# On quasivarieties of nilpotent Moufang loops. I

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*Abstract.* In this part the smallest non-abelian quasivarieties for nilpotent Moufang loops are described.

*Keywords:* loop, associator, commutator, nilpotent, quasivarieties, quasiidentities, identities

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## Introduction

The theory of quasivarieties is one of the most important domains of universal algebra. The base of this theory was set by A.I. Mal'cev ([1], [2], [3], [4], [5], [6]).

Special attention is paid to two important problems:

1) the description of the lattice of quasivarieties of algebras;

2) when an algebra with a finite signature has a finite basis of quasiidentities. The study of these problems in the class of nilpotent Moufang loops is the goal of this paper.

In Section 1 we explain the basic notations and describe the identities that hold true in 2-nilpotent Moufang loops, obtained in [7]. In Section 2 we describe all minimal non-abelian quasivarieties for nilpotent Moufang loops, namely,

- minimal non-associative quasivarieties of commutative Moufang loops;
- minimal non-associative and non-commutative quasivarieties of Moufang $A$-loops with one proper minimal non-associative sub-quasivariety of commutative Moufang loops and one proper minimal non-commutative sub-quasivariety of groups;
- minimal non-associative and non-commutative quasivarieties of Moufang loops with the only proper non-commutative subquasivariety of groups;
- minimal non-commutative quasivarieties of groups.

For some of these quasivarieties, examples of non-associative Moufang loops are constructed. For instance, the smallest non-associative and non-commutative nilpotent Moufang loop has 16 elements (basic elements of Cayley–Dixon algebra and their opposite).

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## 1. Definitions, preliminary results, observations and notation

We shall use some notions and results from the monograph of R.H. Bruck [8].

A Moufang Loop (ML) is an algebra $\langle L, \cdot, ^{-1} \rangle$ of type $\langle 2, 1\rangle$ whose operations and elements satisfy the following identities:

$$x\,(y \cdot xz) = (xy \cdot x)z, \tag{1}$$

$$x^{-1} \cdot xy = y = yx \cdot x^{-1}, \tag{2}$$

where by $x^{-1}$ we denote the result of the unary operation applied to the element $x$.

We observe that (2) implies the identity $y \cdot (x^{-1})^{-1} = yx$, which in turn implies the identity $(x^{-1})^{-1} = x$. This helps to deduce the identity

$$x \cdot x^{-1}y = y = yx^{-1} \cdot x \tag{3}$$

For an arbitrary element $x \in L$ we denote $e = x^{-1} \cdot x$. Then, according to the identities (1)–(3), we will have

$$ye = x^{-1} \cdot x\,(ye) = x^{-1}\left[x \cdot y\left(xx^{-1}\right)\right] = x^{-1}\left[(xy \cdot x)x^{-1}\right] = x^{-1} \cdot xy = y$$

for any $y \in L$. It follows that $e = y^{-1} \cdot y$ and, therefore, $e$ does not depend on the element $x$. Then, taking (3) into consideration,

$$e \cdot y = yy^{-1} \cdot y = y$$

for any $y \in L$ and it follows that $e$ is a unit element of the ML $L$. Further on ML $L$ will be studied with the signature $\langle \cdot, ^{-1}, e\rangle$ made up of three operational symbols, which will be simply noted as $L$.

A ML is dissociative, in the sense that any of its subloops generated by two elements is associative (Moufang theorem [8]).

For elements $x$, $y$ and $z$ in a ML $L$ the associator $[x, y, z]$ and the commutator $[x, y]$ are defined by the equalities $[x, y, z] = (x \cdot yz)^{-1} \cdot (xy \cdot z)$ and $[x, y] = x^{-1} \cdot y^{-1}(xy)$, respectively.

For any subloop $H$ of $L$ we shall let $[H, L]$ denote the subloop generated by all of the elements of the forms $[h, x, y]$ and $[h, x]$, where $h \in H$ and $x, y \in L$.

The associant-commutant of the ML $L$ is the subloop generated in $L$ by all the associators and commutators of $L$ and we shall denote it as $L'$ or $[L, L]$. The set

$$Z(L) = \{x \in L \mid [x, y, z] = e,\ [x, y] = e \ \text{ for any } \ y, z \in L\}$$

is called the center of the ML $L$.

The subloop $H$ of the ML $L$ is called normal in $L$ if $xH = Hx$ and $x \cdot yH = xy \cdot H$ for any $x, y \in L$. It is easy to verify that the associant-commutant $L'$ is normal in $L$. Likewise, any subloop of the ML $L$ that is contained in the center $Z(L)$ is also normal in $L$.

Special associator-commutators of multiplicity $n$ are defined inductively: $x_1$ is a special associator-commutator of multiplicity 1; if $u$ is a special associant of multiplicity $n$ which includes exactly $i_n$ variables, then $[u, x_{i_n+1}]$, $[u, x_{i_n+1}, x_{i_n+2}]$ is a special associator-commutator of multiplicity $n+1$.

A ML $L$ is called (central-)nilpotent (NML) of class $n$ or $n$-nilpotent if for any values of the variables in $L$ the value of any special associator-commutator of multiplicity $n+1$ is equal to the unit element $e \in L$, but the value of at least one special associator-commutator of multiplicity $n$ is different from $e$.

According to [7], in any nilpotent Moufang loop of class 2 the following identities are true:

$$[x,y,z] = [y,z,x] = [y,x,z]^{-1}, \tag{4}$$

$$[x \cdot y, z, t] = [x,z,t]\,[y,z,t], \tag{5}$$

$$[x^m, y, z] = [x,y,z]^m, \tag{6}$$

$$[x,y,z]^6 = e, \tag{7}$$

$$[x \cdot y, z] = [x,z]\,[y,z]\,[x,y,z]^3, \tag{8}$$

and

$$[x^m, y] = [x,y]^m, \tag{9}$$

$$[x,y] = [y,x]^{-1}, \tag{10}$$

because Moufang loops are dissociative.

We shall also use the following notation:

$F_n(K)$ – free ML of rank $n$ of quasivariety $K$;
$v(L)$ – variety generated by loop $L$;
$q(L)$ – quasivariety generated by loop $L$.

## 2. The smallest nilpotent non-abelian quasivarieties of Moufang loops

The following varieties are defined in the class of all 2-nilpotent Moufang loops:

$$K_{1,0,0} = \mathrm{mod}\{[x,y,z] = e\},$$
$$K_{1,p,0} = \mathrm{mod}\{[x,y,z] = e,\ [x,y]^p = e\},$$
$$K_{1,p,p^m} = \mathrm{mod}\{[x,y,z] = e,\ [x,y]^p = e,\ x^{p^m} = e\},$$

where $m = 2,3,\ldots$ for $p = 2$ and $m = 1,2,\ldots$ for any prime number $p \geq 3$,

$$K_{2,0,0} = \mathrm{mod}\{[x,y,z]^2 = e\},$$
$$K_{2,2,0} = \mathrm{mod}\{[x,y,z]^2 = e,\ [x,y]^2 = e\},$$
$$K_{2,2,2^m} = \mathrm{mod}\{[x,y,z]^2 = e,\ [x,y]^2 = e,\ x^{2^m} = e\},\ m = 2,3,\ldots,$$
$$K_{3,0,0} = \mathrm{mod}\{[x,y,z]^3 = e\},$$
$$K_{3,1,0} = \mathrm{mod}\{[x,y,z]^3 = e,\ [x,y] = e\},$$
$$K_{3,1,3^m} = \mathrm{mod}\{[x,y,z]^3 = e,\ [x,y] = e,\ x^{3^m} = e\},\ m = 1,2,\ldots,$$
$$K_{3,3,0} = \mathrm{mod}\{[x,y,z]^3 = e,\ [x,y]^3 = e\},$$

$$K_{3,3,3^m} = \mathrm{mod}\{[x,y,z]^3 = e,\ [x,y]^3 = e,\ x^{3^m} = e\},\ m = 1,2,\dots$$

Denote by $\Re$ the set of all varieties defined above.

**Lemma 1.** *If a 2-nilpotent Moufang loop $N$ is finite, then there exists a variety $K \in \Re$ such that $F_3(K) \in q(N)$.*

Proof: Since $N$ is nilpotent we can regard $N$ as a $p$-loop. Let $\exp(N) = p^m$. We consider the following possible cases.

*Case* 1: *$N$ is non-associative and $p = 2$.* In this case $m > 1$. Then, according to the identity (7), the identity $[x,y,z]^2 = e$ holds true in $N$. For a certain integer $k$, $1 \le k \le m$, the identity $[x,y]^{2^k} = e$ also holds in $N$. Let $F_3 = F_3(x,y,z)$ be a $v(N)$-free loop of rank 3 with free generators $x, y, z$, and $H = \langle a,b,c\rangle$ be the subloop of $F_3^4 = F_3 \times F_3 \times F_3 \times F_3$ generated by the elements

$$a = (x,x,e,e),\ b = (e, y^{2^{k-1}}, y, e),\ c = (e, z^{2^{k-1}}, z^{2^{k-1}}, z).$$

Then it is obvious that

$$a^{2^m} = b^{2^m} = c^{2^m} = e,\ [a,b] = (e, [x,y]^{2^{k-1}}, e, e),\ [a,c] = (e, [x,z]^{2^{k-1}}, e, e),$$
$$[b,c] = (e,\ [y,z]^{2^{2(k-1)}},\ [y,z]^{2^{k-1}}, e),\ [a,b,c] = (e,\ [x,y,z]^{2^{2(k-1)}},\ e,\ e).$$

From here it follows that for $k = 1$ the loop $H$ is both non-associative and non-commutative and the identities

$$[x_1,x_2,x_3]^2 = e,\quad [x_1,x_2]^2 = e \quad \text{and} \quad H \in K_{2,2,2^m}$$

hold. Also, for $k > 1$, $H$ is a non-commutative group and the identity holds true

$$[x_1,x_2]^2 = e \quad \text{and} \quad H \in K_{1,2,2^m}.$$

We will show that any equality relation in $H$ between the elements $a, b$ and $c$ is a trivial equality. Indeed, let

$$(a^\alpha b^\beta \cdot c^\gamma) \cdot [a,b]^\delta [a,c]^\lambda [b,c]^\mu [a,b,c]^\nu = e \tag{11}$$

be such an equality relation in $H$. Then we have

$$\Big(x^\alpha, (x^\alpha y^{2^{k-1}\beta} \cdot z^{2^{k-1}\gamma}) \cdot [x,y]^{2^{k-1}\delta}[x,z]^{2^{k-1}\lambda}[y,z]^{2^{2(k-1)}\mu}[x,y,z]^{2^{2(k-1)}\nu},$$
$$y^\beta z^{2^{k-1}\gamma}[y,z]^{2^{k-1}\mu}, z^\gamma\Big) = (e,e,e,e),$$

from where it follows that the equality relations

$$x^\alpha = e,\ y^\beta[y,z]^{2^{k-1}\mu} = e,\ z^\gamma = e, \tag{12}$$

$$[x,y]^{2^{k-1}\delta}[x,z]^{2^{k-1}\lambda}[x,y,z]^{2^{2(k-1)}\nu} = e, \tag{13}$$

hold true in the $\nu(N)$-free loop $F_3$. But any equality relation between the free generators $x, y, z$ is a true identity in $F_3$. Therefore (12) and (13) are true identities in $F_3$. But the first and the last identity from (12) are true in $F_3$ only if

$$\alpha \equiv 0 \pmod{2^m},\ \gamma \equiv 0 \pmod{2^m}.$$

From the second identity of (12), substituting in it $z = e$, and from identity (13), substituting in it alternatively $z = e$ and $y = e$, we obtain

$$y^\beta = e,\ [y,z]^{2^{k-1}\mu} = e,\ [x,y]^{2^{k-1}\delta} = e,\ [x,z]^{2^{k-1}\lambda} = e, \tag{14}$$

and

$$[x,y,z]^{2^{2(k-1)}\nu} = e. \tag{15}$$

But the identities from (14) are true in $F_3(x,y,z)$ only if

$$\beta \equiv 0 \bmod 2^m,\quad \mu \equiv 0 \bmod 2,\quad \delta \equiv 0 \bmod 2,\quad \lambda \equiv 0 \bmod 2.$$

When $k = 1$, the identity (15) holds true in $F_3(x,y,z)$ only if $\nu \equiv 0 \bmod 2$ and when $k > 1$ it holds true for any positive integer $\nu$. From this we can easily conclude that (11) is a trivial equality. Therefore, for $k = 1$ in the variety $K_{2,2,2^m}$, and for $k > 1$ in the variety $K_{1,2,2^m}$, the loop $H$ has a finite representation formed by three generators without any equality relation. Hence for $k = 1$ the loop $H$ is $K_{2,2,2^m}$-free and for $k > 1$ the loop $H$ is $K_{1,2,2^m}$-free of the third rank with $H \in q(N)$.

*Case* 2: *$N$ is non-associative and $p = 3$.* In this case the identity $(x,y,z)^3 = e$ holds true in $N$. Assume that for a certain integer $k$, $0 \le k \le m$, the identity $[x,y]^{3^k} = e$ holds true in $N$.

If $k = 0$, then in $N$ the identity $[x,y] = e$ holds true and thus $N$ is a commutative Moufang loop. Then the $\nu(N)$-free commutative Moufang loop $F_3(x,y,z)$ is free in any variety of Moufang loops with the exponent $3^m$. Hence $F_3(K_{3,1,3^m}) \cong F_3(x,y,z) \in q(N)$.

Let $k \ge 1$, $F_3 = F_3(x,y,z)$ be a $\nu(N)$-free loop of the third rank with free generators $x, y, z$, and $H = \langle a,b,c\rangle$ be the subloop generated in $F_3^4$ by the elements

$$a = (x,x,e,e),\ b = (e, y^{3^{k-1}}, y, e),\ c = (e, z^{3^{k-1}}, z^{3^{k-1}}, z).$$

Then, obviously

$$a^{3^m} = b^{3^m} = c^{3^m} = (e,e,e,e),\ [a,b] = (e,[x,y]^{3^{k-1}},e,e),\ [a,c] = (e,[x,z]^{3^{k-1}},e,e),$$
$$[b,c] = (e,[y,z]^{3^{2(k-1)}},[y,z]^{3^{m-1}},e),\ [a,b,c] = (e,[x,y,z]^{3^{2(k-1)}},e,e).$$

From here it follows that for $k = 1$ the loop $H$ is non-associative and non-commutative, and the following identities hold true in it

$$[x_1,x_2,x_3]^3 = e,\ [x_1,x_2]^3 = e \ \text{ and } \ H \in K_{3,3,3^m}.$$

For $k > 1$, $H$ is a non-commutative group and the identities

$$[x_1, x_2]^3 = e \quad \text{and} \quad H \in K_{1,3,3^m}$$

hold true in $H$. By analogy with Case 1 we show that for $k = 1$ the loop $H$ is $K_{3,3,3^m}$-free of rank 3 and for $k > 1$ the loop $H$ is $K_{1,3,3^m}$-free of rank 3 with $H \in q(N)$.

*Case* 3: *$N$ is associative and $p$ is any prime number.* Similar to the previous cases, it can be shown that if, in the group $N$, the identity $[x, y]^{p^k}$ holds true for a certain natural number $k$, $1 \leq k \leq m$, then for $k = 1$ $F_3(K_{1,p,p^m}) \in q(N)$. $\square$

**Lemma 2.** *If the 2-nilpotent Moufang loop $N$, generated by three elements, is infinite, then there exists a variety $K \in \Re$ such that $F_3(K) \in q(N)$.*

Proof: Since the loop $N$ is not finite, we have $\exp(N) = 0$. We will consider the following possible cases.

*Case* 1: *$N$ is non-associative, in $N$ the identity $[x, y, z]^2 = e$ holds true and $\exp(\langle [u, v] \mid u, v \in N \rangle) = 2^m s$, where $m$ is a non-negative integer and 2 does not divide $s$.*

We will first show that $m > 0$. So assume that $m = 0$. Then, according to (8) and the identities $[x, y, z]^2 = e$, $[x, y]^s = e$ we can deduce $e = [xy, z]^s = ([x, z][y, z][x, y, z]^3)^s = ([x, z][y, z][x, y, z])^s = [x, z]^s [y, z]^s [x, y, z]^s = [x, y, z]^s$. Hence, in $N$, the identity $[x, y, z]^s = e$ holds true and, since 2 does not divide $s$, we conclude that the identity $[x, y, z] = e$ is also true in $N$. That is, $N$ is associative, a contradiction.

Hence, $m \geq 1$. Now let $F_3 = F_3(x, y, z)$ be a $\nu(N)$-free loop of the third rank with free generators $x, y, z$, and $H = \langle a, b, c \rangle$ be a subloop generated in $F_3^4$ by the elements

$$a = (x, x, e, e),\ b = (e, y^{2^{m-1}s}, y, e),\ c = (e, z^{2^{m-1}s}, z^{2^{m-1}s}, z).$$

Then, obviously, $\exp(H) = 0$ and the following equalities hold true:

(16)
$$[a, b] = (e, [x, y]^{2^{m-1}s}, e, e),\ [a, c] = (e, [x, z]^{2^{m-1}s}, e, e),$$
$$[b, c] = (e, [y, z]^{2^{2(m-1)}s^2}, [y, z]^{2^{m-1}s}, e),\ [a, b, c] = (e, [x, y, z]^{2^{2(m-1)}s^2}, e, e).$$

From here it follows that for $m = 1$ the loop $H$ is both non-associative and non-commutative and the identities $(x_1, x_2, x_3)^2 = e$, $[x_1, x_2]^2 = e$ hold true in it. For $m > 1$ $H$ is a non-commutative group and the identity $[x_1, x_2]^2 = e$ holds true in it. Therefore, for $m = 1$ the loop $H \in K_{2,2,0}$, and for $m > 1$ the loop $H \in K_{1,2,0}$.

We will now show that any equality relation in $H$ between the elements $a, b$ and $c$ is a trivial equality. Indeed, let

$$(a^\alpha b^\beta \cdot c^\gamma) \cdot [a, b]^\delta [a, c]^\lambda [b, c]^\mu [a, b, c]^\nu = e \tag{17}$$

be such an equality relation. Then we have

$$\Big(x^{\alpha}, (x^{\alpha}y^{2^{m-1}s\,\beta} \cdot z^{2^{m-1}s\,\gamma}) \cdot [x,y]^{2^{m-1}s\,\delta}[x,z]^{2^{m-1}s\,\lambda}[y,z]^{2^{2(m-1)}s^2\mu} [x,y,z]^{2^{2(m-1)}s^2\nu}, y^{\beta}z^{2^{m-1}s\,\gamma}[y,z]^{2^{m-1}s\,\mu}, z^{\gamma}\Big) = (e,e,e,e). \tag{18}$$

Like in Lemma 1 we can show the identities

$$x^{\alpha} = e,\ y^{\beta} = e,\ z^{\gamma} = e, \tag{19}$$

$$[x,y]^{2^{m-1}s\,\delta} = e,\ [x,z]^{2^{m-1}s\,\lambda} = e,\ [y,z]^{2^{m-1}s\,\mu} = e, \tag{20}$$

$$[x,y,z]^{2^{2(m-1)}s^2\nu} = e. \tag{21}$$

Because $\exp(N) = \exp(F_3) = 0$, the identities from (19) hold true in $F_3(x,y,z)$ only if

$$\alpha = 0,\ \beta = 0,\ \gamma = 0.$$

The identities from (20) are true only if $\delta \equiv 0 \bmod (2)$, $\lambda \equiv 0 \bmod 2$ and $\mu \equiv 0 \bmod 2$ and the identity (21), when $m = 1$, is true in $F_3$ only if $\nu \equiv 0 \bmod 2$ and when $m > 1$ — for any positive integer $\nu$. We can easily conclude that (17) is a trivial equality. Therefore, for $m = 1$ in the variety $K_{2,2,0}$ and for $m > 1$ in the variety $K_{1,2,0}$, the Moufang loop $H$ has a finite representation formed of three generators without any equality relation. Hence, for $m = 1$ the loop $H$ is $K_{2,2,0}$-free of the third rank and for $m > 1$ the loop $H$ is $K_{1,2,0}$-free of the third rank with $H \in q(N)$.

*Case* 2: *$N$ is non-associative, the identities $[x,y,z]^3 = e$ and $\exp(\langle [u,v] \mid u,v \in N\rangle) = 3^m s$ hold true in it, where $m$ is a non-negative integer and* 3 *does not divide $s$.*

Let $m = 0$, then we consider the subloop $H = \langle a,b,c\rangle$ generated in the $\nu(H)$-free loop $F_3(x,y,z)$ by the elements $a = x$, $b = y^s$, $c = z^s$. We notice that in the loop $F_3(x,y,z)$ the following equalities hold true

$$[a,b,c] = [x,y,z]^{s^2},\ [a,b] = [x,y]^s = e,\ [a,c] = [x,z]^s = e,\ [b,c] = [y,z]^{s^2} = e,$$

which implies that $H$ is a commutative Moufang loop. As $\exp(H) = 0$, it results that $H$ is a free 2-nilpotent commutative Moufang loop, which is contained in the variety $K_{3,1,0}$. Therefore $F_3(K_{3,1,0}) \cong H \in q(N)$.

Now assume that $m \geq 1$. Let $F_3 = F_3(x,y,z)$ be a $\nu(N)$-free loop of the third rank and $H = \langle a,b,c\rangle$ be the subloop generated in $F_3^4$ by the elements

$$a = (x,x,e,e),\ b = (e, y^{3^{m-1}s}, y, e),\ c = (e, z^{3^{m-1}s}, z^{3^{m-1}s}, z).$$

Then, obviously, $\exp(H) = 0$ and the following equalities hold true

$$[a,b] = (e, [x,y]^{3^{m-1}s}, e, e),\quad [a,c] = (e, [x,z]^{3^{m-1}s}, e, e),$$

$$[b,c] = (e, [y,z]^{3^{2(m-1)}s^2}, [y,z]^{3^{m-1}s}, e),\quad [a,b,c] = (e, [x,y,z]^{3^{2(m-1)}s^2}, e, e).$$

From here it follows that for $m = 1$ the loop $H$ is both non-associative and non-commutative and that the identities $(x_1, x_2, x_3)^3 = e$, $[x_1, x_2]^3 = e$ hold true in it. However, for $m > 1$, $H$ is a non-commutative group and the identity $[x_1, x_2]^3 = e$ holds in it. Therefore, for $m = 1$ the loop $H \in K_{3,3,0}$ and for $m = 1$ the loop $H \in K_{1,3,0}$. Then, similar to Case 1, we can show that for $m = 1$ the loop $H$ is $K_{3,3,0}$-free of rank 3 and for $m > 1$ the loop $H$ is $K_{1,3,0}$-free of rank 3 with $H \in q(N)$.

*Case* 3: *$N$ is non-associative, the identities $[x, y, z]^3 = e$ (respectively, $[x, y, z]^2 = e$) and* $\exp(\langle [u, v] \mid u, v \in N \rangle) = 0$ *hold true in it.*

Let $F_3(x, y, z)$ be a $\nu(N)$-free loop with free generators $x, y$ and $z$. It is clear that $F_3(x, y, z) \in K_{3,0,0}$ (respectively, $F_3(x, y, z) \in K_{2,0,0}$).

Let an arbitrary equality relation hold true in the $\nu(N)$-free loop $F_3(x, y, z)$

$$(x^\alpha y^\beta \cdot z^\gamma) \cdot [x, y]^\delta [x, z]^\lambda [y, z]^\mu (x, y, z)^\nu = e. \tag{22}$$

This equality relation is the identity true in $F_3(x, y, z)$. Then we can easily deduce that it implies the identities

$$x^\alpha = e,\ y^\beta = e,\ y^\gamma = e,\ [x, y]^\delta = e,\ [x, z]^\lambda = e,\ [y, z]^\mu = e,\ [x, y, z]^\nu = e,$$

which are true in $F_3(x, y, z)$ only if

$$\alpha = 0,\ \beta = 0,\ \gamma = 0,\ \delta = 0,\ \lambda = 0,\ \mu = 0,\ \nu \equiv 0 \bmod 3$$
$$(\nu \equiv 0 \bmod 2,\ \text{respectively}).$$

From here we obtain that (22) is a trivial equality in $F_3(x, y, z)$. Therefore, $F_3(x, y, z)$ is a free loop in the variety $K_{3,0,0}$ ($K_{2,0,0}$, respectively). It then follows that $F_3(x, y, z) \in q(N)$.

*Case* 4: *$N$ is non-associative, the identities $[x, y, z]^2 = e$ and $[x, y, z]^3 = e$ do not hold true in it.*

We consider one of the non-associative subloops $N_1 = \langle u^2 \mid u \in N \rangle$, $N_2 = \langle u^3 \mid u \in N \rangle$. The loops $N_1$ and $N_2$ are non-associative subloops of $N$. Since the identity $[x, y, z]^6 = e$ holds true in $N$, the identities $[x, y, z]^3 = e$ and $[x, y, z]^2 = e$, respectively, hold true in the non-associative loops $N_1$ and $N_2$, respectively. Thus we obtain one of the situations studied above.

*Case* 5: *$N$ is associative and* $\exp(\langle [u, v] \mid u, v \in N \rangle) = p^m s$, *where $p$ is a prime number not dividing $s$ and $m \geq 1$.*

In this case we consider in the $\nu(N)$-free group $F_3(x, y, z)$ the elements $a = x^s$, $b = y^{p^{m-1}s}$, $c = z^{p^{m-1}s}$ and $H = \langle a, b, c \rangle$. Then it is obvious that the loop $H$ with exponent zero is non-commutative and the following equalities hold true

$$[a, b]^p = e, \quad [a, c]^p = e, \quad [b, c]^p = e.$$

Then in the non-commutative group $H$ the identity $[x, y]^p = e$ is true. Applying the same reasoning as in Case 1 or 2 we obtain $F_3(K_{1,p,0}) \cong H \in q(N)$.

*Case* 6: *$N$ is associative and* $\exp(\langle [u,v] \mid u,v \in N\rangle) = 0$.

Similar to the previous cases we can easily deduce that $F_3(K_{1,0,0}) \in q(N)$. $\square$

**Lemma 3.** *For any variety $K \in \Re$ the following equalities $q(F_3(K)) = q(F_\omega(K))$, $q(F_3(K)) = q(F_n(K))$, $n = 4,5,\dots$, hold.*

Proof: It is enough to show that for any natural number $n$ the $K$-free loop $F_n(K)$, of finite rank $n$, belongs to the quasivariety $Q$. Since $F_1, F_2, F_3 \in Q$, we assume that $n > 3$. Let $F_n = F_n(x_1,\dots,x_n)$ be a $K$-free loop of rank $n$ with free generators $x_1,\dots,x_n$. We will first show that the $K$-free loop $F_n$ is approximated by the subloops of the $K$-free loop $F_3(x,y,z)$, i.e., for any element $u \neq e$ from $F_n$ there exists a homomorphism $\varphi$ from $F_n$ to $F_3$ such that $\varphi(u) \neq e$. If we admit that it is impossible, then in $F_n$ there exists an element $u = u(x_1,\dots,x_n) \neq e$ such that for any homomorphism $\varphi$ from $F_n$ to $F_3$ we have $\varphi(u) \neq e$. We will represent the element $u$ in its canonical form

$$u = (x_1^{\alpha_1},\dots,x_n^{\alpha_n}) \cdot \prod_{1\le i<j\le n} [x_i,x_j]^{\beta_{ij}} \prod_{1\le i<j<k\le n} [x_i,x_j,x_k]^{\gamma_{ijk}},$$

where the multiplication of factors from parenthesis is performed in a certain established order, for instance, from left to right. Assume that for a certain index $i$, $1 \le i \le n$, one has $x_i^{\alpha_i} \neq e$. The mapping $x_j \mapsto e$, $j \in \{1,\dots,n\} \setminus \{i\}$, $x_i \mapsto x$ extends to a homomorphism $\psi$ from $F_n$ to $F_3$. Then $\psi(u) = \psi(x_i)^{\alpha_i} = x^{\alpha_i}$ and in $F_3$ we get the equality $x^{\alpha_i} = e$. But the last expression is a true identity in the $K$-free loop $F_n(x,y,z)$, hence in $F_n$ as well. But in this case we came to a contradiction with $x_i^{\alpha_i} \neq e$. Hence, we can suppose that $x_1^{\alpha_1} = e, \dots, x_n^{\alpha_n} = e$ and

$$u = \prod_{1\le i<j\le n} [x_i,x_j]^{\beta_{ij}} \prod_{1\le i<j<k\le n} [x_i,x_j,x_k]^{\gamma_{ijk}}.$$

Assume that $[x_i,x_j]^{\beta_{ij}} \neq e$ for a certain pair $(i,j)$, $1 \le i < j \le n$. The mapping $x_k \mapsto e$, $k \in \{1,\dots,n\} \setminus \{i,j\}$, $x_i \mapsto x$, $x_j \mapsto y$ extends to a homomorphism $\psi$ from $F_n$ to $F_3$. Then $\psi(u) = [\psi(x_i),\psi(x_j)]^{\beta_{ij}} = [x,y]^{\beta_{ij}}$ and we get that the identity $[x,y]^{\beta_{ij}} = e$ holds true in $F_3$. But then this identity also holds true in $F_n$, which contradicts the inequality $[x_i,x_j]^{\beta_{ij}} \neq e$. Hence, we can say that $\prod_{1\le i<j\le n}[x_i,x_j]^{\beta_{ij}} = e$ and

$$u = \prod_{1\le i<j<k\le n} [x_i,x_j,x_k]^{\gamma_{ijk}}.$$

Now assume that $[x_i,x_j,x_k]^{\gamma_{ijk}} \neq e$ for a certain triple $(i,j,k)$, $1 \le i < j < k \le n$. The mapping $x_l \to e$, $l \in \{1,\dots,n\}\setminus\{i,j,k\}$, $x_i \to x$, $x_j \to y$, $x_k \to z$ extends to a homomorphism $\psi$ from $F_n$ to $F_3$. Then

$$\psi(u) = [\psi(x_i),\psi(x_j),\psi(x_k)]^{\gamma_{ijk}} = [x,y,z]^{\gamma_{ijk}}$$

and we get that the identity $[x,y,z]^{\gamma_{ijk}} = e$ holds true in $F_3$. But then this identity is also true in $F_n$, which contradicts the inequality $[x_i,x_j,x_k]^{\gamma_{ijk}} \neq e$. Then we can say that $\prod_{1\le i<j<k\le n}[x_i,x_j,x_k]^{\gamma_{ijk}} = e$ and $u = e$. We came to a contradiction with the assumption that $u \neq e$. From here we can conclude that the loop $F_n$ is approximated by the subloops of the loop $F_3$, hence it is included isomorphically in a Cartesian product of subloops of the loop $F_3$. Therefore, $F_n$ belongs to the quasivariety $q(F_3)$ and, hence, $F_n$ also belongs to the quasivariety $Q$. □

According Lemmas 1, 2 and 3 we can formulate the following theorem.

**Theorem 1.** *If $Q$ is a quasivariety that contains a nilpotent non-associative or non-commutative Moufang loop, then there exists at least one variety $K \in \Re$ so that $F_\omega(K) \in Q$.*

**Corollary 1.** *For any variety $K \in \Re$ the following statements are true:*

(a) *if $q(F_\omega(K))$ contains a non-associative and non-commutative loop $H$, then $q(H) = q(F_\omega(K))$;*
(b) *if $q(F_\omega(K))$ contains only commutative Moufang loops (respectively, groups) and $H$ is a non-associative (respectively, non-commutative) loop, then $q(H) = q(F_\omega(K))$.*

**Remark 1.** Since the following inclusions hold true

$$K_{3,1,0} \subset K_{3,3,0},\ K_{1,3,0} \subset K_{3,3,0},\ K_{3,1,3^m} \subset K_{3,3,3^m},\ m = 1,2,\ldots,$$

each of the quasivarieties $q(F_\omega(K_{3,3}))$, $q(F_\omega(K_{3,3,3^m}))$, $m = 1,2,\ldots$, contains only two non-abelian subquasivarieties: one formed of commutative Moufang loops and another formed of groups.

**Remark 2.** According to identity (5) and (8) inner permutations of the multiplication group of any loop of $K_{3,0,0}$ are automorphisms. Loops of these varieties are A-loops (see the research on nilpotent A-loops in [9]).

**Remark 3.** Each quasivariety of the set $\{q(F_\omega(K_{2,2,0})),\ q(F_\omega(K_{2,2,2^m})),\ m = 2,3,\ldots\}$ has only one non-abelian own subquasivariety being generated by a free group of rank 2 of this quasivariety.

From Theorem 1, Corollary 1 and Remarks 1–3 one gets the following.

**Theorem 2.** *Non-abelian minimal quasivarieties of the lattice of quasivarieties of nilpotent Moufang loops are:*

– *minimal non-associative quasivarieties of commutative Moufang loops*

$$q(F_\omega(K_{3,1,0})),\ q(F_\omega(K_{3,1,3^m}))\ \ (m = 1,2,\ldots);$$

– *minimal non-associative and non-commutative quasivarieties of Moufang A-loops with one proper minimal non-associative subquasivariety of commutative Moufang loops and one proper minimal non-commutative subquasivariety of groups;*

$$q(F_\omega(K_{3,0,0})),\ q(F_\omega(K_{3,3,0})),\ q(F_\omega(K_{3,3,3^m}))\quad (m=1,2,\dots);$$

– *minimal non-associative and non-commutative quasivarieties of Moufang loops with the only proper non-commutative subquasivariety of groups*

$$q(F_\omega(K_{2,0,0})),\ q(F_\omega(K_{2,2,0})),\ q(F_\omega(K_{2,2,2^m}))\quad (m=2,3,\dots);$$

– *minimal non-commutative quasivarieties of groups*

$$q(F_\omega(K_{1,0,0})),\ q(F_\omega(K_{1,p,0}))\quad (p=2,3,\dots),$$
$$q(F_\omega(K_{1,2,2^m}))\,(m=2,3,\dots),\ q(F_\omega(K_{1,p,p^m}))\,(p\geq 3,\ m=2,3,\dots).$$

Further, we will show a few concrete examples of nilpotent Moufang loops. First, we will prove the following important statement.

**Theorem 3.** *If the alternative ring $K$ with a unit element contains a nilpotent sub-ring $R$ with index $n\geq 2$ (i.e., any product of $n$ factors $a_1a_2\cdots a_n=0$ for any $a_1,\dots,a_n\in K$), then the set $L$ of all elements of the form $1+x$, where $x\in R$, forms a nilpotent Moufang loop of class $n-1$.*

Proof: The equality

$$(1+x)(1-x+x^2-\cdots+(-1)^{n-1}x^{n-1})=1$$

where $x\in R$, shows that any element from $L$ is invertible and, therefore, $L$ is a Moufang loop. Now let $R^k$ be the set of all linear combinations of all products of $k\leq n-1$ elements from $R$. Note that the following inclusions are true:

$$R^k\cdot R^l\subseteq R^{k+l},\ R^{k+1}\subseteq R^k. \tag{23}$$

Then for any $x\in R^k$ we have the equality

$$(1+x)^{-1}=1-x+x^2-\cdots+(-1)^{n-1}x^{n-1},$$

that is,

$$(1+x)^{-1}=1+x^*, \tag{24}$$

where $x^*=-x+x'$ and $x'=x^2-x^3-\cdots+(-1)^{n-1}x^{n-1}$. Because $x\in R^k$, then is clear that $x^2,x^4,\dots,x^{n-1}\in R^{2k}$ and it follows that

$$x'\in R^{2k}. \tag{25}$$

Now, if $x \in R^k$ and $y, z \in R$, then, according to Moufang's Theorem and the equality (24):

$$\begin{aligned}
&[1+z,\, 1+y,\, 1+x] = ((1+z)\cdot(1+y)(1+x))^{-1}\cdot((1+z)(1+y)\cdot(1+x))\\
&= ((1+x)^{-1}(1+y)^{-1}\cdot(1+z)^{-1})\cdot((1+z)(1+y)\cdot(1+x))\\
&= ((1+x^*)(1+y)^{-1}\cdot(1+z)^{-1})\cdot((1+z)(1+y)\cdot(1+x))\\
&= (((1+z)(1+y))^{-1} + x^*(1+y^*)\cdot(1+z^*))\cdot((1+z)(1+y)\cdot(1+x))\\
&= 1 + x + (x^*(1+y^*)\cdot(1+z^*))\cdot((1+z)(1+y)\cdot(1+x))\\
&= 1+x+(x^*+x^*y^*+x^*z^*+x^*y^*\cdot z^*)\cdot(1+z+y+x+zy+zx+yx+zy\cdot x)\\
&= 1+x+x^*+x^*y^*+x^*z^*+x^*y^*\cdot z^*\\
&\quad +(x^*+x^*y^*+x^*z^*+x^*y^*\cdot z^*)\cdot(z+y+x+zy+zx+yx+zy\cdot x);
\end{aligned}$$

similarly we can deduce

$$\begin{aligned}
&[1+x,\, 1+y] = ((1+y)(1+x))^{-1}\cdot(1+x)(1+y)\\
&= ((1+x)^{-1}(1+y)^{-1})\cdot((1+x)(1+y))\\
&= (1+x^*)(1+y)^{-1}\cdot((1+y)+x(1+y))\\
&= 1+x^*+(1+x^*)(1+y)^{-1}\cdot x(1+y)\\
&= 1+x^*+(1+x^*)(1+y^*)\cdot(x+xy)\\
&= 1+x^*+(1+x^*+y^*+x^*y^*)(x+xy)\\
&= 1+x+x^*+xy+(x^*+y^*+x^*y^*)(x+xy).
\end{aligned}$$

We note that

$$\begin{aligned}
x_1 &= x+x^*+[x^*y^*+x^*z^*+x^*y^*\cdot z^*\\
&\quad + (x^*+x^*y^*+x^*z^*+x^*y^*\cdot z^*)\cdot(z+y+x+zy+zx+yx+zy\cdot x)],\\
x_2 &= x+x^*+[xy+(x^*+y^*+x^*y^*)(x+xy)].
\end{aligned}$$

Because $x, x^* \in R^k$, according to (25) $x + x^* = x' \in R^{2k}$, and according to (23) items from square brackets from the last two equalities belong to $R^{k+1}$. Thus we have:

$$[1+z,\, 1+y,\, 1+x] = 1 + x_1 \in 1 + R^{k+1}, \tag{26}$$

$$[1+x,\, 1+y] = 1 + x_2 \in 1 + R^{k+1}. \tag{27}$$

Further, from the fact that $x \in R^k$ and $y, z \in R$ it follows that $x^* \in R^k$ and $y^*, z^* \in R$, which in view of (26) implies that $[1+x^*,\, 1+y^*,\, 1+z^*] \in 1 + R^{k+1}$.

Then according to (26) and (27) we have

$$
\begin{aligned}
&(1+x)\cdot(1+y)(1+z) = (1+x^*)^{-1}\cdot(1+y^*)^{-1}(1+z^*)^{-1}\\
&= ((1+z^*)(1+y^*)\cdot(1+x^*))^{-1}\\
&= (((1+z^*)\cdot(1+y^*)(1+x^*))\cdot[1+z^*,\,1+y^*,\,1+x^*])^{-1}\\
&= [1+z^*,\,1+y^*,\,1+x^*]^{-1}\cdot((1+z^*)\cdot(1+y^*)(1+x^*))^{-1}\\
&= [1+z^*,\,1+y^*,\,1+x^*]^{-1}\cdot((1+x^*)^{-1}(1+y^*)^{-1}\cdot(1+z^*)^{-1})\\
&= [1+z^*,\,1+y^*,\,1+x^*]^{-1}\cdot((1+x)(1+y)\cdot(1+z))\\
&= ((1+x)(1+y)\cdot(1+z))\cdot[1+z^*,\,1+y^*,\,1+x^*]^{-1}\\
&\quad\cdot[[1+z^*,\,1+y^*,\,1+x^*]^{-1},\,(1+x)(1+y)\cdot(1+z)]\\
&\in ((1+x)(1+y)\cdot(1+z))\cdot(1+R^{k+1})(1+R^{k+1})\\
&\subseteq ((1+x)(1+y)\cdot(1+z))\cdot(1+R^{k+1})
\end{aligned}
$$

which shows that the associator

$$[1+x,\,1+y,\,1+z]\in 1+R^{k+1}. \tag{28}$$

Now according to the definition of the special associator-commutator and the formulas (27), (28) by simple induction it shows that values in ML $L$ of any special associator-commutator of multiplicity $k$, $1\le k\le n$ are contained in $1+R^k$. In particular that values in ML $L$ of any special associator-commutator of multiplicity $n$ are contained in $1+R^n=\{1\}$. This means that $L$ is nilpotent of class $n-1$. □

**Example 1.** Let $R$ be an alternative $n$-nilpotent ring and $\mathbb{Z}$ the ring of integers. On the set $K=R\times\mathbb{Z}$ we define operations $+$ and $\cdot$ as follows:

$$
\begin{aligned}
(a,k)+(b,l) &= (a+b,\,k+l),\\
(a,k)\cdot(b,l) &= (a\cdot b+la+kb,\,k\cdot l),
\end{aligned}
$$

where $(a,k)$, $(b,l)\in K$. It is easy to see that $K$ together with the operations defined above is an alternative ring with the unit $e=(0,1)$ and that the set $L'$ of all elements of the form $(a,0)$ is $a$ subring $K$ isomorphic to $R$. Therefore, due to Theorem 3, the set $L=e+L'$ forms an $(n-1)$-nilpotent Moufang loop.

In particular, if $R$ is a free alternating ring of characteristic 3 (or zero), then $L$ is a $(n-1)$-nilpotent Moufang loop with exponent 3 (or zero).

**Example 2.** A basis of a Cayley–Dixon algebra $K$ (see [6]) over the field of real numbers $\mathbb{R}$ consists of the elements $e_1=1$, $e_2=i$, $e_3=j$, $e_4=k$, $e_5=e$, $e_6=ie$, $e_7=je$, $e_8=ke$, the first of which is a unit for algebra $K$ and the first four of which form a basis of the sub-algebra of quaternions. Multiplication is defined on these elements by the relations:

$$
\begin{aligned}
&i^2 = j^2 = k^2 = e^2 = -1,\\
&ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \qquad (29)\\
&eq = \overline{q}e,\ p \cdot qe = qp \cdot e,\ pe \cdot q = p\overline{q},\ pe \cdot qe = -\overline{q}p,
\end{aligned}
$$

where $\overline{q} = -q$, $p, q \in \{i, j, k\}$. The Cayley numbers $K$ are multiplied according to the distributive laws and relations (29). It is easy to verify that

$$
[e_i, e_j] = 1 \text{ or } [e_i, e_j] = -1, \ \ [e_i, e_j, e_k] = 1 \ \text{ or } \ [e_i, e_j, e_k] = -1. \qquad (30)
$$

From (29) and (30) we can see that the subsets

$$
\begin{gathered}
L_1 = \{\pm 1,\ \pm i,\ \pm j,\ \pm k,\ \pm e,\ \pm ie,\ \pm je,\ \pm ke\},\\
L_2 = \overline{\mathbb{R}} \cup \overline{\mathbb{R}}i \cup \overline{\mathbb{R}}j \cup \overline{\mathbb{R}}e \cup \overline{\mathbb{R}}ie \cup \overline{\mathbb{R}}je \cup \overline{\mathbb{R}}ke \ (\overline{\mathbb{R}} = \mathbb{R} \setminus \{0\})
\end{gathered}
$$

with respect to the multiplication are Moufang loops with the associators and commutators equal to 1 or $-1$, hence, they belong to the center of this loop. Therefore, the Moufang loops $L_1$ and $L_2$ are non-associative, non-commutative and 2-nilpotent. It is easy to verify that the exponent of $L_1$ is 4, the exponent of $L_2$ is infinite, and in both loops the following identities hold

$$
[x, y, z]^2 = 1,\ [x, y]^2 = 1.
$$

Therefore, $L_1 \in K_{2,2,2^2}$ and $L_2 \in K_{2,2,0}$.

**Example 3.** In the ring of all square matrices of order $n \geq 3$ over the Cayley–Dixon algebra we study the set $L$ of all matrices of the form $q \cdot A$, where $q$ is an element of the Moufang loop $L_1$ (or $L_2$) from Example 2 and $A$ is a lower (or upper) triangular matrix of order $n$ that has 1s along the main diagonal and the other elements above it are arbitrary real numbers (it is well known that these matrices $A$ form a nilpotent group relative to the usual multiplication [10]).

It is easy to check that for any elements $pA, qB, rC \in L$ we have

$$
\begin{gathered}
[pA, qB, rC] = [p, q, r] \cdot [A, B, C] \in \{-E, E\},\\
[pA, qB] = [p, q] \cdot [A, B] \in \{-[A, B], [A, B]\},
\end{gathered}
$$

where $E$ is the unit matrix. From this it follows that $L$ forms a nilpotent Moufang loop of class $(n-1)$ relative to the multiplication. In particular, for $n = 3$, $L \in K_{2,0,0}$.

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