0.$$

Hence $t_0 = s_0$, as desired. □

Let me note that in an earlier version of this paper, I had a proof of Theorem 6.1 based on the Quackenbush's axiomatization of quasi-affine algebras [8]. In [16], we claim that our axiomatization is larger but much easier to handle (and provide some evidence by finding easy proofs of some older results). Based on my experience from proving Theorem 6.1, I have to confirm our bold statement.

7. Reducts of modules

Our main result answers the question when a differential mode is a *subreduct* of a module. When does it admit a stronger representation, as a *reduct* of a module? The final section contains several observations and remarks with respect to this question.

Similarly as in Example 2.3, every quasi-affine n-ary differential mode can be represented over the ring $\mathbf{R}_n = \mathbb{Z}[x_2, \dots, x_n]/(x_2^2, \dots, x_n^2)$ with

$$a *_i b = (1 - x_i)a + x_i b.$$

Since $(1-x_i)(1+x_i) = 1 - x_i^2 = 1$, the element $1 - x_i$ is invertible. Consequently, if $\mathbf{A}$ is a *reduct* of a module over the ring $\mathbf{R}_n$, every operation $*_i$ forms a *right quasigroup* (it means, all right translations are permutations). This is a stronger condition than (A1), but not sufficiently strong for a differential mode to be a reduct of a module.

Example 7.1. Let $\mathbf{R} = \mathbb{Z}_3[x]/(x^2)$, let $a * b = (1-x)a + xb$ and put $A = \{ux + v : u, v \in \mathbb{Z}_3, v \neq -1\}$. Then $\mathbf{A} = (A, *)$ is a six-element subalgebra of $(R, *)$, and it is a quasi-affine differential mode which also is a right quasigroup. However, the only six-element abelian group is $\mathbb{Z}_6$ and it has only two binary reducts which are right quasigroups: $a \circ_1 b = a$ and $a \circ_2 b = -a + 2b$. None of the reducts is isomorphic to $\mathbf{A}$.

Is there a nice condition characterizing reducts of modules within the variety of differential modes?

Our final remark says, forget about affine algebras. An algebra is called *affine*, if it is polynomially equivalent to a module. In particular, affine algebras have a Mal'tsev polynomial. But every differential mode has a non-trivial factor which is a left projection subalgebra, so it cannot have a Mal'tsev operation. Actually, there is an independence result. (Similar and more general results for binary modes are in Section 8.5 of [11].)

Proposition 7.2. *The variety of differential modes is independent of any variety with a Mal'tsev term.*

PROOF: First, note that if $t(x, \dots)$ is a term where x is the leftmost variable, then the reduct (A, t) of a differential mode $\mathbf{A}$ is a differential mode again: if λ is the congruence on $\mathbf{A}$ such that all blocks and the factor are left projection algebras, it also is a congruence on (A, t), with the same property.

Let $p(x,y,z)$ be a Mal'tsev term in a variety $\mathcal{V}$. If z is the leftmost variable, consider the Mal'tsev term $p(z,y,x)$ instead, hence, without loss of generality, we can assume that x or y is the leftmost variable of p. Let $t(x,y)=p(x,x,p(x,x,y))$. Then, in differential modes, $t(x,y)=p(x,x,x)=x$ using (R) and (I), however in $\mathcal{V}$, we get $t(x,y)=p(x,x,y)=y$. Hence t proves independence of the two varieties. $\square$

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References

[1] Gumm H.P., *Algebras in permutable varieties: geometrical properties of affine algebras*, Algebra Universalis **9** (1979), no. 1, 8–34.
[2] Kearnes K., *The structure of finite modes*, manuscript, 1990's.
[3] Kearnes K., *Subdirectly irreducible modes*, Discuss. Math. Algebra Stochastic Methods **19** (1999), no. 1, 129–145.
[4] Kearnes K., Szendrei Á., *The relationship between two commutators*, Internat. J. Algebra Comput. **8** (1998), no. 4, 497–531.
[5] Kravchenko A., Pilitowska A., Romanowska A., Stanovský D., *Differential modes*, Internat. J. Algebra Comput. **18** (2008), no. 3, 567–588.
[6] Padmanabhan R., Penner P., *An implication basis for linear forms*, Algebra Universalis **55** (2006), no. 2–3, 355–368.
[7] Pilitowska A., Romanowska A., Stanovský D., *Varieties of differential modes embeddable into semimodules*, Internat. J. Algebra Comput. **19** (2009), no. 5, 669–680.
[8] Quackenbush R., *Quasi-affine algebras*, Algebra Universalis **20** (1985), 318–327.
[9] Romanowska A., *Semi-affine modes and modals*, Sci. Math. Jpn. **61** (2005), 159–194.
[10] Romanowska A.B., Smith J.D.H., *Differential groupoids*, Contributions to General Algebra **7** (1991), 283–290.
[11] Romanowska A., Smith J.D.H., *Modes*, World Scientific, River Edge, NJ, 2002.
[12] Smith J.D.H., *Mal'cev varieties*, Lecture Notes in Mathematics, 554, Springer, Berlin, 1976.
[13] Stanovský D., *Idempotent subreducts of semimodules over commutative semirings*, Rend. Semin. Mat. Univ. Padova **121** (2009), 33–43.
[14] Stanovský D., *Subdirectly irreducible differential modes*, Internat. J. Algebra Comput., to appear.
[15] Stronkowski M., *Embedding entropic algebras into semimodules and modules*, Internat. J. Algebra Comput. **19** (2009), no. 8, 1025–1047.
[16] Stronkowski M., Stanovský D., *Embedding general algebras into modules*, Proc. Amer. Math. Soc. **138** (2010), no. 8, 2687–2699.
[17] Szendrei Á., *Modules in general algebra*, Contributions to General Algebra 10 (1998), 41–53.

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