On some convexity properties in the Besicovitch-Musielak-Orlicz space of almost periodic functions with Luxemburg norm

Fazia Bedouhene, Amina Daoui, Hocine Kourat

Abstract. In this article, it is shown that geometrical properties such as local uniform convexity, mid point local uniform convexity, H-property and uniform convexity in every direction are equivalent in the Besicovitch-Musielak-Orlicz space of almost periodic functions ($\widetilde{B}^{\varphi}a.p.$) endowed with the Luxemburg norm.

Keywords: local uniform convexity, uniform convexity in every direction, mid point locally uniform, H-property, strict convexity, approximation, Besicovitch-Musielak-Orlicz space, almost periodic function

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1. Introduction and preliminaries

This article is a continuation of the investigations concerning the geometrical properties in the space of Besicovitch-Orlicz of almost periodic functions (see [1]). Here we are interested in such properties as local uniform convexity, Kadec-Klee property, mid point local uniform convexity and uniform convexity in every direction in the widest class of Besicovitch-Musielak-Orlicz space of almost periodic functions $\widetilde{B}^{\varphi}a.p.$. We are finding criteria for these properties. An approximation property in $\widetilde{B}^{\varphi}a.p.$ is also presented.

Now, we recall the needed definitions and notations.

We say that a Banach space $(X, \|\cdot\|)$ is locally uniformly convex LUC (see [10]) if for each $\varepsilon > 0$ and each $y \in S(X)$ there is a $\delta_X(\varepsilon, y) > 0$ such that if $x \in S(X)$ and $\|x - y\| \geq \varepsilon$, then $\|\frac{1}{2}(x + y)\| \leq 1 - \delta_X(\varepsilon, y)$, where as usual, the notations $S(X)$ and $B(X)$ are used for the unit sphere and unit ball of X respectively.

There are also sequential characterizations of LUC (see [10]): the space $(X, \|\cdot\|)$ is LUC if and only if for each $x \in S(X)$ and every sequence (y_n) in $S(X)$ (or $B(X)$) for which $\|\frac{1}{2}(x + y_n)\| \to 1$, we have $\|y_n - x\| \to 0$.

Let $x \in S(X)$. If $x_n \in X$, $x_n \to x$ weakly ($x_n \stackrel{w}{\to} x$) and $\|x_n\| \to \|x\| = 1$ imply $x_n \to x$ in norm, then we call x an H-point of $B(X)$. If every point in $S(X)$ is an H-point of $B(X)$, then we say that X has the H-property (or satisfy the Kadec-Klee property also called the Radon Riesz property) (see [5]).

The space X is called mid point locally uniformly convex (in short MLUC) when every point $x \in S(X)$ is strongly extreme, i.e., for each sequence (x_n) in X, the conditions $\|x + x_n\| \to 1$ and $\|x - x_n\| \to 1$ implies $\|x_n\| \to 0$.

Now we present the class of Banach spaces introduced by A.G. Garkari, the so-called uniformly convex in every direction (see [4], [15]). We mention that these spaces (among others) are important in approximation theory since they are exactly those Banach spaces in which every bounded set has at most one Cebyshev center. If K is a subset of Banach space X then the Cebyshev centers of K are the elements c in K with the property that

$$\sup_{k\in K} \|c - k\| = \inf_{s\in X} \sup_{t\in K} \|s - t\| .$$

The Banach space X is said to be uniformly convex in every direction (in short UCED) if the following property holds: for every nonzero z in X and $\varepsilon > 0$ there exists $\delta(z, \varepsilon) > 0$ such that $|a| < \varepsilon$ if $\|x\| = \|y\| = 1$, $x - y = az$, and $\|x + y\| = 2[1 - \delta(z, \varepsilon)]$. We mention the following characterization of UCED Banach spaces in terms of sequences: for any $z \in X$, and every sequence (x_n) in X, the conditions $\|x_n\| \to 1$, $\|x_n + z\| \to 1$ and $\|2x_n + z\| \to 2$ imply $z = 0$.

Let us note that the implications LUC $\Rightarrow$ MLUC $\Rightarrow$ SC (strict convexity), LUC $\Rightarrow$ H-property and UCED $\Rightarrow$ SC hold in general Banach spaces (see e.g. [10]).

In the case of Musielak-Orlicz spaces, these geometrical properties are well characterized in [11], [7].

The most important geometrical properties of the space $\widetilde{B}^{\varphi}a.p.$ with respect to the Luxemburg norm are characterized in [8] and [9]. The authors have obtained the following results (see Theorem 3.1 of [9] and Theorem 1 of [8] respectively):

Theorem 1. *The space $\widetilde{B}^{\varphi}a.p.$ endowed with the Luxemburg norm is uniformly convex if and only if φ is uniformly convex and satisfies the $\Delta_2^{B^1}$-condition.*

Theorem 2. *The space $\widetilde{B}^{\varphi}a.p.$ endowed with the Luxemburg norm is strictly convex if and only if φ is strictly convex and satisfies the $\Delta_2^{B^1}$-condition.*

Now, we introduce some notions joined with Besicovitch-Musielak-Orlicz spaces of almost periodic functions. In what follows, let us denote by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ the natural, real and complex numbers respectively.

Let $\varphi : \mathbb{R} \times [0, +\infty[\longrightarrow [0, +\infty[$ be a continuous function on $\mathbb{R} \times [0, +\infty[$ satisfying:

(1) $\forall t \in \mathbb{R}$, $\varphi(t, u) = 0$ iff $u = 0$,
(2) $\forall t \in \mathbb{R}$, $\varphi(t, u)$ is convex with respect to $u \in [0, +\infty[$,
(3) $\forall u \in [0, +\infty[$, $\varphi(t, u)$ is periodic with respect to $t \in \mathbb{R}$, the period τ being fixed and independent of $u \in [0, +\infty[$. Without loss of generality we may suppose that $\tau = 1$.

As a consequence of these assumptions, we get that the function $\phi(\alpha) = \inf_{t\in\mathbb{R}}\{\varphi(t, \alpha)\}$ is strictly positive and convex. This fact will be very useful in our computations.

We denote by $L^\varphi_{\mathrm{loc}}(\mathbb{R})$ the subspace of φ-locally integrable functions, i.e. the subspace of all Lebesgue measurable functions on $\mathbb{R}$ such that for each compact $K \subset \mathbb{R}$, there exists $\lambda_K > 0$ for which $\int_K \varphi(t, \lambda_K |f(t)|)\, dt < +\infty$. The functional

$$\begin{array}{rcl} \rho_{B^\varphi} : L^\varphi_{\mathrm{loc}}(\mathbb{R}) & \longrightarrow & [0, +\infty] \\ f & \longrightarrow & \rho_{B^\varphi}(f) = \limsup\limits_{T \longrightarrow +\infty} \dfrac{1}{2T} \displaystyle\int_{-T}^{+T} \varphi(t, |f(t)|)\, dt, \end{array} \tag{1.1}$$

is a convex pseudomodular (see [12]).

We define the Besicovitch-Musielak-Orlicz space associated to this pseudomodular by

$$\begin{aligned} B^\varphi(\mathbb{R}) &= \{f \in L^\varphi_{\mathrm{loc}}(\mathbb{R}) : \lim_{\alpha \to 0} \rho_{B^\varphi}(\alpha f) = 0\}, \\ &= \{f \in L^\varphi_{\mathrm{loc}}(\mathbb{R}) : \rho_{B^\varphi}(\alpha f) < 0, \quad \text{for some} \ \ \alpha > 0\}. \end{aligned}$$

The space $B^\varphi(\mathbb{R})$ is naturally endowed with the Luxemburg (pseudo)norm

$$\|f\|_{B^\varphi} = \inf\{k > 0 : \rho_{B^\varphi}(\frac{f}{k}) \le 1\}, \qquad f \in B^\varphi(\mathbb{R}).$$

Under the Luxemburg norm, $B^\varphi(\mathbb{R})$ is a Banach space.

Let $\mathcal{A}$ be the set of all generalized trigonometric polynomials, i.e.,

$$\mathcal{A} = \{P_n(t) = \sum_{j=1}^{n} a_j e^{i\lambda_j t},\ a_j \in \mathbb{C},\ \lambda_j \in \mathbb{R},\ n \in \mathbb{N}\}.$$

The Besicovitch-Musielak-Orlicz space of almost periodic functions, denoted $B^\varphi a.p.$, is the closure of the set $\mathcal{A}$ in $B^\varphi(\mathbb{R})$ with respect to the (pseudo)norm $\|\cdot\|_{B^\varphi}$:

$$\begin{aligned} B^\varphi a.p. &= \{f \in B^\varphi(\mathbb{R}) : \exists f_n \in \mathcal{A}, \forall k > 0, \lim_{n \to +\infty} \rho_{B^\varphi}(k(f_n - f)) = 0\}, \\ &= \{f \in B^\varphi(\mathbb{R}) : \exists f_n \in \mathcal{A}, \lim_{n \to +\infty} \|f_n - f\|_{B^\varphi} = 0\}. \end{aligned}$$

We shall also be concerned with the space

$$\widetilde{B}^\varphi a.p. = \{f \in B^\varphi(\mathbb{R}) : \exists f_n \in \mathcal{A}, \exists k_0 > 0, \lim_{n \to +\infty} \rho_{B^\varphi}(k_0(f_n - f)) = 0\},$$

which is defined as the closure of the set $\mathcal{A}$ in $B^\varphi(\mathbb{R})$ with respect to the (pseudo)-modular $\rho_{B^\varphi}(\cdot)$.

Some topological properties (reflexivity and duality properties) of these spaces are considered in [3]. Clearly, we have the following inclusions

$$B^\varphi a.p. \subseteq \widetilde{B}^\varphi a.p. \subseteq B^\varphi(\mathbb{R}).$$

When $\varphi(t, \cdot) = |\cdot|$, we denote by $B^1(\mathbb{R})$ and $B^1 a.p.$ the respective spaces. The notation ρ_1 is used for the associated pseudomodular.

If in addition the Musielak-Orlicz function satisfies the condition that for every $u_0 > 0$ there is a $c > 0$ for which $\frac{\varphi(t,u)}{u} \geq c$ for $u \geq u_0$ and $t \in \mathbb{R}$ (see [12, p. 91, Theorem 13.18]), we get the inclusion $B^\varphi a.p. \subseteq B^1 a.p.$. So, to every f in $B^\varphi a.p.$ we can associate a formal Fourier series. Questions concerning the convergence of the Fourier series are not considered.

Remark 1. To each function $f \in B^\varphi a.p.$, one can associate a Bochner-Fejèr polynomial σ^f as follows:

$$\sigma^f(x) = M(f(x+\cdot)K_B(\cdot)) = \lim_{T\to\infty} \int_{-T}^{+T} f(x+t)K_B(t)\,dt,$$

where $K_B(\cdot)$ is the Bochner-Fejèr kernel (see e.g. [6]). An important question is the approximation property of Bochner-Fejèr, that is, for any $f \in B^\varphi a.p.$ and for each $\varepsilon > 0$, can one find a Bochner-Fejèr polynomial σ^f_ε such that $\|f - \sigma^f_\varepsilon\|_{B^\phi} \leq \varepsilon$? It is still an open problem whether this approximation property is true or not for Besicovitch-Musielak-Orlicz spaces of almost periodic functions $\widetilde{B}^\varphi a.p.$. The only trouble is that, for $f \in \widetilde{B}^\varphi a.p.$ and the associated Bochner-Fejèr's polynomial σ^f, one cannot prove the inequality

$$\rho_{B^\varphi}\left(\sigma^f\right) \leq \rho_{B^\varphi}(f)$$

for any Musielak-Orlicz function φ.

Another fundamental result concerning the functions in $B^\varphi a.p.$ is the fact that if $f \in B^\varphi a.p.$ then $\varphi(\cdot, |f(\cdot)|) \in B^1 a.p.$ (see [8]). This property guaranties the existence of the limit in (1.1).

We say that φ satisfies the $\Delta_2^{B^1}$-condition ($\varphi \in \Delta_2^{B^1}$) if there exists $k > 1$ and a measurable nonnegative function h such that $\rho_1(h) < +\infty$ and $\varphi(t, 2u) \leq k\varphi(t,u) + h(t)$ for almost all $t \in \mathbb{R}$ and all $u \geq 0$.

We say that φ satisfies the $\nabla_2^{B^1}$-condition ($\varphi \in \nabla_2^{B^1}$) if its conjugate ψ given by the formula

$$\psi(t,u) = \sup_{v\geq 0}\{uv - \varphi(t,v)\}, \quad \text{for } \ t \in \mathbb{R} \ \text{ and } \ u \geq 0$$

satisfies the $\Delta_2^{B^1}$-condition.

Let us mention the following important fact (see [8]): φ satisfies the $\Delta_2^{B^1}$-condition if and only if φ satisfies the $\Delta_2^{L^1}$-condition, that is, there exist $k > 0$ and a positive function h with $\int_0^1 h(t)dt < +\infty$ such that

$$\varphi(t, 2u) \leq k\varphi(t,u) + h(t), \quad \text{for almost all } \ t \in [0,1] \ \text{ and } \ u \geq 0.$$

2. Auxiliary results

Let $P(\mathbb{R})$ be the family of subsets of $\mathbb{R}$ and $\Sigma(\mathbb{R})$ the Σ-algebra of its Lebesgue measurable sets. We define the set function

$$\overline{\mu}(A) = \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{+T} \chi_A(t)\,dt = \overline{\lim_{T\to\infty}} \frac{1}{2T}\mu(A\cap[-T,+T])$$

where χ_A denotes the characteristic function of $A \in \Sigma(\mathbb{R})$.

It is easily seen that the set function $\overline{\mu}$ is not σ-additive.

A sequence $\{f_n\} \subset B^{\varphi}(\mathbb{R})$ is said to be $\overline{\mu}$-convergent to some $f \in B^{\varphi}(\mathbb{R})$ when, for every $\alpha > 0$, we have

$$\lim_{n\to\infty} \overline{\mu}\{x \in \mathbb{R} : |f_n(x) - f(x)| > \alpha\} = 0.$$

This convergence concept satisfies the following property:

If $\{f_n\}_{n\geq 1}$ and $\{g_n\}_{n\geq 1}$ are two sequences of Σ-measurable functions $\overline{\mu}$-convergent to f and g respectively, then for all real α and β the sequence $\{\alpha f_n + \beta g_n\}$ is $\overline{\mu}$-convergent to $\alpha f + \beta g$.

Remark 2. We can also see that $\overline{\mu}$ does not satisfy the extraction property. Indeed, let us consider the sequence $(f_n)_n$ of $B^{\phi}(\mathbb{R})$ defined by

$$f_n(t) = \chi_{[-n,n]}(t).$$

It is not difficult to see that f_n is $\overline{\mu}$-convergent to $f \equiv 0$ in $B^{\phi}(\mathbb{R})$. Nevertheless, there is no subsequence which converges $\overline{\mu}$ almost everywhere ($\overline{\mu}$ a.e.) to $f \equiv 0$. More exactly, for any bijection $\theta : \mathbb{N} \longrightarrow \mathbb{N}$, the sequence $(f_{\theta(n)})_n$ converges to 1 with respect to the $\overline{\mu}$ a.e. convergence on $\mathbb{R}$.

We give here some technical results that are the key arguments in proof of the main theorems. First we need the following results (see [8] and [9]):

Lemma 1 ([8], [9]). *Let $\{f_n\}_{n\geq 1}$ be a sequence in $B^{\varphi}(\mathbb{R})$. Then:*

(i) *if $\{f_n\}_{n\geq 1}$ is modular convergent to $f \in B^{\varphi}a.p.$ it is also $\overline{\mu}$-convergent to f;*

(ii) *if $\{f_n\}_{n\geq 1}$ is $\overline{\mu}$-convergent to $f \in B^1a.p.$ and there exists $g \in B^1a.p.$ satisfying $\max(|f_n|, |f|) \leq g$, then*

$$\lim_{n\to\infty} \rho_1(f_n) = \rho_1(f).$$

Lemma 2 ([8], [9]). *Let $f \in B^{\varphi}a.p.$. Then*

(1) *$\|f\|_{B^{\varphi}} \leq 1$ if and only if $\rho_{B^{\varphi}}(f) \leq 1$,*

(2) *$\|f\|_{B^{\varphi}} = 1$ if and only if $\rho_{B^{\varphi}}(f) = 1$.*

Lemma 3 ([8]). *Let $\{f_n\}, \{g_n\}$ be sequences in $B^{\varphi}a.p.$ such that $\rho_{B^{\varphi}}(f_n) \leq 1$, $\rho_{B^{\varphi}}(g_n) \leq 1$ and $\lim_{n\to\infty} \rho_{B^{\varphi}}(\frac{1}{2}(f_n + g_n)) = 1$. Suppose that φ is strictly convex. Then, the sequence $\{f_n - g_n\}_n$ is $\overline{\mu}$-convergent to zero.*

In the following we denote by $\mathcal{M}(\mathbb{R})$ the set of Lebesgue measurable functions on $\mathbb{R}$, and $L^{\varphi}([0,1])$ the usual Musielak-Orlicz class

$$L^{\varphi}([0,1]) = \{f \in \mathcal{M}(\mathbb{R}) : \exists \lambda > 0, \int_0^1 \varphi(t, \lambda |f(t)|)\, dt < +\infty\}.$$

Proposition 1 ([8], [9]). *Let $f \in L^{\varphi}([0,1])$. Then,*

(1) *if $\widetilde{f}$ is the periodic extension of f to the whole $\mathbb{R}$ (with period $\tau = 1$), we have $\widetilde{f} \in \widetilde{B}^{\varphi} a.p.$.*
(2) *The injection map $i : L^{\varphi}([0,1]) \hookrightarrow \widetilde{B}^{\varphi} a.p.$, $i(f) = \widetilde{f}$ is an isometry with respect to the modulars and for the respective Luxemburg norms.*

We are ready now to present our results.

Lemma 4. *Let $f \in B^{\varphi}(\mathbb{R})$. Then $\lim_{n \to +\infty} \overline{\mu}\{t \in \mathbb{R},\ |f(t)| \geq n\} = 0$.*

Proof: For f being in $B^{\varphi}(\mathbb{R})$ there exists $\alpha > 0$ for which $\rho_{B^{\varphi}}(\alpha f) < \infty$. For an integer N, let f_N be the truncation of f, i.e.,

$$f_N(t) = \begin{cases} f(t) & \text{if } \ |f(t)| \leq N, \\ N & \text{if } \ |f(t)| > N. \end{cases}$$

Putting $E_N = \{t \in \mathbb{R}, |f(t)| \geq N\}$ and taking into account the convexity of ϕ we will have for each $N \in \mathbb{N}$,

$$\begin{aligned} \rho_{B^{\varphi}}(\alpha f) &\geq \rho_{B^{\varphi}}(\alpha f_N) \\ &\geq \rho_{B^{\varphi}}(\alpha f_N \chi_{E_N}) \\ &= \rho_{B^{\varphi}}(\alpha N \chi_{E_N}) \\ &\geq \phi(\alpha N) \overline{\mu}(E_N). \end{aligned}$$

Then, letting N tend to infinity, it follows directly that $\lim_{N \to \infty} \overline{\mu}(E_N) = 0$. □

Lemma 5. *Let $f \in B^{\varphi} a.p.$. Then the following equivalence holds:*

$$\rho_{B^{\varphi}}(f) = 0 \ \text{ iff } \ f = 0 \quad \overline{\mu} \ a.e.$$

Proof: The assertion that $\rho_{B^{\varphi}}(f) = 0$ implies $f = 0 \quad \overline{\mu}$ *a.e.* is a direct consequence of (i) in Lemma 1.

Let us show that if $\rho_{B^{\varphi}}(f) > 0$ then there exist real numbers $\alpha, \theta > 0$ such that

$$\overline{\mu}\{t \in \mathbb{R}, |f(t)| \geq \alpha\} > \theta.$$

In the contrary case, we will have for all $n \geq 1$

$$\overline{\mu}\{G_n\} \leq \frac{1}{n}$$

with $G_n = \{t \in \mathbb{R}, |f(t)| \geq \frac{1}{n}\}$. We will denote by G_n^c its complement.

Since $\lim_{n\to\infty} \overline{\mu}\{G_n\} = 0$, by using Lemma 4 in [8], we get

$$\lim_{n\to\infty} \rho_{B^{\varphi}}\left(f\chi_{G_n}\right) = 0.$$

On the other hand,

$$\rho_{B^{\varphi}}\left(f\chi_{G_n^c}\right) \leq \sup_{t\in\mathbb{R}} \varphi\left(t, \frac{1}{n}\right) \overline{\mu}\left(G_n^c\right) \leq \sup_{t\in\mathbb{R}} \varphi\left(t, \frac{1}{n}\right). \tag{2.1}$$

Letting n tend to infinity in (2.1), it follows

$$\lim_{n\to+\infty} \rho_{B^{\varphi}}\left(f\chi_{G_n^c}\right) = 0.$$

Otherwise, we have for all $n \geq 1$

$$\rho_{B^{\varphi}}(f) \leq \rho_{B^{\varphi}}\left(f\chi_{G_n}\right) + \rho_{B^{\varphi}}\left(f\chi_{G_n^c}\right). \tag{2.2}$$

Finally, by choosing n sufficiently large, the last term of inequality (2.2) can be made smaller than any $\varepsilon > 0$ from which we get $\rho_{B^{\varphi}}(f) = 0$. This is a contradiction, which finishes the proof. □

Lemma 6. *Let $\{f_n\}$ and f be in $B^{\varphi}(\mathbb{R})$ such that f_n is $\overline{\mu}$-convergent to f, then the sequence $(\varphi(\cdot, |f_n(\cdot)|))_n$ is $\overline{\mu}$-convergent to $\varphi(\cdot, |f(\cdot)|)$ in $B^1(\mathbb{R})$.*

Proof: Let us mention that the continuity of φ is sufficient to show the desired result. The method developed here is influenced by the proof of Proposition 1 in [8]. In view of Lemma 4, for each $\theta \in]0, 1[$ there is an $M > 0$ such that

$$\overline{\mu}\{t \in \mathbb{R}, |f(t)| \geq M\} < \theta.$$

Let now $\varepsilon > 0$. We define the set

$$G_n = \{t \in \mathbb{R}, |f(t)| \geq M\} \cup \{t \in \mathbb{R}, |f_n(t) - f(t)| \geq \varepsilon\}.$$

The function φ being continuous on $\mathbb{R} \times [0, +\infty[$ is also uniformly continuous on $[0, 1] \times [0, M+\varepsilon]$. Moreover, using the periodicity of $\varphi(t, u)$ with respect to $t \in \mathbb{R}$, it follows that φ is uniformly continuous on $\mathbb{R} \times [0, M + \varepsilon]$.
Then, there exists $\eta > 0$ for which the following implication holds:

$$|\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| > \varepsilon \Rightarrow |f_n(t) - f(t)| > \eta, \ \forall t \in G_n^c.$$

On the other hand, since $\{f_n\}$ is $\overline{\mu}$-convergent to f, we have

$$\lim_{n\to+\infty} \overline{\mu}\left\{t \in G_n^c, |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| > \varepsilon\right\} = 0 \tag{2.3}$$

and then

$$\begin{aligned}
& \overline{\mu}\left\{t \in \mathbb{R}, \left|\varphi\left(t, \left|f_n\left(t\right)\right|\right) - \varphi\left(t, \left|f\left(t\right)\right|\right)\right| \geq \varepsilon\right\} \\
\leq\; & \overline{\mu}\left\{t \in G_n, \left|\varphi\left(t, \left|f_n\left(t\right)\right|\right) - \varphi\left(t, \left|f\left(t\right)\right|\right)\right| \geq \varepsilon\right\} \\
& + \overline{\mu}\left\{t \in G_n^c, \left|\varphi\left(t, \left|f_n\left(t\right)\right|\right) - \varphi\left(t, \left|f\left(t\right)\right|\right)\right| \geq \varepsilon\right\} \\
\leq\; & \overline{\mu}\left(G_n\right) + \overline{\mu}\left\{t \in G_n^c, \left|\varphi\left(t, \left|f_n\left(t\right)\right|\right) - \varphi\left(t, \left|f\left(t\right)\right|\right)\right| \geq \varepsilon\right\} \\
\leq\; & \overline{\mu}\left\{t \in \mathbb{R}, \left|f\left(t\right)\right| \geq M\right\} + \overline{\mu}\left\{t \in \mathbb{R}, \left|f_n\left(t\right) - f\left(t\right)\right| \geq \varepsilon\right\} \\
& + \overline{\mu}\left\{t \in G_n^c, \left|\varphi\left(t, \left|f_n\left(t\right)\right|\right) - \varphi\left(t, \left|f\left(t\right)\right|\right)\right| \geq \varepsilon\right\}.
\end{aligned}$$

Now, letting n tend to infinity and in view of (2.3) we get:

$$\lim_{n \to +\infty} \overline{\mu}\left\{t \in \mathbb{R}, \left|\varphi\left(t, |f_n(t)|\right) - \varphi\left(t, |f(t)|\right)\right| \geq \varepsilon\right\} \leq \theta.$$

Since θ is arbitrary, it follows that the sequence $\{\varphi(\cdot, |f_n|)\}_n$ is $\overline{\mu}$-convergent to $\varphi(\cdot, |f|)$. □

Corollary 1. *If $\{f_n\}_{n\geq 1} \subset B^{\varphi}(\mathbb{R})$ is $\overline{\mu}$-convergent to $f \in B^{\varphi}a.p.$ and there exists $g \in B^{\varphi}a.p.$ satisfying $\max(|f_n|, |f|) \leq g$, then*

$$\lim_{n \to \infty} \rho_{B^{\varphi}}(f_n) = \rho_{B^{\varphi}}(f).$$

Proof: First, remark that in the proof of (ii) of Lemma 1 (see Lemma 4 of [8] and Lemma 2.6. of [9]) we can assume that $\{f_n\}_{n\geq 1}$ and f are in $B^1(\mathbb{R})$ instead of $B^1a.p.$.

Now, let us show the corollary. Let $\{f_n\}_{n\geq 1}$ be a sequence in $B^{\varphi}(\mathbb{R})$ *convergent to* f in the sense of $\overline{\mu}$-convergence. Then in view of Lemma 6, we get that the sequence $\varphi(\cdot, f_n(\cdot))$ is $\overline{\mu}$-convergent to $\varphi(., f(\cdot)) \in B^1(\mathbb{R})$ and satisfies the following fact:

$$\max\left(\varphi\left(., |f_n(\cdot)|\right), \varphi\left(., |f(\cdot)|\right)\right) \leq \varphi\left(., |g(\cdot)|\right) \in B^1 a.p.$$

Consequently, using Lemma 1, we deduce that

$$\lim_{n \to \infty} \rho_1(\varphi(\cdot, |f_n(\cdot)|)) = \rho_1(\varphi\left(\cdot, |f(\cdot)|\right)),$$

which means that

$$\lim_{n \to \infty} \rho_{B^{\varphi}}\left(f_n\right) = \rho_{B^{\varphi}}(f). \qquad \square$$

We now give an adapted version of Fatou's Lemma in $B^{\varphi}a.p.$.

Lemma 7. *Let $\{f_n\}_{n\geq 1}$ be a sequence in $B^{\varphi}(\mathbb{R})$ $\overline{\mu}$-convergent to $f \in B^{\varphi}a.p.$, then we have*

$$\underline{\lim}_{n \to +\infty} \rho_{B^{\varphi}}\left(f_n\right) \geq \rho_{B^{\varphi}}(f).$$

Proof: Consider the following sequence

$$g_n(t) = f(t)\chi_{E_n}(t) + f_n(t)\chi_{E_n^c}(t), \quad t \in \mathbb{R}$$

where $E_n = \{t \in \mathbb{R}, |f_n(t)| > |f(t)|\}$ and E_n^c is its complement. It is clear that for each $n \in \mathbb{N}$, g_n belongs to $B^\varphi(\mathbb{R})$ and satisfies

$$|g_n(t) - f(t)| = \begin{cases} 0 & \text{if } |f_n(t)| > |f(t)|, \\ |f_n(t) - f(t)| & \text{if } |f_n(t)| \leq |f(t)|. \end{cases}$$

It follows that $|g_n(t) - f(t)| \leq |f_n(t) - f(t)|$ and consequently the sequence $\{g_n\}_n$ is $\overline{\mu}$-convergent to f.

Now, since $|g_n(t)| \leq |f(t)|$ and $f \in B^\varphi a.p.$, using Corollary 1 we deduce that $\lim_{n\to+\infty} \rho_{B^\varphi}(g_n) = \rho_{B^\varphi}(f)$. Hence,

$$\rho_{B^\varphi}(f) = \lim_{n\to+\infty} \rho_{B^\varphi}(g_n) \leq \varliminf_{n\to+\infty} \rho_{B^\varphi}(f_n). \qquad \square$$

Lemma 8. *Let $\{f_n\}_{n\geq 1}$ be a sequence in $B^\varphi a.p.$. Suppose that $\{f_n\}$ is $\overline{\mu}$-convergent to $f \in B^\varphi(\mathbb{R})$ and* $\lim_{n\to+\infty} \rho_{B^\varphi}(f_n) = \rho_{B^\varphi}(f)$. *Then,*

$$\lim_{n\longrightarrow+\infty} \rho_{B^\varphi}\left(\frac{f_n - f}{2}\right) = 0.$$

If in addition, $\varphi \in \Delta_2^{B^1}$ then $\lim_{n\longrightarrow+\infty} \|f_n - f\|_{B^\varphi} = 0$.

Proof: In view of Lemma 6, we deduce that $\{\varphi(\cdot, \frac{|f_n-f|}{2})\}_n$ is $\overline{\mu}$-convergent to 0 and consequently the sequence $g_n = \frac{\varphi(\cdot,|f_n|)+\varphi(\cdot,|f|)}{2} - \varphi(\cdot, \frac{|f_n-f|}{2})$ is also $\overline{\mu}$-convergent to $g = \varphi(\cdot, |f|)$. Then, by using Lemma 7, we get that

$$\varliminf_{n\to+\infty} \rho_1(g_n) \geq \rho_1(g).$$

Consequently, in virtue of the existence of the limit in the expression of $\rho_1(\cdot)$, we obtain

$$\begin{aligned} \rho_\varphi(f) &= \rho_1(g) \\ &\leq \varliminf_{n\to+\infty} \rho_1\left(\frac{\varphi(|f_n|) + \varphi(|f|)}{2} - \varphi\left(\frac{|f_n - f|}{2}\right)\right) \\ &\leq \varliminf_{n\to+\infty} \left\{\frac{1}{2}\rho_{B^\varphi}(f_n) + \frac{1}{2}\rho_{B^\varphi}(f) - \rho_{B^\varphi}\left(\frac{f_n - f}{2}\right)\right\} \\ &\leq \rho_{B^\varphi}(f) - \varlimsup_{n\to+\infty} \rho_{B^\varphi}\left(\frac{f_n - f}{2}\right). \end{aligned}$$

Finally, we get $\lim_{n\to+\infty} \rho_{B^\varphi}(\frac{f_n-f}{2}) = 0$. $\square$

3. Main results

Theorem 3. *The following properties are equivalent to each other:*

(1) $\widetilde{B}^{\varphi}a.p.$ *is LUC,*
(2) $\widetilde{B}^{\varphi}a.p.$ *has the H-property,*
(3) φ *is strictly convex and* φ *satisfies the* $\Delta_2^{B^1}$*-condition.*

Proof: We will show the following implications: (3) $\Longrightarrow$ (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). Observe that the implication (1) $\Longrightarrow$ (2) holds in general Banach spaces.

To prove (3) $\Longrightarrow$ (1), let f_n, f be in $\widetilde{B}^{\varphi}a.p.$ such that

$$\|f_n\|_{B^\varphi} = \|f\|_{B^\varphi} = 1 \quad \text{and} \quad \left\|\frac{f+f_n}{2}\right\|_{B^\varphi} \to 1 \quad \text{as} \quad n \to +\infty.$$

Recall that since φ satisfies the $\Delta_2^{B^1}$-condition, we have $B^{\varphi}a.p. = \widetilde{B}^{\varphi}a.p.$ and from Lemma 2, we have $\rho_{B^\varphi}(f_n) = \rho_{B^\varphi}(f) = 1$. Following analogous arguments to those of [14, Lemma 2], it is possible to show the following assertion:

$$\rho_{B^\varphi}\left(\frac{f+f_n}{2}\right) \to 1 \quad \text{as} \quad n \to +\infty$$

whenever

$$\left\|\frac{f+f_n}{2}\right\|_{B^\varphi} \to 1 \quad \text{as} \quad n \to +\infty.$$

Indeed, suppose the assertion is false. Then, there exists $\varepsilon > 0$ such that the following inequalities hold for all $n \geq 1$: $\rho_{B^\varphi}(\frac{f+f_n}{2}) \leq 1-\varepsilon$ or $\rho_{B^\varphi}(\frac{f+f_n}{2}) \geq 1+\varepsilon$. In both cases, we will obtain a contradiction. In the first case, by using the $\Delta_2^{B^1}$-condition, we get $\sup_n \rho_{B^\varphi}(f+f_n) < \infty$, and consequently

$$\begin{aligned}
1 &= \rho_{B^\varphi}\left(\frac{f+f_n}{\|f+f_n\|_{B^\varphi}}\right) = \rho_{B^\varphi}\left(\left(\frac{2}{\|f+f_n\|_{B^\varphi}} - 1\right)(f+f_n)\right. \\
&\quad \left. + \left(2 - \frac{2}{\|f+f_n\|_{B^\varphi}}\right)\left(\frac{f+f_n}{2}\right)\right) \\
&\leq \left(\frac{2}{\|f+f_n\|_{B^\varphi}} - 1\right)\rho_{B^\varphi}(f+f_n) + \left(2 - \frac{2}{\|f+f_n\|_{B^\varphi}}\right)\rho_{B^\varphi}\left(\frac{f+f_n}{2}\right) \\
&\leq \left(\frac{2}{\|f+f_n\|_{B^\varphi}} - 1\right)\sup_n \rho_{B^\varphi}(f+f_n) + \left(2 - \frac{2}{\|f+f_n\|_{B^\varphi}}\right)(1-\varepsilon).
\end{aligned}$$

Passing to the limit for $n \to +\infty$, we obtain $1 \leq 1 - \varepsilon$, that is, a contradiction.

If $\rho_{B^\varphi}(\frac{f+f_n}{2}) \geq 1+\varepsilon$, the $\Delta_2^{B^1}$-condition implies that $\sup_n \rho_{B^\varphi}(2\frac{f+f_n}{\|f+f_n\|_{B^\varphi}}) < \infty$, and then

$$\begin{aligned}
1+\varepsilon \quad &\leq \quad \rho_{B^\varphi}\left(\frac{f+f_n}{2}\right) = \rho_{B^\varphi}\left(\left(2-\left\|\frac{f+f_n}{2}\right\|_{B^\varphi}\right)\left(\frac{f+f_n}{\|f+f_n\|_{B^\varphi}}\right)\right. \\
&\quad + \left.\left(\left\|\frac{f+f_n}{2}\right\|_{B^\varphi}-1\right)\left(2\frac{f+f_n}{\|f+f_n\|_{B^\varphi}}\right)\right) \\
&\leq \quad \left(2-\left\|\frac{f+f_n}{2}\right\|_{B^\varphi}\right)\rho_{B^\varphi}\left(\frac{f+f_n}{\|f+f_n\|_{B^\varphi}}\right) \\
&\quad + \left(\left\|\frac{f+f_n}{2}\right\|_{B^\varphi}-1\right)\rho_{B^\varphi}\left(2\frac{f+f_n}{\|f+f_n\|_{B^\varphi}}\right) \\
&\leq \quad \left(2-\left\|\frac{f+f_n}{2}\right\|_{B^\varphi}\right) + \left(\left\|\frac{f+f_n}{2}\right\|_{B^\varphi}-1\right)\sup_n \rho_{B^\varphi}\left(2\frac{f+f_n}{\|f+f_n\|_{B^\varphi}}\right).
\end{aligned}$$

Letting n tend to infinity, we get $1+\varepsilon \leq 1$, a contradiction. This completes the proof of the previous assertion.

Hence, in view of Lemma 3, it follows that the sequence $\{f_n\}_n$ is $\overline{\mu}$-convergent to f. Then using Lemma 8 and the $\Delta_2^{B^1}$-condition on φ, we conclude that

$$\|f_n - f\|_{B^\varphi} \to 0 \;\text{ as }\; n \to +\infty.$$

(2) $\Longrightarrow$ (3): Suppose that $\widetilde{B}^\varphi a.p.$ has the H-property. Using Proposition 1 and the same techniques as in [1] (see the proof of Theorem 1) we will show that the Musielak-Orlicz space $L^\varphi([0,1])$ has also the H-property. We repeat this justification for the clarity of the proof. Let $\{f_n\}$ be a sequence in $L^\varphi([0,1])$ such that:

- $\{f_n\}$ converge weakly to some f in $L^\varphi([0,1])$,
- $\|f_n\|_\varphi \longrightarrow \|f\|_\varphi$ (here, the notation $\|\cdot\|_\varphi$ is used to designate the Luxemburg norm associated to the Musielak-Orlicz space $L^\varphi([0,1])$).

Then, for each G in the dual space $(\widetilde{B}^\varphi a.p.)^*$, we have $G \circ i \in (L^\varphi([0,1]))^*$. Moreover, since $f_n \longrightarrow f$ weakly in $L^\varphi([0,1])$, we get

$$G \circ i\,(f_n) \longrightarrow G \circ i(f)$$

or equivalently $G(\widetilde{f_n}) \longrightarrow G(\widetilde{f})$. Thus $\widetilde{f_n} \longrightarrow \widetilde{f}$ weakly in $\widetilde{B}^\varphi a.p.$.

It is clear that $\|\widetilde{f_n}\|_{B^\varphi} \longrightarrow \|\widetilde{f}\|_{B^\varphi}$ and since $\widetilde{B}^\varphi a.p.$ has the H-property, we can write $\|\widetilde{f_n - f}\|_{B^\varphi} \longrightarrow 0$ and finally $\|f_n - f\|_\varphi \longrightarrow 0$. This means that the Musielak-Orlicz space $L^\varphi([0,1])$ has the H-property.

It follows from [11] that φ is strictly convex and satisfies the $\Delta_2^{L^1}$-condition. Since it satisfies also the $\Delta_2^{B^1}$-condition, the proof is finished. □

Theorem 4. *The following properties are equivalent to each other:*

(1) $\widetilde{B}^\varphi a.p.$ *is UCED;*

(2) *φ is strictly convex and φ satisfies the $\Delta_2^{B^1}$-condition.*

Proof: Since $\widetilde{B}^{\varphi}a.p.$ is a pseudonormed space, we will adapt the definition of UCED property to this space as follows: for any $g \in \widetilde{B}^{\varphi}a.p.$, and every sequence (f_n) in $\widetilde{B}^{\varphi}a.p.$, the conditions $\|f_n\| \to 1$, $\|f_n + g\| \to 1$ and $\|2f_n + g\| \to 2$ imply $\|g\| = 0$. Remark that this definition is equivalent to that of UCED property of a normed space.

(2) $\Longrightarrow$ (1): Let $\|f_n\|_{B^\varphi} \to 1$, $\|f_n + g\|_{B^\varphi} \to 1$ and $\|2f_n + g\|_{B^\varphi} \to 2$. Assume that φ is strictly convex and φ satisfies the $\Delta_2^{B^1}$-condition. Then, we have also $\rho_{B^\varphi}(f_n) \to 1$, $\rho_{B^\varphi}(f_n + g) \to 1$ and $\rho_{B^\varphi}(\frac{2f_n+g}{2}) \to 1$. Now, applying Lemma 3 to the sequences $(f_n)_n$ and $(f_n + g)_n$, we get that $g = 0$ $\overline{\mu}$ *a.e.* and in view of Lemma 5 we deduce that $\rho_{B^\varphi}(g) = 0$ and using again the $\Delta_2^{B^1}$-condition it follows that $\|g\|_{B^\varphi} = 0$.

(1) $\Longrightarrow$ (2): Using Proposition 1, and since the UCED property of $\widetilde{B}^{\varphi}a.p.$ implies the UCED property of $L^\varphi([0,1])$, we get the necessity of the strict convexity of φ and the $\Delta_2^{L^1}$-condition (see [7]) and then the necessity of the $\Delta_2^{B^1}$-condition. □

Corollary 2. *The following properties are equivalent to each other:*

(1) *$\tilde{B}^{\varphi}a.p.$ is LUC;*
(2) *$\tilde{B}^{\varphi}a.p.$ is MLUC;*
(3) *$\tilde{B}^{\varphi}a.p.$ has the H-property;*
(4) *$\tilde{B}^{\varphi}a.p.$ is UCED;*
(5) *$\tilde{B}^{\varphi}a.p.$ is SC;*
(6) *φ is strictly convex and φ satisfies the $\Delta_2^{B^1}$-condition.*

Now, we apply the previous results to give an application in best approximation.

Let $(X, \|\cdot\|_X)$ be a Banach space, C be a subset of X and $x \in X$. Let us consider the metric projection

$$P_C : x \to d(x, C) = \inf\{\|x - y\|_X, y \in C\}.$$

In the paper [3], the authors have shown that, under the additional conditions on φ:

$$\forall t \in \mathbb{R}, \quad \lim_{u\to\infty} \frac{\varphi(t,u)}{u} = +\infty, \quad \lim_{u\to 0} \frac{\varphi(t,u)}{u} = 0, \tag{3.1}$$

the space $\widetilde{B}^{\varphi}a.p.$ is reflexive if and only if $\varphi \in \Delta_2^{B^1} \cap \nabla_2^{B^1}$.

Since reflexive strictly convex Besicovitch-Musielak-Orlicz spaces of almost periodic functions are LUC, and so they have the H-property, we get the following corollary which is a generalization of Doob Theorem:

Corollary 3. *Assume that φ is strictly convex, $\varphi \in \Delta_2^{B^1} \cap \nabla_2^{B^1}$ and φ satisfies the conditions* (3.1), *then for any closed convex sets $C_1 \supset C_2 \supset \cdots \supset C_\infty = \overline{\cap_n C_n}$*

in $\widetilde{B}^{\varphi}a.p.$ *and any* $x \in \widetilde{B}^{\varphi}a.p.$,

$$\|P_{C_n}(x) - P_{C_\infty}(x)\| \to \infty, \quad as \;\; n \to \infty.$$

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Department of Mathematics, Faculty of Sciences, Mouloud Mammeri University, Tizi-Ouzou, Algeria

E-mail: fbedouhene@yahoo.fr
amina-daouidz@yahoo.fr
hkourat@gmail.com

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