Further remarks on formal power series

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Abstract. In this paper, we present a considerable simplification of the proof of a theorem by Gan and Knox, stating a sufficient and necessary condition for existence of a composition of two formal power series. Then, we consider the behavior of such series and their (formal) derivatives at the boundary of the convergence circle, obtaining in particular a theorem of Bugajewski and Gan concerning the structure of the set of points where a formal power series is convergent with all its derivatives.

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1. Introduction

The existence of superposition of two functions depends on the relation of the range of the interior function and the domain of the exterior function. If the functions in question are represented by power series, the domain is obviously circular, and it is not obvious what the range might look like. However, it is well-known that the superposition must exist if the leading coefficient of the interior series vanishes (see e.g. [5, p. 66, Theorem 3.4]).

In 2002, Gan and Knox showed in [3] that this criterion may be generalized to series with nonvanishing leading coefficient, and that there exists a simple necessary and sufficient condition for the existence of superposition of two power series. The proof, however, is far from simple. The first part of this paper consists of a simpler (although based on a similar idea) proof of this fact.

It is well-known that a power series is convergent in the interior of its convergence circle (or interval) and divergent in its exterior. However, it is not at all obvious whether it converges or diverges on the boundary of its convergence circle. It is easily shown that both cases may occur; moreover, convoluted examples were devised (see for example [4]) to show that the structure of the set of convergence points of a power series can be quite complicated. Since power series can be differentiated term-by-term infinitely many times, it seems a natural question to determine whether any relationship exists between the convergence of the power series and the convergence of its derivatives. In order to fully answer such question, it also seems natural to consider *formal power series*, i.e., sequences of (real or complex) coefficients giving rise to series which are not necessarily convergent, since such series can be (formally) differentiated (even at points of divergence).

Such series are also useful in applications, in particular in the field of differential equations (see e.g. [2] and references therein).

Recently, it was proved in [1] that if there exists a point at the boundary of the convergence circle of a given (formal) power series such that all its derivatives are convergent at this point, then each point on this boundary possesses such a property. However, the proof given therein seems overly complicated; moreover, it does not clearly reveal the reason for such a property, namely that if a formal power series is not absolutely convergent at some point at the boundary of the convergence circle, then its second derivative (and all subsequent ones) must diverge on the *whole* boundary. Also, the proof given therein uses the abovementioned theorem on the existence of the superposition of two formal power series shown in [3]. In this paper, we give a short proof of the above property, together with a similar theorem on the first derivative, not using that result.

2. Preliminaries

By $\mathbb{K}$ we will denote the field of real or complex numbers. Any sequence $(b_n)_{n=0}^{\infty}$ of numbers in $\mathbb{K}$ will be called a (*formal*) *power series* (with *coefficients* $b_0, b_1, \dots$); the set of all such sequences will be denoted by $\mathbb{X}(\mathbb{K})$ or just $\mathbb{X}$, and instead of writing $f = (b_0, b_1, \dots)$, we will use the notation $f(z) := \sum_{n=0}^{\infty} b_n z^n$. (This notation will not necessarily mean that the sum is convergent in any sense.) We define addition and multiplication by scalars in $\mathbb{X}$ in the usual way. Moreover, *multiplication* of two formal power series is defined by the Cauchy formula:

$$\Big(\sum_{n=0}^{\infty} b_n z^n\Big) \cdot \Big(\sum_{n=0}^{\infty} a_n z^n\Big) = \sum_{n=0}^{\infty} c_n z^n \quad \text{iff} \quad c_n = \sum_{k=0}^{n} b_k a_{n-k} \text{ for } n = 0, 1, \dots$$

If $f(z) := \sum_{n=0}^{\infty} b_n z^n$ is a formal power series, we will denote by $\deg(f)$ its *degree*, that is, the highest index of a nonzero coefficient; by $r(g)$, its *radius of convergence*; by $I(g)$, its *domain of convergence*, that is, the set of all points $z \in \mathbb{K}$ such that $f(z)$ is convergent. (Notice that we will adopt the usual convention that $z^0 = 1$ even if $z = 0$.) The (*first*) *formal derivative* of $f(z)$ will be defined as $f'(z) := \sum_{n=1}^{\infty} n b_n z^{n-1}$; the second and subsequent (formal) derivatives are defined accordingly. The set of points of convergence of f and all its derivatives will be denoted by $D(f)$.

Let $g(z) := \sum_{n=0}^{\infty} a_n z^n$ be a formal power series. We will define the set $\mathbb{X}_g \subset \mathbb{X}$ by the formula

$$\mathbb{X}_g = \Big\{ f(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathbb{X} \;\Big|\; \text{for each } k = 0, 1, \dots, \text{ the series } \sum_{n=0}^{\infty} a_n b_k^{(n)} \text{ converges} \Big\},$$

where $b_k^{(n)}$ are the coefficients of the series f^n, the nth power (in the sense of Cauchy multiplication) of f. For all $f \in \mathbb{X}_g$ we can then define the mapping $T_g\colon \mathbb{X}_g \to \mathbb{X}$ such that $T_g(f)(z) := g \circ f(z) := \sum_{k=0}^{\infty} c_k z^k$, where $c_k := \sum_{n=0}^{\infty} a_n b_k^{(n)}$; $g \circ f$ is called the *superposition* of g and f. Finally, given a power series $g(z)$, the set of all power series $f(z) = \sum_{n=0}^{\infty} b_n z^n$ such that $b_0 \in D(g)$ will be denoted by $\overline{\mathbb{X}}_g$.

3. Existence of a superposition of two formal power series

In this section, we provide a shorter variant of the proof of the following theorem.

Theorem 1 ([3, p. 766, Theorem 3.1])**.** *Denote* $f(z) := \sum_{n=0}^{\infty} b_n z^n$ *and* $g(z) := \sum_{n=0}^{\infty} a_n z^n$ *and assume that* $\deg(f) \neq 0$. *Then the composition* $g \circ f$ *exists if and only if*

$$\sum_{n=k}^{\infty} \binom{n}{k} a_n b_0^{n-k} \text{ converges} \tag{1}$$

for $k = 0, 1, 2, \ldots$.

Proof: The case when $b_0 = 0$ is trivial, so suppose that $b_0 \neq 0$. By definition (cf. [3, p. 763, equation (2.3)]), $g \circ f$ exists if and only if all series

$$c_k = \sum_{n=0}^{\infty} a_n \left(\sum \frac{n!}{r_0! \ldots r_k!} b_0^{r_0} \ldots b_k^{r_k} \right) \tag{2}$$

converge, where the inner sum is taken over nonnegative integer solutions of

$$\begin{cases} r_0 + r_1 + \cdots + r_k = n \\ r_1 + 2r_2 + \cdots + kr_k = k, \end{cases} \quad k = 0, 1, \ldots \tag{3}$$

(of course, we adhere to the usual conventions that summing over an empty set gives zero and that $b_i^0 = 1$ even if $b_i = 0$).

Since $c_0 = \sum_{n=0}^{\infty} a_n b_0^n$, existence of c_0 is equivalent to convergence of (1) for $k = 0$.

Now fix $k > 0$. Since we are dealing only with convergence of series (2), it is enough to consider $n \geq k$. Notice that under this assumption, due to the equality $r_0 = n - (r_1 + \cdots + r_k)$, there is a one-to-one correspondence between the nonnegative integer solutions of (3) and the nonnegative integer solutions of

$$r_1 + 2r_2 + \cdots + kr_k = k. \tag{3'}$$

Remark that the solution set of (3′) is finite; let p denote the number of solutions of (3′) (given the fixed $k > 0$). For any $j = 1, \ldots, p$, denote the jth solution of $(3')$ by $(r_1^{(j)},\dots,r_k^{(j)})$ and (for $s=1,\dots,k$) put:

$$\begin{aligned} W_s &:= \{j\in\{1,\dots,p\} : \sum_{i=1}^k r_i^{(j)} = s\}\\ m_j &:= b_1^{r_1^{(j)}}\dots b_k^{r_k^{(j)}}/\,r_1^{(j)}!\dots r_k^{(j)}!\\ d_s^{(k)} &:= \sum_{j\in W_s} m_j. \end{aligned}$$

Now let us consider the part of the sum (2) from the kth term to the qth term. For brevity, we will denote $r_0^{(j)}(n) := n-(r_1^{(j)}+\dots+r_k^{(j)})$.

$$\begin{aligned} \sum_{n=k}^q a_n\Big(\sum \frac{n!}{r_0!\dots r_k!} b_0^{r_0}\dots b_k^{r_k}\Big) &= \sum_{n=k}^q a_n\Big(\sum_{j=1}^p \frac{n!}{r_0^{(j)}(n)!} b_0^{r_0^{(j)}(n)} m_j\Big)\\ &= \sum_{n=k}^q a_n\Big(\sum_{s=1}^k\sum_{j\in W_s} \frac{n!}{(n-s)!} b_0^{n-s} m_j\Big)\\ &= \sum_{n=k}^q a_n\Big(\sum_{s=1}^k \frac{n!}{(n-s)!} b_0^{n-s} d_s^{(k)}\Big)\\ &= \sum_{s=1}^k d_s^{(k)}\Big(\sum_{n=k}^q a_n \frac{n!}{(n-s)!} b_0^{n-s}\Big)\\ &= \sum_{s=1}^k s!\, d_s^{(k)}\Big(\sum_{n=k}^q \binom{n}{s} a_n b_0^{n-s}\Big). \end{aligned} \tag{4}$$

We can see that if the series (1) converge for $k=1,2,\dots$, then the above sum (and hence c_k) is well-defined.

Suppose now that the composition $g\circ f$ exists, i.e., all the series (2) converge. As we have seen, this implies that (1) converges for $k=0$. We will proceed by induction on k. Let l be the smallest positive number for which $b_l\neq 0$ and let us fix some integer $m>0$. We will first show that $d_k^{(ml)}=0$ for $k>m$, and $d_m^{(ml)}=\frac{b_l^m}{m!}$. Indeed, assume that nonnegative integers $r_1,\dots,r_{ml}$ solve the system $r_1+2r_2+\dots+mlr_{ml}=ml$ and $r_1+\dots+r_{ml}=k$. If $l=1$ and $k>m$ then $d_k^{(ml)}=0$, since there are no r_i's satisfying the system. If $k>m$ and $l>1$, it cannot be true that $r_1=\dots=r_{l-1}=0$, since then we would have $ml=lr_l+\dots+mlr_{ml}\ge l(r_l+\dots+r_{ml})=kl>ml$, which is a contradiction. Therefore, if $l>1$, then in each component of the sum $d_k^{(ml)}$, where $k>m$, there is some $1\le h<l$ such that $r_h>0$ and $b_h^{r_h}=0$, so $d_k^{(ml)}=0$. Consider now the case when $k=m$. If for some $1\le h<l$ we have $r_h>0$, the component containing $b_h^{r_h}$ adds nothing to $d_m^{(ml)}$, so we can restrict ourselves to the case where only $r_l,\dots,r_{ml}$ can be positive. If there were some $lm\ge h>l$ such that

$r_h > 0$, we would have $r_l + \cdots + r_{ml} = m$ and $lr_l + \cdots + mlr_{ml} = ml$; but $ml = l(r_l + \cdots + r_{ml}) < lr_l + \cdots + mlr_{ml} = ml$ — again a contradiction. This means that the only nonzero component of $d_m^{(ml)}$ is $b_l^m/m!$, for any fixed m.

Now we obtain the following formula for the part of the sum (4) defining c_{ml} from the mlth to the qth term:

$$\sum_{s=1}^{ml} s!\, d_s^{(ml)} \Big(\sum_{n=ml}^{q} \binom{n}{s} a_n b_0^{n-s} \Big) = \sum_{s=1}^{m-1} s!\, d_s^{(ml)} \Big(\sum_{n=ml}^{q} \binom{n}{s} a_n b_0^{n-s} \Big) + b_l^m \sum_{n=ml}^{q} \binom{n}{m} a_n b_0^{n-m}.$$

Putting $m = k+1$ in the above equality we get that existence of c_{ml} implies convergence of the left-hand side (when q tends to infinity) and the inductive hypothesis guarantees the convergence of the first component of the right-hand side sum, so the last component must converge, too. So, since $b_l \neq 0$, then $\sum_{n=(k+1)l}^{\infty} \binom{n}{k+1} a_n b_0^{n-(k+1)}$ converges, which finishes the proof. □

4. Boundary behavior of power series

In this section we describe the behavior of formal power series (in particular, its derivatives) on the boundary of the convergence circle.

Lemma 1. *Let $g \in \mathbb{X}(\mathbb{K})$. Assume that $r(g) \in (0, +\infty)$. If g is not absolutely convergent at some point in the boundary of $I(g)$, then the (formal) second derivative of g is divergent on the whole boundary of $I(g)$.*

Proof: Let $g(a) = \sum_{n=0}^{\infty} b_n a^n$, where $|a| = r = r(g)$, be not absolutely convergent. Then, there exists an increasing sequence $(n_k)_{k=1}^{\infty}$ of natural numbers such that $n_1 \geq 2$ and $|b_{n_k}| r^{n_k} > \frac{1}{n_k^2}$ for each $k \in \mathbb{N}$. We have therefore:

$$n_k(n_k-1)|b_{n_k}| r^{n_k-2} > \frac{n_k(n_k-1)}{r^2} \cdot \frac{1}{n_k^2} = \tfrac{1}{r^2}(1 - \tfrac{1}{n_k}) \xrightarrow[k\to\infty]{} \tfrac{1}{r^2} \neq 0,$$

so that $g''(z)$ is divergent for any z such that $|z| = r$. □

Corollary 1 (see [1, Corollary 1])**.** *Let $g \in \mathbb{X}(\mathbb{K})$ and $a \in \partial I(g)$. If $a \in D(g)$, then $\partial I(g) \subset D(g)$.*

Corollary 2. *Let $g \in \mathbb{X}(\mathbb{K})$. Then either $D(g) = \{z \in \mathbb{K} : |z| < r(g)\}$ or $D(g) = \{z \in \mathbb{K} : |z| \leq r(g)\}$.*

Corollary 3 (see [1, Lemma 2])**.** *Let $g \in \mathbb{X}(\mathbb{K})$. If there exists some a with $|a| = r(g)$ such that for each $k \in \{0, 1, 2, \dots\}$, the series $g^{(k)}(a)$ converges, then each $g^{(k)}$ converges absolutely on the closed disc $D(g) = \{z \in \mathbb{K} : |z| \leq r(g)\}$.*

Lemma 2. *Let $g \in \mathbb{X}(\mathbb{K})$. Assume that $r(g) \in (0, +\infty)$. If g is not absolutely convergent at some point in the boundary of $I(g)$, then the (formal) first derivative of g is not absolutely convergent on the whole boundary of $I(g)$.*

PROOF: Let $g(a) = \sum_{n=0}^{\infty} b_n a^n$, where $|a| = r = r(g)$, be not absolutely convergent, i.e. $\sum_{n=0}^{\infty} |b_n| r^n = +\infty$. It is obvious that for any $k \in N$: $\sum_{n=k}^{\infty} |b_n| r^n = +\infty$. For all large n it holds $n|b_n|r^{n-1} = \frac{n}{r}|b_n|r^n \geq |b_n|r^n$ and the divergence follows. $\square$

Corollary 4. *If $a \in I(g)$ and $g(a)$ is not absolutely convergent then neither is $g^{(n)}(a)$ for $n = 1, 2, \ldots$.*

Corollary 5. *Let $g \in \mathbb{X}(\mathbb{K})$. Assume that $r(g) = 1$. If $g^{(1)}(1)$ is absolutely convergent, then g is absolutely convergent over $I(g)$.*

Combining Lemmas 1 and 2, we obtain the following result.

Lemma 3. *Let $g \in \mathbb{X}(\mathbb{K})$. Assume that $r(g) \in (0, +\infty)$. If g is not absolutely convergent at some point in the boundary of $I(g)$, then the (formal) nth derivative of g is divergent on the whole boundary of $I(g)$ for $n = 2, 3, \ldots$.*

From the above lemma we obtain a simple criterion deciding which part of the alternative from Corollary 2 holds.

Corollary 6. *$D(g) = \operatorname{Int} I(g)$ iff $g^{(k)}(r(g))$ is not absolutely convergent for some $k = 0, 1, \ldots$.*

5. Appendix

Finally, we show variants of two proofs contained in [1].

Proposition 2 (see [1, Proposition 4.1])**.** *Let $g(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathbb{X}(\mathbb{C})$. Then T_g maps $\overline{\mathbb{X}}_g$ into itself if and only if g maps $D(g)$ into itself.*

PROOF: Assume that $g(a) \in D(g)$ for $a \in D(g)$. Let $f \in \overline{\mathbb{X}}_g$, i.e., $f = a + \tilde{f}$ for some $a \in D(g)$ and $\tilde{f} \in m(\mathbb{X})$. Using the notation of [1, Definition 1.2] we have $c_0 = \sum_{n=0}^{\infty} b_n a^n = g(a) \in D(g)$ and $T_g(f) = g \circ f \in c_0 + m(\mathbb{X}) \in \overline{\mathbb{X}}_g$. On the other hand, if $T_g \colon \overline{\mathbb{X}}_g \to \overline{\mathbb{X}}_g$, then for $f = a$ for some $a \in D(g)$ we have $T_g(f) = g \circ f = g(a) \in \overline{\mathbb{X}}_g$; but the only constants in $\overline{\mathbb{X}}_g$ are in $D(g)$ and the proof is finished. $\square$

Proposition 3 (see [1, Corollary 4.2])**.** *Let $g(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathbb{X}(\mathbb{C})$. Then T_g maps $\mathbb{X}_g$ into itself if and only if g maps $I(g)$ into itself.*

PROOF: If there exists any $a \in D(g)$ such that $|a| = r(g)$, we infer from Corollaries 1 and 3 that $D(g) = I(g)$ and $\overline{\mathbb{X}}_g = \mathbb{X}_g$, so it is enough to apply Proposition 2.

Assume now that there exists no such a; this means that $D(g)$ is an open disk $\{z : |z| < r(g)\}$. Assume that g maps $I(g)$ into itself. It follows easily from the maximum principle that either g is constant (and there is nothing to prove), or for $a \in D(g) = \operatorname{Int} I(g)$, also $g(a) \in D(g)$. Now, if $f \in \mathbb{X}_g$, then by Theorem 1, either $f \in \overline{\mathbb{X}}_g$, or $f = a$ for some $a \in I(g) \setminus D(g)$. In the former case we apply Proposition 2; in the latter, we have $T_g(f) = g \circ f = g(a) \in I(g) \subset \mathbb{X}_g$.

Let now T_g map $\mathbb{X}_g$ into itself. Put $f = a$ for some $a \in I(g)$; then $g(a) = g \circ f = T_g(f) \in \mathbb{X}_g$; but the only constants in $\mathbb{X}_g$ are those from $I(g)$ and thus the proof is finished. $\square$

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References

[1] Bugajewski D., Gan X.-X., *A note on formal power series*, Comment. Math. Univ. Carolin. **51** (2010), no. 4, 595–604.

[2] Bugajewski D., Gan X.-X., *On formal Laurent series*, Bull. Braz. Math. Soc., New Series **42** (2011), no. 3, 415–437.

[3] Gan X.-X., Knox N., *On composition of formal power series*, Int. J. Math. Math. Sci. **30** (2002), no. 12, 761–770.

[4] Herzog F., Piranian, G., *Sets of convergence of Taylor series I*, Duke Math. J. **16** (1949), 529–534.

[5] Lang S., *Complex Analysis*, Springer, 4th edition, New York, 1999.

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