Affine regular icosahedron circumscribed around the affine regular octahedron in GS–quasigroup

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Abstract. The concept of the affine regular icosahedron and affine regular octahedron in a general GS-quasigroup will be introduced in this paper. The theorem of the unique determination of the affine regular icosahedron by means of its four vertices which satisfy certain conditions will be proved. The connection between affine regular icosahedron and affine regular octahedron in a general GS-quasigroup will be researched. The geometrical representation of the introduced concepts and relations between them will be given in the GS-quasigroup $\mathbb{C}(\frac{1}{2}(1+\sqrt{5}))$.

Keywords: GS-quasigroup, affine regular icosahedron, affine regular octahedron

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1. Introduction

In [1] the concept of GS-quasigroup is defined. A quasigroup $(Q,\cdot)$ is said to be golden section quasigroup or shortly GS-quasigroup if it satisfies the (mutually equivalent) identities

$$a(ab\cdot c)\cdot c=b, \qquad a\cdot(a\cdot bc)c=b \tag{1, 1'}$$

and the identity of idempotency

$$aa=a. \tag{2}$$

GS-quasigroups are medial quasigroups, i.e. the identity

$$ab\cdot cd=ac\cdot bd \tag{3}$$

is valid.

As a consequence of the identity of mediality the considered GS-quasigroup $(Q,\cdot)$ satisfies the identities of elasticity and the left and right distributivity, i.e. we have the identities

$$a\cdot ba=ab\cdot a, \tag{4}$$

$$a\cdot bc=ab\cdot ac, \qquad ab\cdot c=ac\cdot bc. \tag{5, 5'}$$

Further, the identities

(6) $a(ab \cdot b) = b,$ $(b \cdot ba)a = b,$ (6)'

(7) $a(ab{\cdot}c) = b{\cdot}bc,$ $(c{\cdot}ba)a = cb{\cdot}b,$ (7)'

(8) $a(a{\cdot}bc) = b(b{\cdot}ac),$ $(cb{\cdot}a)a = (ca{\cdot}b)b$ (8)'

are also valid in any GS-quasigroup.

Let $\mathbb{C}$ be the set of points of the Euclidean plane. For any two different points a, b we define $ab = c$ if the point b divides the pair a, c in the ratio of golden section. In [1] it is proved that $(\mathbb{C}, \cdot)$ is a GS-quasigroup. We shall denote that quasigroup by $\mathbb{C}(\frac{1}{2}(1+\sqrt{5}))$ because we have $c = \frac{1}{2}(1+\sqrt{5})$ if $a = 0$ and $b = 1$. The figures in this quasigroup $\mathbb{C}(\frac{1}{2}(1+\sqrt{5}))$ can be used for illustration of "geometrical" relations in any GS-quasigroup.

From now on let $(Q, \cdot)$ be any GS-quasigroup. The elements of the set Q are said to be *points*.

The points a, b, c, d are said to be the vertices of a *parallelogram* and we write $\mathrm{Par}(a, b, c, d)$ if the identity $a \cdot b(ca \cdot a) = d$ holds. In [1] some properties of the quaternary relation Par on the set Q are proved. We shall mention only the two following properties which we shall use afterwards.

Lemma 1.1. *If* (e, f, g, h) *is any cyclic permutation of* (a, b, c, d) *or of* (d, c, b, a)*, then* $\mathrm{Par}(a, b, c, d)$ *implies* $\mathrm{Par}(e, f, g, h)$*.*

Lemma 1.2. *From* $\mathrm{Par}(a, b, c, d)$ *and* $\mathrm{Par}(c, d, e, f)$ *follows* $\mathrm{Par}(a, b, f, e)$*.*

We shall say that b is the *midpoint* of the pair of points a, c and write $M(a, b, c)$ if $\mathrm{Par}(a, b, c, b)$. In [1] the following statement is proved.

Lemma 1.3. *The statement* $M(a, b, c)$ *holds if and only if* $c = ba \cdot b$*.*

In [2] the concept of the GS-trapezoid is defined. The points a, b, c, d successively are said to be the vertices of the *golden section trapezoid* and it is denoted by $\mathrm{GST}(a, b, c, d)$ if the identity $a \cdot ab = d \cdot dc$ holds. The following properties of the relation GST are also proved in [2].

Lemma 1.4. *The statements* $\mathrm{GST}(a, b, c, d)$ *and* $\mathrm{GST}(a, b, c', d')$ *imply the statement* $\mathrm{GST}(d, c, c', d')$*.*

Lemma 1.5. *Let the statements* $\mathrm{GST}(a, b, c, d)$ *and* $\mathrm{GST}(a', b, c, d')$ *be valid and let* b' *be a given point. Then there is one and only one point* c' *such that* $\mathrm{GST}(a, b', c', d)$ *and* $\mathrm{GST}(a', b', c', d')$ *are valid.*

Lemma 1.6. *Any three of the four statements* $\mathrm{GST}(a, b, c, d)$, $\mathrm{GST}(a', b, c, d')$, $\mathrm{GST}(a', a, e, c)$ *and* $\mathrm{GST}(d', d, e, b)$ *imply the remaining statement.*

Lemma 1.7. *Any two of the three statement* $\mathrm{GST}(a,b,c,d)$, $\mathrm{GST}(a,b',c',d)$, $\mathrm{Par}(b,b',c',c)$ *imply the remaining statement.*

In [2] it is proved that any two of the five statements

(9) $\mathrm{GST}(a,b,c,d), \mathrm{GST}(b,c,d,e), \mathrm{GST}(c,d,e,a), \mathrm{GST}(d,e,a,b), \mathrm{GST}(e,a,b,c)$

imply the remaining statement.

In [3] the concept of the affine regular pentagon is defined. The points a,b,c,d,e successively are said to be the vertices of the *affine regular pentagon* and it is denoted by $\mathrm{ARP}(a,b,c,d,e)$ if any two (and then all five) of the five statements (9) are valid (Figure 1).

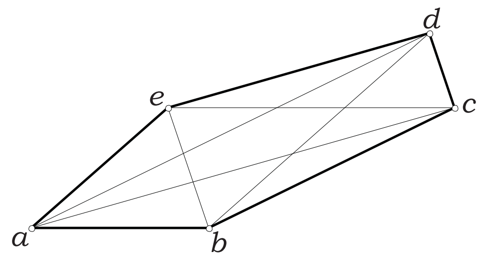

FIGURE 1. Affine regular pentagon in $\mathbb{C}(\frac{1}{2}(1+\sqrt{5}))$

2. The concept of the affine regular icosahedron and affine regular octahedron in GS-quasigroup

In this chapter we are going to introduce the concept of the affine regular icosahedron and affine regular octahedron in a general GS-quasigroup.

Definition 1. We shall say that the points $a,b,c,d,e,f,a',b',c',d',e',f'$ are the vertices of an *affine regular icosahedron* (Figure 2) and we shall write $\mathrm{ARI}(a,b,c,d,e,f,a',b',c',d',e',f')$ if the following twelve statements are valid

$$\begin{array}{lll}
\mathrm{ARP}(b,c,f,a',e), & \mathrm{ARP}(c,a,d,b',f), & \mathrm{ARP}(a,b,e,c',d),\\
\mathrm{ARP}(b',c',f',a,e'), & \mathrm{ARP}(c',a',d',b,f'), & \mathrm{ARP}(a',b',e',c,d'),\\
\mathrm{ARP}(b,c,e',d,f'), & \mathrm{ARP}(c,a,f',e,d'), & \mathrm{ARP}(a,b,d',f,e'),\\
\mathrm{ARP}(b',c',e,d',f), & \mathrm{ARP}(c',a',f,e',d), & \mathrm{ARP}(a',b',d,f',e).
\end{array}$$

If the statement $\mathrm{ARI}(a,b,c,d,e,f,a',b',c',d',e',f')$ is valid then vertices a,a'; b,b'; c,c'; d,d'; e,e' respectively f,f' are called the *opposite* vertices. For the opposite vertices of the ARI the following statement is valid.

Theorem 2.1. *If x and x', respectively y and y' are opposite vertices of the* ARI *then* $\mathrm{Par}(x,y,x',y')$ *is valid (Figure 2).*

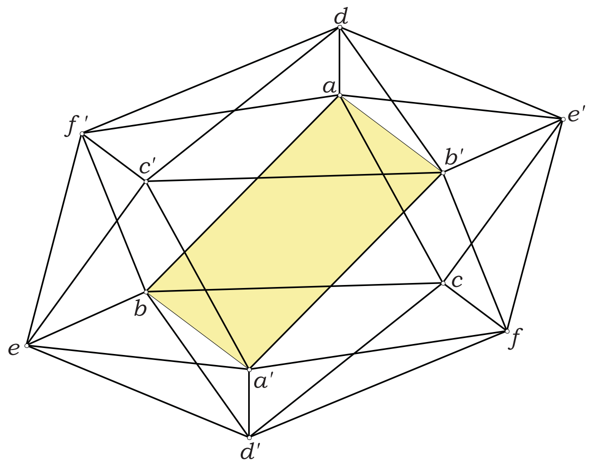

FIGURE 2. Affine regular icosahedron in $\mathbb{C}(\frac{1}{2}(1+\sqrt{5}))$

PROOF: It is sufficient to prove that, along with $\mathrm{ARI}(a,b,c,d,e,f,a',b',c',d',e',f')$ the statement $\mathrm{Par}(a,b,a',b')$ is valid. However, the statements $\mathrm{ARP}(a,b,e,c',d)$ and $\mathrm{ARP}(a',b',d,f',e)$ imply $\mathrm{GST}(d,a,b,e)$ and $\mathrm{GST}(d,b',a',e)$ wherefrom according to Lemma 1.7 the statement $\mathrm{Par}(a,b',a',b)$ follows. □

It is possible to prove that the affine regular icosahedron is uniquely determined by its any four independent vertices, i.e. vertices which are not in the relation Par or GST. For the illustration we shall prove the following theorem.

Theorem 2.2. *For any points a,b,c,d the points e,f,a',b',c',d',e',f' are uniquely determined so that $\mathrm{ARI}(a,b,c,d,e,f,a',b',c',d',e',f')$ is valid.*

PROOF: Let e,f be such points that $\mathrm{GST}(d,a,b,e)$, $\mathrm{GST}(d,a,c,f)$ are valid, and then according to Lemma 1.4 the statement $\mathrm{GST}(e,b,c,f)$ is valid too. Let a',b',c' be such points that statements $\mathrm{ARP}(b,c,f,a',e)$, $\mathrm{ARP}(c,a,d,b',f)$, $\mathrm{ARP}(a,b,e,c',d)$ are valid.
Because of that we have the statements $\mathrm{GST}(b',d,a,c)$ and $\mathrm{GST}(c',d,a,b)$, and then by Lemma 1.5 there is the point d' such that the statements $\mathrm{GST}(b',a',d',c)$ and $\mathrm{GST}(c',a',d',b)$ are valid. Similarly there is the point e' such that the statements $\mathrm{GST}(c',b',e',a)$ and $\mathrm{GST}(a',b',e',c)$ are valid and the point f' such that $\mathrm{GST}(a',c',f',b)$ and $\mathrm{GST}(b',c',f',a)$ are valid. However, from $\mathrm{GST}(b',c',f',a)$ and $\mathrm{GST}(c',b',e',a)$ follows $\mathrm{ARP}(b',c',f',a,e')$, and analogously the statements $\mathrm{ARP}(c',a',d',b,f')$ and $\mathrm{ARP}(a',b',e',c,d')$ are valid. Then the statements $\mathrm{GST}(c',d,a,b)$, $\mathrm{GST}(b',d,a,c)$ and $\mathrm{GST}(b',c',f',a)$ are also valid wherefrom by Lemma 1.6 follows $\mathrm{GST}(c,b,f',d)$, and similarly the statement $\mathrm{GST}(b,c,e',d)$ is valid too. From these two statements $\mathrm{ARP}(b,c,e',d,f')$ follows. The remaining statements $\mathrm{ARP}(c,a,\ f',e,d')$, $\mathrm{ARP}(a,b,d',f,e')$, $\mathrm{ARP}(b',c',e,d',f)$, $\mathrm{ARP}(c',a',f,e',d)$ and $\mathrm{ARP}(a',b',d,f',e)$ can be proved analogously. □

Definition 2. We shall say that the points a, b, c, a', b', c' are the vertices of an affine regular octahedron with the center o if the statements $M(a, o, a')$, $M(b, o, b')$, $M(c, o, c')$ are valid (Figure 3).

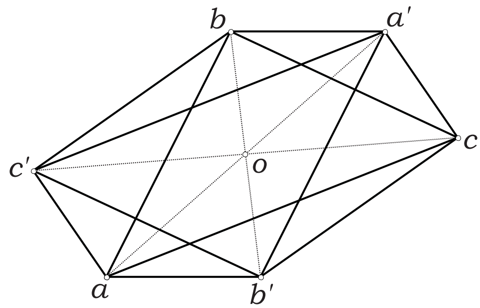

FIGURE 3. Affine regular octahedron in $\mathbb{C}(\frac{1}{2}(1+\sqrt{5}))$

3. Affine regular icosahedron circumscribed around the affine-regular octahedron

Now, we will obtain the vertices of an affine regular icosahedron starting from the vertices of an affine regular octahedron in a general GS-quasigroup. So, we have the following statement.

Theorem 3.1. *If a, b, c, a', b', c' are the vertices of an affine regular octahedron with the center o then $\mathrm{ARI}(a_b, a_{b'}, a'_b, a'_{b'}, b_c, b_{c'}, b'_c, b'_{c'}, c_a, c_{a'}, c'_a, c'_{a'})$ is valid, where $a_b = o(ab \cdot b)$, $a_{b'} = o(ab' \cdot b')$, $\ldots$, $c'_{a'} = o(c'a' \cdot a')$ (Figure 4).*

PROOF: It is enough to prove the statements $\mathrm{GST}(c_a, c'_a, b_{c'}, a'_b)$, $\mathrm{GST}(c'_a, c_a, b_c, a'_b)$, i.e., $c_a \cdot c_a c'_a = a'_b \cdot a'_b b_{c'}$.

$$\begin{aligned}
c_a \cdot c_a c'_a &= o(ca \cdot a) \cdot [o(ca \cdot a) \cdot o(c'a \cdot a)] \overset{(5)}{=} o[(ca \cdot a) \cdot (ca \cdot a)(c'a \cdot a)] \\
&\overset{(4)}{=} o[(ca)(ca \cdot a) \cdot a(c'a \cdot a)] \overset{(5')}{=} o[(c \cdot ca)a \cdot a(c'a \cdot a)] \\
&\overset{(6')}{=} o[c \cdot a(c'a \cdot a)] \overset{(8')}{=} o[c \cdot (ac' \cdot a)a] \overset{(4)}{=} o[c \cdot (aa \cdot c')c'] \overset{(2)}{=} o \cdot c(ac' \cdot c'), \\
a'_b \cdot a'_b b_{c'} &= o(a'b \cdot b) \cdot [o(a'b \cdot b) \cdot o(bc' \cdot c')] \overset{(5)}{=} o[(a'b \cdot b) \cdot (a'b \cdot b)(bc' \cdot c')] \\
&\overset{(3)}{=} o[(a'b)(a'b \cdot b) \cdot b(bc' \cdot c')] \overset{(6)}{=} o[(a'b)(a'b \cdot b) \cdot c'] \\
&\overset{(5')}{=} o[(a' \cdot a'b)b \cdot c'] \overset{(6')}{=} o \cdot a'c'.
\end{aligned}$$

It remains to prove the equality $c(ac' \cdot c') = a'c'$. Because of $M(a, o, a')$, $M(c, o, c')$, by Lemma 1.3, we get $c = oc' \cdot o$, $oa \cdot o = a'$.

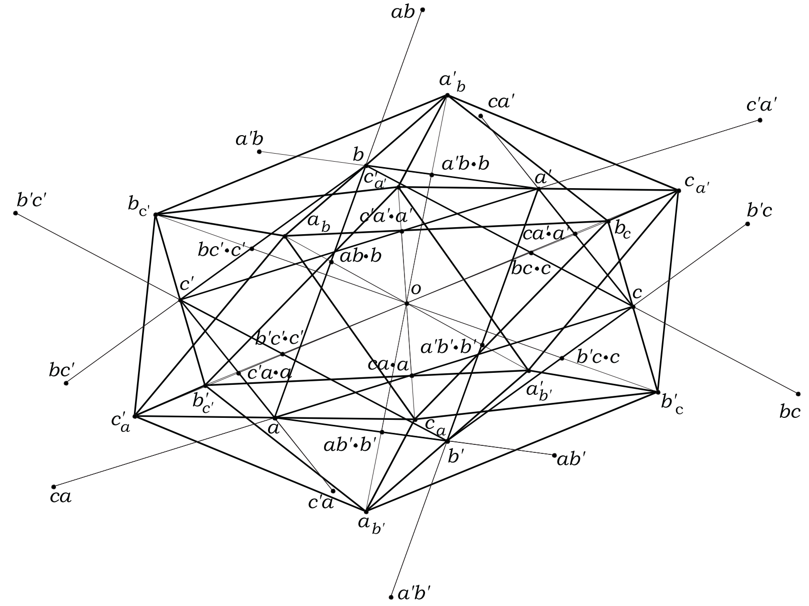

FIGURE 4.

According to the previous equalities we obtain

$$c(ac' \cdot c') = (oc' \cdot o)(ac' \cdot c') \overset{(4)}{=} (oc' \cdot ac') \cdot oc' \overset{(5)'}{=} (oa \cdot o)c' = a'c'$$

which proves the assertion of theorem. □

In this case we shall say that affine regular icosahedron is *circumscribed* around a given affine regular octahedron (Figure 5).

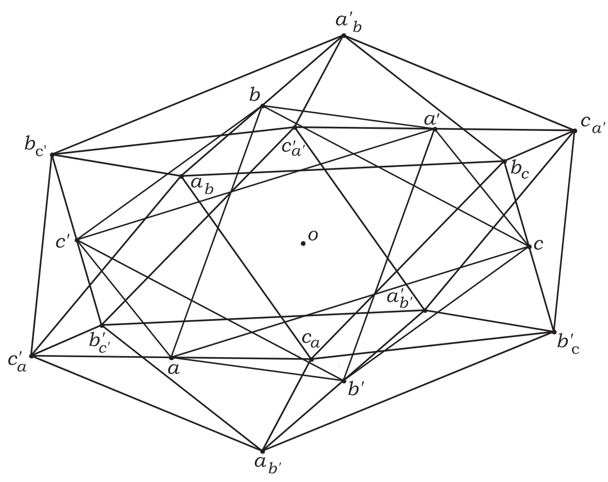

FIGURE 5. Affine regular octahedron and icosahedron in $\mathbb{C}(\frac{1}{2}(1+\sqrt{5}))$

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