Orlicz spaces associated with a semi-finite von Neumann algebra

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Abstract. Let M be a von Neumann algebra, let φ be a weight on M and let Φ be N-function satisfying the (δ_2, Δ_2)-condition. In this paper we study Orlicz spaces, associated with M, φ and Φ.

Keywords: Orlicz spaces, von Neumann algebra, weight

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Introduction

Construction and investigation of various classes of symmetric spaces of measurable operators affiliated with a von Neumann algebra M is one of important applications of the non commutative integration theory for a faithful normal semi-finite trace τ on the von Neumann algebra M. Examples of such spaces are given, in particular by non commutative L_p-spaces $L_p(M,\tau)$ [15] and by Orlicz spaces $L_\Phi(M,\tau)$ associated with an N-function Φ [5], [6], [7]. All these spaces are realized as ideal subspaces of the $*-$algebra $S(M)$ of measurable operators affiliated with M.

Investigations based on the modular theory for von Neumann algebras enable to consider non commutative versions of L_p-spaces associated with states and weights (see e.g. the survey [13]). But in these cases in general L_p-spaces cannot be realized as ideal subspaces of $S(M)$. This fact explains in particular why in their attempt to introduce non commutative Orlicz spaces for states in [1] as a subspaces of $S(M)$, the authors were unable to prove the completeness of these spaces with respect to the Luxemburg norm.

In the present paper we introduce a certain class of non commutative Orlicz spaces, associated with arbitrary faithful normal locally-finite weight φ on a semi-finite von Neumann algebra M. We show that the introduced Orlicz space $L_{\Phi,\alpha}(M,\varphi,\tau), \alpha \in [0,1]$, as a Banach space, is isometrically isomorphic to the Orlicz space $L_\Phi(M,\tau)$ considered in [5], [6], [7]. In particular, this implies that Orlicz spaces $L_\Phi(M,\tau_1)$ and $L_\Phi(M,\tau_2)$ are isomorphic for arbitrary faithful normal semi-finite traces τ_1 and τ_2 on M. We describe the dual spaces for such Orlicz spaces and, in the case of regular weights, we show that they can be realized as linear subspaces of the algebra of $LS(M)$ of locally measurable operators affiliated with M.

For the terminology and notations from the von Neumann algebra theory we refer to [10] and from theory of measurable and locally measurable operators refer to [8], [14].

Preliminaries

Let M be a von Neumann algebra acting on a Hilbert space H with $\mathbf{1}$-the identity operator on H, and let $P(M) = \{p \in M : p = p^2 = p^*\}$ be the lattice of all projection from M. Denote by $S(M)$ (respectively by $LS(M)$) the *-algebra of all measurable (respectively, locally-measurable) operators affiliated with M. It is well-known that $S(M)$ is a *-subalgebra in $LS(M)$, and M is a *-subalgebra of $S(M)$ ([8, Chapter 2]).

If $x \in LS(M)$ and $x = u|x|$ is its polar decomposition, where $|x| = (x^*x)^{1/2}$ and u is a partial isometry such that u^*u is the right support of x, then we have that $u \in M$ and $|x| \in LS(M)$. It is also known that the spectral family of projections $\{e_\lambda(x)\}_{\lambda\in\mathbf{R}}$ for a self adjoint operator $x \in LS(M)$, always belongs to $P(M)$, where $\mathbf{R}$ is the field of all real numbers.

Given a subset $A \subset LS(M)$, put $A_h = \{x \in A : x = x^*\}$ and $A_+ = \{x \in A : (x\xi, \xi) \geq 0$ for all $\xi \in D(x)\}$, where $D(x)$ is the domain of the operator $x \in LS(M)$, and $(\cdot,\cdot)$ is the inner product in the Hilbert space H.

Let τ be a faithful normal semi-finite trace on M. For each real number $p \geq 1$ consider the set

$$L_p(M,\tau) = \left\{x \in S(M) : \int_0^\infty \lambda^p \, d\tau(e_\lambda(|x|)) < \infty\right\}.$$

It known [15] that $L_p(M,\tau)$ is a linear subspace in $S(M)$ and the function $\|x\|_p = (\int_0^\infty \lambda^p \, d(\tau(e_\lambda(|x|)))^{1/p}$ is a norm, which turns $L_p(M,\tau)$ into a Banach space.

A map $\varphi : M_+ \to [0,\infty]$ is said to be *a weight* if

$$\varphi(x+y) = \varphi(x) + \varphi(y), \quad \varphi(\lambda x) = \lambda\varphi(x), \quad (x, y \in M_+, \lambda \geq 0, \text{ where } 0\cdot\infty = 0).$$

A weight φ is said to be

— *normal*, if $\varphi(x) = \sup \varphi(x_i)(x_i \nearrow x;\ x_i, x \in M_+)$;

— *faithful*, if $\varphi(x) = 0$, $x \in M_+$ implies that $x = 0$;

— *semi-finite*, if the linear span m_φ of the cone $\{x \in M_+ : \varphi(x) < \infty\}$ is dense in M with respect to the ultra-weak topology;

— *locally finite*, if

$$\forall x \in M_+ \ \ (x \neq 0) \ \ \exists y \in M_+ : y \leq x, 0 < \varphi(y) < \infty;$$

— *regular*, if

$$\forall \omega \in (M_*)_+ \ (\omega \neq 0) \ \exists \omega' \in (M_*)_+ \ (\omega' \neq 0) : \omega' \leq \omega, \omega' \leq \varphi,$$

where $(M_*)_+$ is the set of all positive ultra-weakly continuous linear functionals on M.

If the weight φ is a trace, i.e. when $\varphi(x^*x) = \varphi(xx^*)$ for all $x \in M$, the properties of semi-finiteness and locally finiteness (and respectively of faithfulness and regularity) of φ coincide with each other [12].

For a faithful normal semi-finite weight φ on M there exists a uniquely defined non singular self-adjoint positive operator h, affiliated with M such that $\varphi(\cdot) = \tau(h\cdot)$, and which is called the Radon-Nikodym derivative of the weight φ with respect to the trace τ [9].

Recall the following result

Theorem 1 ([12]). *Let τ be a faithful normal semi-finite trace on M and let $\varphi = \tau(h\cdot)$ be a faithful normal semi-finite weight on M, where h is the Radon-Nikodym derivative of φ with respect to τ. Then*

(i) *the weight φ is locally finite if and only if the operator h is locally measurable;*

(ii) *the weight φ is regular if and only if the operator h^{-1} is locally measurable.*

Now let $\varphi(\cdot) = \tau(h\cdot)$ be a faithful normal locally finite weight on M. For real numbers $p \geq 1$ and $\alpha \in [0, 1]$ put

$$m_\alpha^{1/p} = \{x \in M : h^{\alpha/p} x h^{(1-\alpha)/p} \in L_p(M, \tau)\};$$

$$\|x\|_{p,\alpha} = \|h^{\alpha/p} x h^{(1-p)/p}\|_p.$$

In [11] it has been proved that $m_\alpha^{1/p}$ is a linear subspace in M, and $\|\cdot\|_{p,\alpha}$ is a norm on $m_\alpha^{1/p}$. The completion of the normed space $(m_\alpha^{1/p}, \|\cdot\|_{p,\alpha})$ is denoted by $L_p(M, \varphi)$. In [11] it is proved that the Banach space $(L_p(M, \varphi), \|\cdot\|_{p,\alpha})$ is isometrically isomorphic to the space $(L_p(M, \tau), \|\cdot\|_p)$ for all $\alpha \in [0, 1]$.

In order to define the Orlicz space associated with a weight, we need the notion of N-function.

A continuous non-negative convex monotone increasing function Φ on the set of real numbers $\mathbf{R}$ is called *N-function* [4], if

$$\Phi(t) = \int_0^{|t|} p(s)\, ds,$$

where $p(s)$ is a non-decreasing function, positive for $s > 0$ and right continuous for $s \geq 0$, which satisfies the conditions

$$p(0) = 0, \quad p(\infty) = \lim_{s \to \infty} p(s) = \infty.$$

For each N-function $\Phi(t)$ a complementary N-function $\Psi(t)$ is defined as

$$\Psi(t) = \int_0^{|t|} q(s)\, ds,$$

where $q(s) = \sup\{t \geq 0 : p(t) \leq s\}$. It is clear that the complementary N-function for the N-function $\Psi(t)$ coincides with the initial function $\Phi(t)$, and moreover the following Young inequality is valid

$$ts \leq \Phi(t) + \Psi(s) \quad \text{for all} \quad t, s \geq 0.$$

We say that an N-function $\Phi(t)$ satisfies the (δ_2, Δ_2)-condition, if given any real $k > 0$ there exists a positive number $r(k)$ such that $\Phi(kt) \leq r(k)\Phi(t)$ for all $t \geq 0$. Examples of N-function which satisfy the (δ_2, Δ_2)-condition are given by the function $\Phi(t) = \frac{1}{p}|t|^p$, $p > 1$.

Let $\Phi(t)$ be an N-function and let $x \in LS_h(M)$ and let $\{e_\lambda\}_{\lambda \in \mathbf{R}}$ be the spectral family of projections for x, i.e. $x = \int_{-\infty}^{\infty} \lambda \, de_\lambda(x)$. It is known ([8, §2.3]), that one can define a self-adjoint operator $\Phi(x) = \int_{-\infty}^{\infty} \Phi(\lambda) \, de_\lambda(x)$, and moreover $\Phi(x) \in LS(M)$.

Let us extend the faithful normal semi-finite trace τ from M_+ to operators from $LS_+(M)$ as

$$\tau(x) = \sup_{h \geq 1} \tau \left(\int_0^h \lambda \, de_\lambda(x) \right) = \int_0^\infty \lambda \, d\tau(e_\lambda(x)).$$

It is known (e.g. [8, §4.1]), that

$$\tau(x) = \sup\{\tau(y) : y \in M_+, y \leq x\}$$

for all $x \in LS_+(M)$.

It is clear that $\tau(|x|) < \infty$ for $x \in LS(M)$ if and only if $x \in L_1(M, \tau)$; in this case $\tau(\mathbf{1} - e_\lambda(|x|)) < \infty$ for all $\lambda > 0$. Further we shall need the following result.

Proposition 1 ([3])**.** *If $x, y \in LS_+(M)$, then*

(i) $\tau(f(x)) \leq \tau(f(y))$ *for $x \leq y$ and each continuous monotone increasing function $f : [0, \infty) \to \mathbf{R}$ with $f(0) = 0$;*

(ii) $\tau(f(\lambda x + (1-\lambda)y)) \leq \lambda \tau(f(x)) + (1-\lambda)\tau(f(y))$ *for all $\lambda \in [0, 1]$ and each convex monotone increasing function f with $f(0) = 0$.*

Let Φ be an N-function. The set $K_\Phi = \{x \in S(M) : \tau(\Phi(|x|)) \leq 1\}$ is an absolutely convex subset in $S(M)$ [5]. The linear subspace $L_\Phi(M, \tau) = \bigcup_{n=1}^{\infty} nK_\Phi$ equipped with the norm

$$\|x\|_\Phi = \inf\left\{\lambda > 0 : \frac{x}{\lambda} \in K_\Phi\right\}, \tag{1}$$

is a Banach space [5] which is called the Orlicz space associated with M, τ and Φ. If the N-function Φ satisfies the (δ_2, Δ_2)-condition, then

$$L_\Phi(M, \tau) = \{x \in LS(M) : \tau(\Phi(|x|)) < \infty\},$$

moreover the linear subspace $m_\Phi^\tau = \{x \in M : \tau(\Phi(|x|)) < \infty\}$ is dense in $(L_\Phi(M, \tau), \| \cdot \|_\Phi)$.

Note that

$$m_\tau = \{x \in M : \tau(|x|) < \infty\} \subset m_\Phi^\tau. \tag{2}$$

Indeed, from the equalities

$$\lim_{t\downarrow 0} \frac{\Phi(t)}{t} = \lim_{t\downarrow 0} p(t) = 0$$

it follows that $\Phi(t) \le t$ for sufficiently small $t > 0$.

Therefore for $x \in m_\tau$ there exists $t_0 > 0$ such that

$$\begin{aligned}\tau(\Phi(|x|e_{t_0}(|x|))) &= \int_0^{t_0} \Phi(\lambda)\, d\tau(e_\lambda(|x|)) \\ &\le \int_0^{t_0} \lambda\, d\tau(e_\lambda(|x|)) = \tau(|x|e_{t_0}(|x|)) < \infty.\end{aligned}$$

Since $\tau(\mathbf{1} - e_{t_0}(|x|)) < \infty$, we have that

$$\tau(\Phi(|x|(\mathbf{1} - e_{t_0}))) \le \Phi(\|x\|_M)\tau(\mathbf{1} - e_{t_0}(|x|)) < \infty,$$

where $\|\cdot\|_M$ is the C^*-norm on M. Therefore $\tau(\Phi(|x|)) < \infty$, i.e. $x \in m_\Phi^\tau$.

Proposition 2. *If the N-function Φ satisfies the (δ_2, Δ_2)-condition, then m_τ is dense in $L_\Phi(M,\tau)$.*

Proof: Since $m_\tau \subset m_\Phi^\tau$ (see (2)) and m_Φ^τ is dense in $L_\Phi(M,\tau)$, it is sufficient to prove that m_τ is dense in m_Φ^τ. Moreover since each element of m_ϕ^τ is a finite linear combination of positive elements from m_Φ^τ it is sufficient to show that every element from $x \in (m_\Phi^\tau)_+$ belongs to the closure of m_τ in $L_\Phi(M,\tau)$. First, let us show that

$$x_n = x(\mathbf{1} - e_{\frac{1}{n}}) \in m_\tau,$$

where $e_\lambda = e_\lambda(x)$ is the spectral family of projections for x. From

$$\begin{aligned}\Phi\left(\frac{1}{n}\right)\tau\left(\mathbf{1} - e_{\frac{1}{n}}\right) &= \tau\left(\Phi\left(\frac{1}{n}\left(\mathbf{1} - e_{\frac{1}{n}}\right)\right)\right) \\ &\le \tau\left(\Phi\left(x\left(\mathbf{1} - e_{\frac{1}{n}}\right)\right)\right) \le \tau(\Phi(x)) < \infty,\end{aligned}$$

it follows that $\tau(\mathbf{1} - e_{\frac{1}{n}}) < \infty$ and the inequality $0 \le x(\mathbf{1} - e_{\frac{1}{n}}) \le \|x\|_M(\mathbf{1} - e_{\frac{1}{n}})$ implies that

$$x_n = x\left(\mathbf{1} - e_{\frac{1}{n}}\right) \in m_\tau.$$

Since $0 \le xe_{\frac{1}{n}} \downarrow 0$ when $n \to \infty$, it follows that $\tau(\Phi(\frac{1}{\varepsilon}xe_{\frac{1}{n}})) \downarrow 0$ for any $\varepsilon > 0$. In particular, there exists $n(\varepsilon)$ such that $\tau(\Phi(\frac{1}{\varepsilon}xe_{\frac{1}{n}})) < 1$ for $n \ge n(\varepsilon)$, i.e. $\|xe_{\frac{1}{n}}\|_\Phi < \varepsilon$. This means that $\|x - x_n\|_\Phi \to 0$, i.e. m_τ is dense in m_Φ^τ. □

Let Ψ be the complementary N-function for the N-function Φ satisfying the (δ_2, Δ_2)-condition. In this case given any $y \in L_\Psi(M,\tau)$ the function $f_y(x) = \tau(xy), x \in L_\Phi(M,\tau)$, defines the general form of continuous linear functionals on $L_\Phi(M,\tau)$ [5], moreover

$$\|f_y\| = \sup\{|\tau(xy)| : x \in L_\Phi(M,\tau), \|x\|_\Phi \leq 1\} = \|y\|_\Psi.$$

Further we shall need also two inequalities from the following proposition.

Proposition 3. *Let τ be a faithful normal semi-finite trace on a von Neumann algebra M. Then*

(i) ([8, §3.4]). *Given any $x, y \in LS(M)$ there exist two partial isometries $u, v \in M$ such that*

$$|x + y| \leq u^*|x|u + v^*|y|v.$$

(ii) [2]. *For every N-function Φ, arbitrary operator $z \in M$ with $\|z\|_M \leq 1$, and for each $x \in LS_+(M)$ we have the following inequality*

$$\tau(\Phi(z^*xz)) \leq \tau(z^*\Phi(x)z).$$

Orlicz spaces associated with a weight

In this section an approach is suggested for the construction of Orlicz spaces associated with a faithful normal locally finite weight on a semi-finite von Neumann algebra for an N-function satisfying the (δ_2, Δ_2)-condition. For these spaces the dual spaces are described. In the case of regular locally finite normal weights the constructed Orlicz spaces are represented as spaces of locally measurable operators.

Let τ be a faithful normal semi-finite trace on a von Neumann algebra M. From now on φ denotes a faithful normal locally finite weight on M. Therefore the Radon-Nikodym derivative h of the weight φ with respect to τ is a positive locally measurable non singular operator.

Given an N-function Φ and a real number $\alpha \in [0,1]$ put

$$U(x) = U^{\varphi,\tau}_{\Phi,\alpha}(x) = (\Phi^{-1}(h))^\alpha x (\Phi^{-1}(h))^{1-\alpha}, \quad x \in LS(M).$$

It is clear that $U(x) \in LS(M)$ and $\Phi(|U(x)|) \in LS(M)$.

Consider the functional on $LS(M)$ defined by

$$O^{\varphi,\tau}_{\Phi,\alpha}(x) = \tau(\Phi(|U(x)|)),$$

and put

$$m^{\varphi,\tau}_{\Phi,\alpha} = \left\{ x \in M : O^{\varphi,\tau}_{\Phi,\alpha}(x) < \infty \right\}.$$

Consider on the set $m^{\varphi,\tau}_{\Phi,\alpha}$ the functional

$$\|x\|^{\varphi,\tau}_{\Phi,\alpha} = \inf\left\{ \lambda > 0 : O^{\varphi,\tau}_{\Phi,\alpha}\left(\frac{x}{\lambda}\right) \leq 1 \right\}.$$

Theorem 2. *If the N-function Φ satisfies the (δ_2, Δ_2)-condition, then the set $m^{\varphi,\tau}_{\Phi,\alpha}$ is a linear subspace in M.*

In order to prove this theorem we need the following inequality.

Lemma 1. *For the N-function Φ and real number $\lambda \in [0,1]$ the following inequality is valid*

$$\tau(\Phi(|U(\lambda x)|)) \leq \lambda\tau(\Phi(|U(x)|)) \tag{3}$$

for all $x \in m^{\varphi,\tau}_{\Phi,\alpha}$.

Proof: By the linearity of the map U we have

$$\tau(\Phi(|U(\lambda x)|)) = \tau(\Phi(\lambda|U(x)|)).$$

From the inequality (ii) in Proposition 1 with $y = 0$, we obtain

$$\tau(\Phi(|U(\lambda x)|)) \leq \lambda\tau(\Phi(|U(x)|)).$$ □

Proof of Theorem 2: The inequality (3) above implies that $m^{\varphi,\tau}_{\Phi,\alpha}$ is closed under the multiplication by complex number γ with $|\gamma| \leq 1$. Let us show that for $x \in m^{\varphi,\tau}_{\Phi,\alpha}$ and any complex number γ with $|\gamma| > 1$ we have that $\gamma x \in m^{\varphi,\tau}_{\Phi,\alpha}$, i.e. $\tau(\Phi(|U(\gamma x)|)) < \infty$.

Since Φ satisfies the (δ_2, Δ_2)-condition, given any positive number $|\gamma|$ there exists a positive number $r(|\gamma|)$ such that $\Phi(|\gamma|t) \leq r(|\gamma|)\Phi(t)$ for all $t \geq 0$. Therefore (δ_2, Δ_2)-condition implies that

$$\tau(\Phi(|U(\gamma x)|)) = \tau(\Phi(|\gamma||U(x)|)) \leq r(|\gamma|)\tau(\Phi(|U(x)|)) < \infty,$$

i.e. the set $m^{\varphi,\tau}_{\Phi,\alpha}$ is closed under multiplication by any complex number.

Now let us prove that the sum of any two operators from $m^{\varphi,\tau}_{\Phi,\alpha}$ also belongs to $m^{\varphi,\tau}_{\Phi,\alpha}$. Let $x, y \in m^{\varphi,\tau}_{\Phi,\alpha}$, i.e. $\tau(\Phi(|U(x)|)) < \infty$ and $\tau(\Phi(|U(y)|)) < \infty$. The inequalities (i) and (ii) from Proposition 3, the linearity of the operator U, the convexity of Φ, the tracial property of τ and the fact that $m^{\varphi,\tau}_{\Phi,\alpha}$ is closed under the multiplication by complex numbers imply:

$$\begin{aligned}
\tau(\Phi(|U(x+y)|)) &= \tau(\Phi(|U(x)+U(y)|)) \leq \tau(\Phi(u^*|U(x)|u + v^*|U(y)|v)) \\
&\leq \tau\left(\frac{1}{2}(\Phi(2u^*|U(x)|u) + \Phi(2v^*|U(y)|v))\right) \\
&= \frac{1}{2}(\tau\left(\Phi(u^*2|U(x)|u)\right) + \tau(\Phi(v^*2|U(y)|v)) \\
&\leq \frac{1}{2}\left(\tau(u^*\Phi(2|U(x)|)u) + \tau(v^*\Phi(2|U(y)|)v)\right) \\
&\leq \frac{1}{2}(\tau(\Phi(2|U(x)|)) + \tau(\Phi(2|U(y)|)))
\end{aligned}$$

$$= \frac{1}{2}(\tau(\Phi|U(2x)|) + \tau(\Phi|U(2y)|)) < \infty,$$

i.e. $x + y \in m_{\Phi,\alpha}^{\varphi,\tau}$. □

Remark 1. If M is a non-atomic commutative von Neumann algebra and $\varphi(\mathbf{1}) < \infty$ then the condition of linearity of the set $m_{\Phi,\alpha}^{\varphi,\tau}$ implies the (δ_2, Δ_2)-condition for Φ ([4, Chapter II, §8]). Therefore, if there exists a projection $p \in P(M)$ with $\varphi(p) < +\infty$ such that the von Neumann algebra pMp has a non-atomic commutative von Neumann subalgebra, then the set $m_{\Phi,\alpha}^{\varphi,\tau}$ is a linear space if and only if the N-function Φ satisfies the (δ_2, Δ_2)-condition.

Theorem 3. *The set*

$$K_{\Phi,\alpha}^{\varphi,\tau} = \{x \in M : O_{\Phi,\alpha}^{\varphi,\tau}(x) \leq 1\}$$

is absolutely convex and absorbing in $m_{\Phi,\alpha}^{\varphi,\tau}$.

Proof: Let us prove the convexity of $K_{\Phi,\alpha}^{\varphi,\tau}$. Let $x, y \in K_{\Phi,\alpha}^{\varphi,\tau}$ and $\lambda \in [0, 1]$. In view of Proposition 3(i) there exist partial isometries u and v in M such that

$$|\lambda U(x) + (1 - \lambda)U(y)| \leq \lambda u^*|U(x)|u + (1 - \lambda)v^*|U(y)|v.$$

From the inequalities of Propositions 1 and 3 and from the tracial property of τ we obtain

$$\begin{aligned}\tau(\Phi|\lambda U(x) + (1 - \lambda)U(y)|) &\leq \lambda\tau(\Phi(u^*|U(x)|u)) + (1 - \lambda)\tau(\Phi(v^*|U(y)|v)) \\ &\leq \lambda\tau(u^*\Phi(|U(x)|u)) + (1 - \lambda)\tau(v^*\Phi(|U(y)|)v) \\ &\leq \lambda\tau(\Phi(|U(x)|)) + (1 - \lambda)\tau(\Phi(|U(y)|)),\end{aligned}$$

i.e.

$$O_{\Phi,\alpha}^{\varphi,\tau}(\lambda x + (1 - \lambda)y) \leq \lambda O_{\Phi,\alpha}^{\varphi,\tau}(x) + (1 - \lambda)O_{\Phi,\alpha}^{\varphi,\tau}(y),$$

which implies the convexity of $K_{\Phi,\alpha}^{\varphi,\tau}$.

The inequality (3) shows that the set $K_{\Phi,\alpha}^{\varphi,\tau}$ is balanced, and hence is absolutely convex.

Finally let us prove that $K_{\Phi,\alpha}^{\varphi,\tau}$ is absorbing in $m_{\Phi,\alpha}^{\varphi,\tau}$. If $x \in m_{\Phi,\alpha}^{\varphi,\tau}$, then there exists $t > 1$ such that $O_{\Phi,\alpha}^{\varphi,\tau}(x) < t$. Let γ be a complex number and $|\gamma| \geq t$. By Lemma 1 we have that

$$O_{\Phi,\alpha}^{\varphi,\tau}(\frac{x}{\gamma}) \leq \frac{1}{|\gamma|}O_{\Phi,\alpha}^{\varphi,\tau}(x) \leq \frac{1}{t}O_{\Phi,\alpha}^{\varphi,\tau}(x) < 1,$$

i.e. $\frac{x}{\gamma} \in K_{\Phi,\alpha}^{\varphi,\tau}(x)$. □

Corollary 1. *The Minkowski functional of the set* $K_{\Phi,\alpha}^{\varphi,\tau}$ *defined as*

$$\|x\|_{\Phi,\alpha}^{\varphi,\tau} = \inf\left\{\lambda > 0 : \frac{x}{\lambda} \in K_{\Phi,\alpha}^{\varphi,\tau}\right\}, \tag{4}$$

is a norm on the linear space $m^{\varphi,\tau}_{\Phi,\alpha}$.

PROOF: It is sufficient to prove that $\|x\|^{\varphi,\tau}_{\Phi,\alpha} = 0$ implies that $x = 0$. Indeed, if $\|x\|^{\varphi,\tau}_{\Phi,\alpha} = 0$ then $O^{\varphi,\tau}_{\Phi,\alpha}(\frac{x}{\lambda}) \leq 1$ for all $\lambda \in (0,1)$. By Lemma 1 we obtain that $\frac{1}{\lambda}O^{\varphi,\tau}_{\Phi,\alpha}(x) \leq O^{\varphi,\tau}_{\Phi,\alpha}(\frac{x}{\lambda}) \leq 1$ for all $\lambda \in (0,1)$, i.e. $O^{\varphi,\tau}_{\Phi,\alpha}(x) = 0$. Faithfulness of τ then implies that $\Phi^{-1}(h)^{\alpha}x\Phi^{-1}(h)^{1-\alpha} = 0$. Since $h \in LS_+(M)$ (see Theorem 1(i)) and h is a non singular operator, we have that $\Phi^{-1}(h)^{\alpha}$, $\Phi^{-1}(h)^{1-\alpha} \in LS_+(M)$ and $\Phi^{-1}(h)^{\alpha}$, $\Phi^{-1}(h)^{1-\alpha}$ are non singular operators too. Let $\{e_\lambda\}_{\lambda\in\mathbf{R}}$ be the spectral family of projectors for $\Phi^{-1}(h)$, i.e. $\Phi^{-1}(h) = \int_0^\infty \lambda\, de_\lambda$.

Put

$$x_n = \int_{\frac{1}{n}}^{n} \left(\frac{1}{\lambda}\right)^{\alpha} \lambda\, de_\lambda,\ y_n = \int_{\frac{1}{n}}^{n} \left(\frac{1}{\lambda}\right)^{1-\alpha} \lambda\, de_\lambda.$$

It is clear that $x_n, y_n \in M$ for all n. Using

$$x_n\Phi^{-1}(h)^{\alpha} = e_n - e_{\frac{1}{n}} = \Phi^{-1}(h)^{1-\alpha}y_n$$

we see that

$$\left(e_n - e_{\frac{1}{n}}\right)x\left(e_n - e_{\frac{1}{n}}\right) = 0.$$

Since $\Phi^{-1}(h)$ is a non singular operator, it follows that $(e_n - e_{\frac{1}{n}}) \uparrow \mathbf{1}$, when $n \to \infty$. Consequently, $x = 0$. $\square$

Denote by $L_{\Phi,\alpha}(M,\varphi,\tau)$ the Banach space obtained as the completion of $m^{\varphi,\tau}_{\Phi,\alpha}$ in the norm $\|\cdot\|^{\varphi,\tau}_{\Phi,\alpha}$ and call this completion *the Orlicz space* constructed by the N-function Φ on the von Neumann algebra M with respect to the faithful normal locally finite weight φ. It is clear that if φ is a trace or M is a commutative von Neumann algebra, then the norm $\|\cdot\|^{\varphi,\tau}_{\Phi,\alpha}$ and the space $L_{\Phi,\alpha}(M,\varphi,\tau)$ do not depend on $\alpha \in [0,1]$.

Note also that in the case where $\Phi(t) = \frac{1}{t}|t|^p$, $p > 1$, the norm $\|\cdot\|^{\varphi,\tau}_{\Phi,\alpha}$ and the space $L_{\Phi,\alpha}(M,\varphi,\tau)$ do not depend on the choice of the faithful normal semi-finite trace τ and of $\alpha \in [0,1]$ [11].

For general N-functions Φ this is not true even in the commutative case.

Example 1. Take $M = l_\infty$, $f_i = \{0,\dots,0,1,0\dots\}$, where 1 is on the i-th position, and put $\Phi(t) = |t|^{\beta}(\ln|t| + 1)$, $t \neq 0$, $\beta > 1$, $\Phi(0) = 0$. In [4, Chapter I, §4] it is proved that Φ is an N-function satisfying the (δ_2, Δ_2)-condition. Consider the trace ν on l_∞ defined as $\nu(f_i) = \frac{1}{i^2}((e^{i^2})^{2\beta}(2i^2+1))^{-1}$. Put

$$h = \{e^{\beta i^2}(i^2+1)\}_{i=1}^{\infty},\ f = \Phi^{-1}(h) = \{e^{i^2}\}_{i=1}^{\infty}.$$

Now define the trace μ on l_∞ as $\mu(\cdot) = \nu(h\cdot)$.

Let us show that in this case the norms $\|\cdot\|^{\mu,\nu}_{\Phi,1}$ and $\|\cdot\|^{\mu,\mu}_{\Phi,1}$ are not equivalent on the ideal E of all finite sequences from l_∞ (it is clear that $E \subset m^{\mu,\nu}_{\Phi,\alpha}$ and $E \subset m^{\mu,\mu}_{\Phi,\alpha}$). For this it is sufficient to find a sequence $\{x_n\}$ of elements from

$(K^{\mu,\nu}_{\Phi,1}) \cap E$ such that $\{x_n\} \not\subset \lambda K^{\mu,\mu}_{\Phi,1}$ for all $\lambda > 0$. Let $x_n = \sum_{i=2}^n e^{i^2} f_i$. It is clear that for commutative algebra $M = l_\infty$ one has

$$O^{\mu,\nu}_{\Phi,1}(x) = \nu(\Phi(|\Phi^{-1}(h)x|))$$

and

$$O^{\mu,\mu}_{\Phi,1}(x) = \mu(\Phi(|x|)). \tag{5}$$

Therefore

$$O^{\mu,\nu}_{\Phi,1}(x_n f_i) = \nu(\Phi(f x_n f_i)) = \nu(\Phi((e^{2i^2} f_i)^2)) = (e^{i^2})^{2\beta}(2i^2+1)\nu(f_i) = \frac{1}{i^2}$$

and

$$O^{\mu,\nu}_{\Phi,1}(x_n) = \nu(\Phi(f \sum_{i=2}^n e^{2i^2} f_i)) = \nu(\sum_{i=2}^n (e^{i^2})^{2\beta}(2i^2+1) f_i) = \sum_{i=2}^n \frac{1}{i^2} < 1, \tag{6}$$

i.e. $x_n \in K^{\mu,\nu}_{\Phi,1}$ for all n. Let us show that $\{x_n\} \not\subset \lambda K^{\mu,\mu}_{\Phi,1}$ for all positive real λ. From (5) we have

$$\begin{aligned} O^{\mu,\mu}_{\Phi,1}(x_n f_i) &= \mu(\Phi(x_n f_i)) = \nu(h\Phi(x_n) f_i) \\ &= (e^{i^2})^{3\beta}(2i^4 + 3i^2 + 1)\nu(f_i) > (e^{i^2})^{2\beta} i(2i^2+1)\nu(f_i) = \frac{1}{i}\,. \end{aligned}$$

Therefore $O^{\mu,\mu}_{\Phi,1}(x_n) > \sum_{i=2}^n \frac{1}{n}$, and hence

$$\{x_n\} \not\subset \lambda K^{\mu,\mu}_{\Phi,1} \tag{7}$$

for all positive λ. From (6) and (7) it follows that the norms $\|\cdot\|^{\mu,\nu}_{\Phi,1}$ and $\|\cdot\|^{\mu,\mu}_{\Phi,1}$ are not equivalent on E. In particular the identity mapping from E into E cannot be extended to an isomorphism between $L_{\Phi,\alpha}(l_\infty, \mu, \nu)$ and $L_{\Phi,\alpha}(l_\infty, \mu, \mu)$.

At the same time by following theorem the Orlicz spaces $(L_{\Phi,\alpha}(M, \varphi, \tau), \|\cdot\|^{\varphi,\tau}_{\Phi,\alpha})$ and $(L_\Phi(M,\tau), \|\cdot\|_\Phi)$ are isometrically isomorphic.

Theorem 4. *Let the N-function Φ satisfy the (δ_2, Δ_2)-condition, $\alpha \in [0,1]$. Then the Banach space $L_{\Phi,\alpha}(M,\varphi,\tau)$ is isometrically isomorphic to the Banach space $L_\Phi(M,\tau) = L_{\Phi,1}(M,\tau,\tau)$.*

Proof: For every $x \in m^{\varphi,\tau}_{\Phi,\alpha}$ we have

$$U(x) = (\Phi^{-1}(h))^\alpha x (\Phi^{-1}(h))^{1-\alpha} \in L_\Phi(M,\tau).$$

Therefore from definitions (1) and (4) of the norms we obtain

$$\|x\|^{\varphi,\tau}_{\Phi,\alpha} = \|(\Phi^{-1}(h))^\alpha x (\Phi^{-1}(h))^{1-\alpha}\|_\Phi.$$

This means that the map U defined as

$$m_{\Phi,\alpha}^{\varphi,\tau} \ni x \xrightarrow{U} (\Phi^{-1}(h))^{\alpha} x (\Phi^{-1}(h))^{1-\alpha} \in L_\Phi(M,\tau) \tag{8}$$

is a linear isometry. Let us show that the $U(m_{\Phi,\alpha}^{\varphi,\tau}) = (\Phi^{-1}(h))^{\alpha} m_{\Phi,\alpha}^{\varphi,\tau} (\Phi^{-1}(h))^{1-\alpha}$ is dense in $L_\Phi(M,\tau)$.

Let $h = \int_0^\infty \lambda\, de_\lambda(h)$ and $q_n = \int_{\frac{1}{n}}^n de_\lambda(h)$, $(n = 1, 2, \dots)$. Consider the set

$$\mathcal{F} = \bigcup_{m,n=1}^{\infty} q_m m_\tau q_n.$$

Since $q_n \le q_{n+1}$, it follows that $\mathcal{F}$ is a linear subspace in m_τ and by (2) $\mathcal{F} \subset L_\Phi(M,\tau)$.

Firstly, let us prove that $\mathcal{F}$ is dense in $L_\Phi(M,\tau)$. From the (δ_2, Δ_2)-condition it follows that for $y \in L_\Psi(M,\tau)$ (where Ψ is the complementary N-function for Φ) the functional $f(x) = \tau(xy)$, $x \in L_\Phi(M,\tau)$, defines the general form of continuous linear functional on $L_\Phi(M,\tau)$.

Let $y \in L_\Psi(M,\tau)$ and suppose that $f(q_m x q_n) = \tau((q_m x q_n)y) = 0$ for all $x \in m_\tau$ and $m, n = 1, 2, \dots$. In order to prove that $\mathcal{F}$ is dense in $L_\Phi(M,\tau)$ it is sufficient to show that $y = 0$.

From the tracial property of τ we have that $\tau(x q_n y q_m) = 0$ for all $x \in m_\tau$. By Proposition 2 m_τ is dense in $L_\Phi(M,\tau)$ and hence $q_n y q_m = 0$ for all $m, n = 1, 2, \dots$. Since $q_n \nearrow \mathbf{1}$ as $n \to \infty$, this implies that $y = 0$. Therefore $\mathcal{F}$ is dense $L_\Phi(M,\tau)$.

Now let us show that $\mathcal{F} \subset U(m_{\Phi,\alpha}^{\varphi,\tau})$. For this it is sufficient to prove that given any $x \in m_\tau$ and $m, n = 1, 2, \dots$, there exists $y \in m_{\Phi,\alpha}^{\varphi,\tau}$ such that $q_m x q_n = U(y)$.

Since the operators $(\Phi^{-1}(h))^{-\alpha} q_m$ and $(\Phi^{-1}(h))^{\alpha-1} q_n$ belong to M, the operator $y = U^{-1}(q_m x q_n) = (\Phi^{-1}(h))^{-\alpha}(q_m x q_n)(\Phi^{-1}(h))^{\alpha-1}$ also belongs to M. From (2) and from $\tau(|q_m x q_n|) < \infty$ we obtain that $\tau(\Phi(|U(y)|)) = \tau(\Phi(|q_m x q_n|)) < \infty$, i.e. $y \in m_{\Phi,\alpha}^{\varphi,\tau}$. This implies that $\mathcal{F} \subset U(m_{\Phi,\alpha}^{\varphi,\tau})$.

Now since $m_{\Phi,\alpha}^{\varphi,\tau}$ is dense in $(L_{\Phi,\alpha}(M,\varphi,\tau), \|\cdot\|_{\Phi,\alpha}^{\varphi,\tau})$ and $U(m_{\Phi,\alpha}^{\varphi,\tau})$ is dense in $(L_\Phi(M,\tau), \|\cdot\|_\Phi)$ the isometry $U : m_{\Phi,\alpha}^{\varphi,\tau} \to L_\Phi(M,\tau)$ defined in (8) can be uniquely extended to an isometric isomorphism between $L_{\Phi,\alpha}(M,\varphi,\tau)$ and $L_\Phi(M,\tau)$. $\square$

Since every faithful normal semi-finite trace τ_1 on M is a locally finite weight [12], Theorem 4 implies the following

Corollary 2. *If τ_1 and τ_2 are faithful normal semi-finite traces on a von Neumann algebra M, Φ is an N-function satisfying the (δ_2, Δ_2)-condition, then the Orlicz spaces $L_\Phi(M,\tau_1)$ and $L_\Phi(M,\tau_2)$ are isometrically isomorphic.*

Theorem 4 and Corollary 2 together imply the following theorem

Theorem 5. *Let τ_1 and τ_2 be faithful normal semi-finite traces on a von Neumann algebra M, and let φ_1, φ_2 be faithful normal locally finite weights on M. Suppose* *that Φ is an N-function satisfying the (δ_2, Δ_2)-condition, $\alpha, \beta \in [0,1]$. Then the Orlicz spaces $L_{\Phi,\alpha}(M, \varphi_1, \tau_1)$ and $L_{\Phi,\beta}(M, \varphi_2, \tau_2)$ are isometrically isomorphic.*

Theorem 4 implies also the following

Corollary 3. *Let Φ be an N-function satisfying the (δ_2, Δ_2)-condition and let Ψ be the complementary N-function for Φ, and $\alpha, \beta \in [0,1]$. Then the dual space $(L_{\Phi,\alpha}(M, \varphi, \tau))^*$ for the Orlicz space $L_{\Phi,\alpha}(M, \varphi, \tau)$ is isometrically isomorphic to the space $L_\Psi(M, \tau)$. If moreover Ψ also satisfies the (δ_2, Δ_2)-condition then $(L_{\Phi,\alpha}(M, \varphi, \tau))^*$ is isometrically isomorphic to $L_{\Psi,\beta}(M, \varphi, \tau)$ and the Banach space $L_{\Phi,\alpha}(M, \varphi, \tau)$ is reflexive.*

Now let us give a representation of the space $L_{\Phi,\alpha}(M, \varphi, \tau)$ by locally measurable operators in the case where φ is a regular locally finite weight, and the N-function Φ satisfies (δ_2, Δ_2)-condition.

Consider the following subset in the algebra $LS(M)$ of locally measurable operators affiliated with the von Neumann algebra M:

$$\mathcal{L}_{\Phi,\alpha}(M, \varphi, \tau) = \{x \in LS(M) : O^{\varphi,\tau}_{\Phi,\alpha}(x) < \infty\},$$

and for each $x \in \mathcal{L}_{\Phi,\alpha}(M, \varphi, \tau)$ put

$$\|x\|^{\varphi,\tau}_{\Phi,\alpha} = \inf\left\{\lambda > 0 : O^{\varphi,\tau}_{\Phi,\alpha}\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

It is clear that

$$m^{\varphi,\tau}_{\Phi,\alpha} = M \bigcap \mathcal{L}_{\Phi,\alpha}(M, \varphi, \tau).$$

Repeating the proof of the Theorems 2 and 3 and of Corollary 1 we obtain that $\mathcal{L}_{\Phi,\alpha}(M, \varphi, \tau)$ is a linear subspace of $LS(M)$ and that $\|\cdot\|^{\varphi,\tau}_{\Phi,\alpha}$ is a norm on $\mathcal{L}_{\Phi,\alpha}(M, \varphi, \tau)$.

Theorem 6. *Let φ be a regular locally finite normal weight on M and suppose that Φ is an N-function satisfying the (δ_2, Δ_2)-condition and $\alpha \in [0,1]$. Then $(\mathcal{L}_{\Phi,\alpha}(M, \varphi, \tau), \|\cdot\|^{\varphi,\tau}_{\Phi,\alpha})$ is a Banach space and $m^{\varphi,\tau}_{\Phi,\alpha}$ is dense in $(\mathcal{L}_{\Phi,\alpha}(M, \varphi, \tau), \|\cdot\|^{\varphi,\tau}_{\Phi,\alpha})$.*

To prove Theorem 6 we need the following criterion for the local measurability of a closed operator affiliated with a von Neumann algebra M (see [8, §2.7, Proposition 2.3.4]).

Lemma 2. *Let x be a closed linear operator affiliated with a von Neumann algebra M and let $\{e_\lambda(|x|)\}_{\lambda\in\mathbf{R}}$ be the spectral family of projections for the operator $|x| = (x^*x)^{1/2}$. Then $x \in LS(M)$ if and only if for any sequence of positive numbers $\lambda_n \uparrow \infty$ there exists an increasing sequence $\{z_n\}_{n=1}^\infty$ of the central projections in M, such that $\sup_{n\geq 1} z_n = \mathbf{1}$ and $z_n(\mathbf{1} - e_{\lambda_n}(|x|))$ are finite projections for all $n = 1, 2, \ldots$.*

Proof of Theorem 6: Let h be the Radon-Nikodym derivate of the weight φ with respect to the trace τ. From Theorem 1 it follows that the operators h and h^{-1} are locally measurable.

Since the N-function Φ is strictly increasing on $[0,\infty), \Phi([0,\infty)) = [0,\infty)$, and the support of the operator h is equal to $\mathbf{1}$ and $\Phi^{-1}(h) \in LS_+(M)$, then for the spectral family of projections $\{e_\lambda(h)\}_{\lambda\in\mathbf{R}}$ and $\{e_\lambda(\Phi^{-1}(h))\}_{\lambda\in\mathbf{R}}$ the following equalities are valid:

$$\begin{aligned}\mathbf{1} &= \sup_{n\geq 1} e_n(h)(\mathbf{1} - e_{\frac{1}{n}}(h)) = \sup_{n\geq 1}\{h \leq n\}\{h > \frac{1}{n}\}\\ &= \sup_{n\geq 1}\{\Phi^{-1}(h) \leq \Phi^{-1}(n)\}\{\Phi^{-1}(h) > \Phi^{-1}(\frac{1}{n})\}\\ &= \sup_{n\geq 1} e_{\Phi^{-1}(n)}(\Phi^{-1}(h))(\mathbf{1} - e_{\Phi^{-1}(\frac{1}{n})}(\Phi^{-1}(h))).\end{aligned}$$

Therefore, there exists the inverse operator $a = (\Phi^{-1}(h))^{-1}$ with dense domain $D(a)$, moreover this operator is a closed self-adjoint positive and affiliated with the von Neumann algebra M.

Let us show that $a \in LS(M)$. Since $h^{-1} \in LS_+(M)$, by Lemma 2, there exists a sequence of central projections $\{z_n\}_{n=1}^{\infty}$ in M such that $z_n \uparrow \mathbf{1}$ and $z_n(\mathbf{1} - e_{\lambda_n}(h^{-1}))$ are finite projections for all $n = 1, 2, \ldots$, where $\lambda_n = \frac{1}{\Phi(\frac{1}{n})} \uparrow \infty$. Since

$$\begin{aligned}\mathbf{1} - e_n(a) &= \{a > n\} = \{(\Phi^{-1}(h))^{-1} > n\} = \{\Phi^{-1}(h)) < \frac{1}{n}\}\\ &= \{h < \Phi(\frac{1}{n})\} = \{h^{-1} > \frac{1}{\Phi(\frac{1}{n})}\} = \mathbf{1} - e_{\lambda_n}(h^{-1}),\end{aligned}$$

Lemma 2 implies that $a \in LS_+(M)$.

Consider the linear map U from $\mathcal{L}_{\Phi,\alpha}(M, \varphi, \tau)$ into $L_\Phi(M, \tau)$ defined by (8), i.e.

$$U(x) = (\Phi^{-1}(h))^\alpha x(\Phi^{-1}(h))^{1-\alpha}, x \in \mathcal{L}_{\Phi,\alpha}(M, \varphi, \tau).$$

As in the proof of Theorem 4, we see that U is an isometry. Now we show that U is a surjection. Let $y \in L_\Phi(M, \tau)$ and

$$x = a^\alpha y a^{1-\alpha} = ((\Phi^{-1}(h))^{-1})^\alpha y((\Phi^{-1}(h))^{-1})^{1-\alpha}.$$

It is clear that $x \in LS(M)$ and

$$O^{\varphi,\tau}_{\Phi,\alpha}(x) = \tau(\Phi(|(\Phi^{-1}(h))^\alpha x(\Phi^{-1}(h))^{1-\alpha}|) = \tau(\Phi(|y|)) < \infty,$$

i.e. $x \in \mathcal{L}_{\Phi,\alpha}(M, \varphi, \tau)$ and $U(x) = y$.

Thus U is a linear isometry from $(\mathcal{L}_{\Phi,\alpha}(M,\varphi,\tau), \|\cdot\|^{\varphi,\tau}_{\Phi,\alpha})$ onto $(L_\Phi(M,\tau), \|\cdot\|_\Phi)$. Therefore, the normed space $(\mathcal{L}_{\Phi,\alpha}(M,\varphi,\tau), \|\cdot\|^{\varphi,\tau}_{\Phi,\alpha})$ is isometrically isomorphic to the Banach space $(L_{\Phi,\alpha}(M,\varphi,\tau), \|\cdot\|^{\varphi,\tau}_{\Phi,\alpha})$, in addition, $m^{\varphi,\tau}_{\Phi,\alpha}$ is dense in $(\mathcal{L}_{\Phi,\alpha}(M,\varphi,\tau), \|\cdot\|^{\varphi,\tau}_{\Phi,\alpha})$. $\square$

Theorem 6 implies that in the case where h and h^{-1} are locally measurable operators and the N-function Φ satisfies (δ_2, Δ_2)-condition, the Orlicz space $L_{\Phi,\alpha}(M,\varphi,\tau)$ can be described by locally measurable operators in the following form

$$L_{\Phi,\alpha}(M,\varphi,\tau) = \mathcal{L}_{\Phi,\alpha}(M,\varphi,\tau) = (\Phi^{-1}(h))^{-\alpha} L_\Phi(M,\tau)(\Phi^{-1}(h))^{\alpha-1} \subset LS(M).$$

In the case of N-function $\Phi(t) = \frac{1}{p}|t|^p$, $p > 1$, the assertions of Theorems 4 and 6 were proved in [13].

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