# Sharp constants for Moser-type inequalities concerning embeddings into Zygmund spaces

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*Abstract.* Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded set. We give a Moser-type inequality for an embedding of the Orlicz-Sobolev space $W_0L^{\Phi}(\Omega)$, where the Young function $\Phi$ behaves like $t^n \log^{\alpha}(t)$, $\alpha < n-1$, for $t$ large, into the Zygmund space $Z_0^{\frac{n-1-\alpha}{n}}(\Omega)$. We also study the same problem for the embedding of the generalized Lorentz-Sobolev space $W_0^m L^{\frac{n}{m},q} \log^{\alpha} L(\Omega)$, $m < n$, $q \in (1, \infty]$, $\alpha < \frac{1}{q'}$, embedded into the Zygmund space $Z_0^{\frac{1}{q'}-\alpha}(\Omega)$.

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## 1. Introduction

Throughout this note, $\Omega$ is an open bounded set in $\mathbb{R}^n$, $n \geq 2$, $\omega_{n-1}$ is the surface of the unit sphere in $\mathbb{R}^n$ and $q' = \frac{q}{q-1}$ (with the standard convention that $1' = \infty$ and $\infty' = 1$).

If $W_0^{1,p}(\Omega)$ denotes the usual completition of $C_0^{\infty}(\Omega)$ in $W^{1,p}(\Omega)$, then the Sobolev Embedding Theorem states that

$$\begin{aligned} &W_0^{1,p}(\Omega) \subset L^{\frac{np}{n-p}}(\Omega) &&\text{if } 1 \leq p < n, \\ &W_0^{1,p}(\Omega) \subset L^{\infty}(\Omega) &&\text{if } n < p. \end{aligned}$$

In the borderline case $p = n$ we have from above

$$W_0^{1,n}(\Omega) \subset L^q(\Omega) \quad \text{for every} \quad q \in [1, \infty),$$

but there can be constructed examples showing that for the space $W_0^{1,n}(\Omega)$ we generally cannot take the limit of the exponent in the Sobolev Embedding Theorem, i.e.

$$W_0^{1,n}(\Omega) \not\subset L^{\infty}(\Omega).$$

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The lack of optimal target space for the Sobolev embedding of $W_0^{1,n}(\Omega)$ among the Lebesgue spaces inspired Trudinger [29] to show that

$$W_0^{1,n}(\Omega) \subset L^{\Phi}(\Omega),$$

where $L^{\Phi}(\Omega)$ is the Orlicz space corresponding to the Young function $\Phi(t) = \exp(t^{n'}) - 1$ (the same results were independently obtained by Yudovich [30] and Pohozhaev [27]). In particular, for any $K \geq 0$ and $u \in W_0^{1,n}(\Omega)$ we have

$$\int_{\Omega} \exp(K|u|^{n'}) < \infty\,. \tag{1.1}$$

Even though the integrals in (1.1) are always finite, they are not bounded by the same constant even if we consider the functions $u$ from the unit ball in $W_0^{1,n}(\Omega)$ only (by the unit ball in $W_0^{1,n}(\Omega)$ we mean the set $\{u \in W_0^{1,n}(\Omega) : \|\nabla u\|_{L^n(\Omega)} \leq 1\}$). This phenomenon is described by the Moser-Trudinger inequality [25]

$$\sup_{u \in W_0^{1,n}(\Omega), \|\nabla u\|_{L^n(\Omega)} \leq 1} \int_{\Omega} \exp(K|u|^{n'}) \begin{cases} \leq C(n, K, |\Omega|) & \text{when} \quad K \leq n\omega_{n-1}^{\frac{1}{n-1}} \\ = \infty & \text{when} \quad K > n\omega_{n-1}^{\frac{1}{n-1}}\,. \end{cases} \tag{1.2}$$

In the last two decades, the Moser-Trudinger inequality became a crucial tool when proving the existence and regularity of nontrivial weak solutions to elliptic partial differential equations with the nonlinearity having the critical growth (see for example the pioneering works [3] and [4] by Adimurthi). Further applications required several versions and generalizations of the Moser-Trudinger inequality such as a version for unbounded domains (see [1]), a version without boundary conditions (see [14]), the Concentration-Compactness Alternative (see [10] and [23]) and others. There are also Moser-type inequalities for higher order Sobolev spaces (see [2]), versions with the space $W_0^{1,n}(\Omega)$ replaced by Orlicz-Sobolev spaces (see [12], [13] and [22]), versions for the Lorentz-Sobolev spaces (see [5], [7] and [11]) and a version with the target exponential space understood as a Zygmund space (see [8]).

The aim of this note is to use several auxiliary estimates from [9], [11] and [22] to find the sharp constants concerning the Moser-type inequalities corresponding to embeddings into Zygmund spaces with the underlying Sobolev-type spaces being either Orlicz-Sobolev spaces with the borderline Sobolev-type embedding, or weighted Lorentz-Sobolev spaces with the borderline Sobolev-type embedding.

**Zygmund space $Z_0^{\eta}(\Omega)$.** The Zygmund space $Z_0^{\eta}(\Omega)$, $\eta > 0$, consists of all measurable functions satisfying

$$\lim_{t \to 0} \frac{u^*(t)}{(1 + \log(\frac{|\Omega|}{t}))^{\eta}} = 0,$$

which is equivalent to

$$\int_\Omega \exp(\lambda |u|^{\frac{1}{\eta}}) < \infty \quad \text{for every} \quad \lambda > 0.$$

Here $u^*$ denotes the non-decreasing rearrangement of $u$ defined as

$$u^*(t) = \inf\big\{s > 0 : |\{x \in \Omega : |u(x)| > s\}| \le t\big\} \quad \text{for} \quad t > 0.$$

The space $Z_0^\eta(\Omega)$ is often equipped with the quasinorm

$$\|u\|_{Z_0^\eta(\Omega)} = \sup_{t\in(0,|\Omega|)} \frac{u^*(t)}{(1+\log(\frac{|\Omega|}{t}))^\eta}. \tag{1.3}$$

This quantity is equivalent to an actual norm obtained by replacing $u^*(t)$ with $u^{**}(t) = \frac{1}{t}\int_0^t u^*(s)\,ds$.

**Orlicz-Sobolev case.** A function $\Phi : [0,\infty) \mapsto [0,\infty)$ is a Young function if $\Phi$ is increasing, convex, $\Phi(0) = 0$ and $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$.

We denote by $L^\Phi(\Omega)$ the Orlicz space corresponding to a Young function $\Phi$ on a set $\Omega$. This space is equipped with the Luxemburg norm

$$\|u\|_{L^\Phi(\Omega)} = \inf\Big\{\lambda > 0 : \int_\Omega \Phi\Big(\frac{|u|}{\lambda}\Big) \le 1\Big\}. \tag{1.4}$$

For an introduction to Orlicz spaces see e.g. [28].

We define the Orlicz-Sobolev space $WL^\Phi(\Omega)$ as the set

$$WL^\Phi(\Omega) := \{u \colon u, |\nabla u| \in L^\Phi(\Omega)\}$$

equipped with the norm

$$\|u\|_{WL^\Phi(\Omega)} := \|u\|_{L^\Phi(\Omega)} + \|\nabla u\|_{L^\Phi(\Omega)}.$$

We put $W_0L^\Phi(\Omega)$ for the closure of $C_0^\infty(\Omega)$ in $WL^\Phi(\Omega)$. In the sequel, we are interested in the spaces with borderline Sobolev-type embedding. These are the spaces $W_0L^\Phi(\Omega)$ with the Young function $\Phi$ satisfying

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^n \log^\alpha(t)} = 1, \qquad \alpha < n-1. \tag{1.5}$$

Since $\Omega$ is bounded, all Young functions $\Phi$ satisfying condition (1.5) (with $n$ and $\alpha$ fixed) give the same Orlicz-Sobolev space.

Next, let us define the constants related to our Moser-type inequality

$$\gamma = \frac{n}{n-1-\alpha} > 0, \qquad B = 1 - \frac{\alpha}{n-1} = \frac{n}{(n-1)\gamma} > 0 \quad \text{and} \quad K_{n,\alpha} = B^{\frac{1}{B}} n \omega_{n-1}^{\frac{\gamma}{n}} . \tag{1.6}$$

By the results from [26], Orlicz-Sobolev spaces with $\Phi$ satisfying (1.5) can be identified as the Lorentz-Sobolev spaces for which we have the embedding results from [15]–[19]. In particular, the space $W_0 L^\Phi(\Omega)$ with $\Phi$ satisfying (1.5) is embedded into an Orlicz space with the Young function that behaves like $\exp(t^\gamma)$ for large $t$. Moreover in the limiting case $\alpha = n-1$ we have the embedding into a double exponential space and for $\alpha > n-1$ we have the embedding into $L^\infty(\Omega)$. For further information we refer the reader to [13], [15], [16], [17], [18], [19], [20], [21] and [26].

The following theorem summarizes known versions of Moser-type inequality for embedding into single exponential spaces (see [9], [15] and [22]). For an information concerning the Moser-type inequalities for the spaces embedded into double and other multiple exponential spaces see [12].

**Theorem 1.1.** *Let $n \geq 2$, $\alpha < n-1$ and $K \geq 0$. Suppose that $\Omega \subset \mathbb{R}^n$ is an open bounded set. Let $\Phi$ be a Young function satisfying* (1.5).
(i) *If $u \in W_0 L^\Phi(\Omega)$, then*

$$\int_\Omega \exp\big(K|u(x)|^\gamma\big) < \infty .$$

(ii) *If $K \neq K_{n,\alpha}$, then*

$$\sup_{u \in W_0 L^\Phi(\Omega), \|\nabla u\|_{L^\Phi(\Omega)} \leq 1} \int_\Omega \exp\big(K|u|^\gamma\big) \begin{cases} \leq C(n,\alpha,\Phi,|\Omega|,K) & \text{when } K < K_{n,\alpha} \\ = \infty & \text{when } K > K_{n,\alpha}. \end{cases}$$

(iii) *Suppose that $K = K_{n,\alpha}$, there are $a \in (0, \min\{1, \frac{1}{\gamma}\})$ and $t_0 \geq \exp(1)$ such that $\Phi$ satisfies*

$$\Phi(t) \geq t^n \log^\alpha(t)\Big(1 + \log^{-a}(t)\Big) \qquad \text{for } t \in [t_0, \infty). \tag{1.7}$$

*Then*

$$\sup_{u \in W_0 L^\Phi(\Omega), \|\nabla u\|_{L^\Phi(\Omega)} \leq 1} \int_\Omega \exp\big(K|u(x)|^\gamma\big) \leq C(n,\alpha,\Phi,|\Omega|).$$

(iv) *Suppose that $K = K_{n,\alpha}$, there are $t_0 \geq \exp(1)$, $a \in (0, \min\{1, B\})$ and $C > 0$ such that*

$$\Phi(t) \leq \begin{cases} Ct^n & \text{for } t \in [0, t_0] \\ t^n \log^\alpha(t)\Big(1 - \log^{-a}(t)\Big) & \text{for } t \in [t_0, \infty). \end{cases} \tag{1.8}$$

*Then*

$$\sup_{u\in W_0L^\Phi(\Omega),\|\nabla u\|_{L^\Phi(\Omega)}\le 1}\int_\Omega \exp\big(K|u(x)|^\gamma\big)\,dx=\infty\,.$$

Notice that even though all Young functions $\Phi$ satisfying condition (1.5) (with $n$ and $\alpha$ fixed) give the same Orlicz-Sobolev space and the same Trudinger-type embedding, the validity of the Moser-type inequality in the case $K=K_{n,\alpha}$ depends on the choice of $\Phi$.

Now, we turn our attention to the embedding into Zygmund spaces. Since we are going to give a careful analysis of the Moser-type inequalities with respect to the behavior of the Young function $\Phi$ for large arguments, it is natural to consider the formula (1.3) with the supremum taken over $(0,\delta)$ with some small $\delta\in(0,|\Omega|)$ dependent on the choice of $\Phi$.

**Theorem 1.2.** *Let $n\ge 2$, $\alpha<n-1$ and $K\ge 0$. Suppose that $\Omega\subset\mathbb{R}^n$ is an open bounded set. Let $\Phi$ be a Young function satisfying* (1.5).
(i) *For every $K>K_{n,\alpha}^{-\frac{1}{\gamma}}$, there is $\delta\in(0,|\Omega|)$ such that*

$$\sup_{t\in(0,\delta),u\in W_0L^\Phi(\Omega),\|\nabla u\|_{L^\Phi(\Omega)}\le 1}\frac{u^*(t)}{(1+\log(\frac{|\Omega|}{t}))^{\frac{1}{\gamma}}}<K\,.$$

(ii) *For every $\delta\in(0,|\Omega|)$ we have*

$$\sup_{t\in(0,\delta),u\in W_0L^\Phi(\Omega),\|\nabla u\|_{L^\Phi(\Omega)}\le 1}\frac{u^*(t)}{(1+\log(\frac{|\Omega|}{t}))^{\frac{1}{\gamma}}}\ge K_{n,\alpha}^{-\frac{1}{\gamma}}\,.$$

(iii) *If $\Phi$ satisfies* (1.7), *then there is $\delta\in(0,|\Omega|)$ such that*

$$\sup_{u\in W_0L^\Phi(\Omega),\|\nabla u\|_{L^\Phi(\Omega)}\le 1}\frac{u^*(t)}{(1+\log(\frac{|\Omega|}{t}))^{\frac{1}{\gamma}}}<K_{n,\alpha}^{-\frac{1}{\gamma}}\qquad \textit{for every}\ \ t\in(0,\delta)$$

*and*

$$\sup_{t\in(0,\delta),u\in W_0L^\Phi(\Omega),\|\nabla u\|_{L^\Phi(\Omega)}\le 1}\frac{u^*(t)}{(1+\log(\frac{|\Omega|}{t}))^{\frac{1}{\gamma}}}=K_{n,\alpha}^{-\frac{1}{\gamma}}\,.$$

(iv) *If $\Phi$ satisfies* (1.8), *then for every $\delta\in(0,|\Omega|)$ we have*

$$\sup_{t\in(0,\delta),u\in W_0L^\Phi(\Omega),\|\nabla u\|_{L^\Phi(\Omega)}\le 1}\frac{u^*(t)}{(1+\log(\frac{|\Omega|}{t}))^{\frac{1}{\gamma}}}>K_{n,\alpha}^{-\frac{1}{\gamma}}\,.$$

**Lorentz-Sobolev case.** There also exist Moser-type inequalities for the generalized Lorentz-Sobolev spaces of the logarithmic-type embedded into exponential (and multiple exponential) spaces. By the generalized Lorentz spaces we mean the weighted Lorentz spaces introduced in [24].

Let $w : (0,\infty) \mapsto (0,\infty)$ be an integrable function. The space $L^{p,q;w}(\Omega)$ is the set of all measurable functions such that the following quantity is finite

$$\|u\|_{L^{p,q;w}(\Omega)} := \left\|u^*(t)t^{\frac{1}{p}-\frac{1}{q}}w(t)\right\|_{L^q(\Omega)} = \begin{cases} \left(\int_0^{|\Omega|}(u^*(t)t^{\frac{1}{p}-\frac{1}{q}}w(t))^q\,dt\right)^{\frac{1}{q}} \\ \qquad\qquad\qquad\qquad \text{if } \ q\in[1,\infty) \\ \sup_{t\in(0,|\Omega|)} u^*(t)t^{\frac{1}{p}}w(t) \\ \qquad\qquad\qquad\qquad \text{if } \ q=\infty\,. \end{cases}$$

Notice that our definition is a bit different from Lorentz's original definition (Lorentz considers $q$ finite only and he has $w^{\frac{1}{q}}$ instead of our $w$). Our version is chosen so that our definition was compatible with the notation in [15]-[19] (these papers provide us with Trudinger-type embedding for the generalized Lorentz-Sobolev spaces we are interested in).

We use the symbol $\nabla^m u$, $m\in\mathbb{N}$, to denote the $m$-th order gradient of $u\in C_0^\infty(\Omega)$, that is,

$$\nabla^m u = \begin{cases} \Delta^{\frac{m}{2}}u & \text{if } m \text{ is even} \\ \nabla\Delta^{\frac{m-1}{2}} & \text{if } m \text{ is odd,} \end{cases}$$

where $\nabla$ is the usual gradient operator and $\Delta$ is the Laplacian. By $|\nabla^m u|$ we denote the usual Euclidean length of the vector $\nabla^m u$. For general $u\in W_0^mL^{p,q;w}(\Omega)$, the $m$-th order gradient $\nabla^m u$ is considered in the distributional sense.

Let $m,n\in\mathbb{N}$, $1\le m<n$, $q\in(1,\infty]$ and $\alpha<\frac{1}{q'}$. Furthermore, we suppose that $w:(0,\infty)\mapsto(0,\infty)$ is a continuous function satisfying

$$\lim_{t\to 0_+}\frac{w(t)}{\log^\alpha(\frac{1}{t})}=1. \tag{1.9}$$

We define

$$D = 1-\alpha q' > 0, \tag{1.10}$$

$$\frac{1}{\gamma}=\frac{1}{q'}-\alpha=\frac{D}{q'}>0, \tag{1.11}$$

$$\beta_{n,m} = \begin{cases} \dfrac{\pi^{\frac{n}{2}}2^m\Gamma(\frac{m}{2})}{\omega_n^{\frac{n-m}{n}}\Gamma(\frac{n-m}{2})} & \text{if } m \text{ is even} \\ \dfrac{\pi^{\frac{n}{2}}2^m\Gamma(\frac{m+1}{2})}{\omega_n^{\frac{n-m}{n}}\Gamma(\frac{n-m+1}{2})} & \text{if } m \text{ is odd,} \end{cases}$$

where $\omega_n=\frac{\omega_{n-1}}{n}$ is the volume of the unit ball in $\mathbb{R}^n$. Further we set

$$K_{n,m,q,\alpha}=\beta_{n,m}^\gamma D^{\frac{1}{D}}\,. \tag{1.12}$$

The space $W_0^mL^{\frac{n}{m},q;w}(\Omega))$ of the Sobolev-type is continuously embedded into the Orlicz space with the Young function that behaves like $\exp(t^\gamma)$ for large $t$. In the limiting case $\alpha = \frac{1}{q'}$ we have the embedding into a double exponential space and so on. We refer the reader to [16], [17], [18], [19], [20] and [26].

Now, let us recall the Moser-type inequality for embedding into exponential Orlicz spaces obtained in [11].

**Theorem 1.3.** *Let $m, n \in \mathbb{N}$, $1 \leq m < n$, $q \in (1, \infty]$, $\alpha < \frac{1}{q'}$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose that the weight $w$ satisfies* (1.9).
(i) *If $K < K_{n,m,q,\alpha}$, then*

$$\sup_{u \in W_0^m L^{\frac{n}{m},q;w}(\Omega), \|\nabla^m u\|_{L^{\frac{n}{m},q;w}(\Omega)} \leq 1} \int_\Omega \exp(K|u|^\gamma) < \infty .$$

(ii) *If $K > K_{n,m,q,\alpha}$, then*

$$\sup_{u \in W_0^m L^{\frac{n}{m},q;w}(\Omega), \|\nabla^m u\|_{L^{\frac{n}{m},q;w}(\Omega)} \leq 1} \int_\Omega \exp(K|u|^\gamma) = \infty .$$

(iii) *Suppose that $K = K_{n,m,q,\alpha}$ and there are $t_0 > 0$ and $\mu \in (0, \min\{\frac{1}{\gamma}, \frac{1}{q'}\})$ such that*

$$w(t) \geq \log^\alpha\Big(\frac{1}{t}\Big)\Big(1 + \log^{-\mu}\Big(\frac{1}{t}\Big)\Big) \qquad \textit{for } \ t \in (0, t_0). \tag{1.13}$$

*Then*

$$\sup_{u \in W_0^m L^{\frac{n}{m},q;w}(\Omega), \|\nabla^m u\|_{L^{\frac{n}{m},q;w}(\Omega)} \leq 1} \int_\Omega \exp(K|u|^\gamma) < \infty .$$

(iv) *Suppose that $K = K_{n,m,q,\alpha}$, $m = 1$, $q \in [n, \infty)$ and there are $t_0 > 0$ and $\mu \in (0, \min\{D, 1\})$ such that the weight $w$ satisfies*

$$w(t) \leq \log^\alpha\Big(\frac{1}{t}\Big)\Big(1 - \log^{-\mu}\Big(\frac{1}{t}\Big)\Big) \qquad \textit{for } \ t \in (0, t_0). \tag{1.14}$$

*Then*

$$\sup_{u \in W_0^1 L^{n,q;w}(\Omega), \|\nabla u\|_{L^{n,q;w}(\Omega)} \leq 1} \int_\Omega \exp(K|u|^\gamma) = \infty .$$

The restrictive assumptions $m = 1$ and $q \in [n, \infty)$ in Theorem 1.3(iv) come from [11] where these assumptions were used to avoid some technical difficulties when constructing a version of Moser's sequence.

Now, we turn our attention to the second result of this paper which is a version of the previous result with the Orlicz target space replaced by a Zygmund space. Our result is a generalization of the results from [6] and [8].

**Theorem 1.4.** *Let $m, n \in \mathbb{N}$, $1 \le m < n$, $q \in (1, \infty]$, $\alpha < \frac{1}{q'}$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Suppose that the weight $w$ satisfies* (1.9)*.*

(i) *If $K > K_{n,m,q,\alpha}^{-\frac{1}{\gamma}}$, then there is $\delta \in (0, |\Omega|)$ such that*

$$\sup_{t\in(0,\delta),u\in W_0^m L^{\frac{n}{m},q;w}(\Omega),\|\nabla^m u\|_{L^{\frac{n}{m},q;w}(\Omega)}\le 1} \frac{u^*(t)}{(1+\log(\frac{|\Omega|}{t}))^{\frac{1}{\gamma}}} < K\,.$$

(ii) *For every $\delta \in (0, |\Omega|)$ we have*

$$\sup_{t\in(0,\delta),u\in W_0^m L^{\frac{n}{m},q;w}(\Omega),\|\nabla^m u\|_{L^{\frac{n}{m},q;w}(\Omega)}\le 1} \frac{u^*(t)}{(1+\log(\frac{|\Omega|}{t}))^{\frac{1}{\gamma}}} \ge K_{n,m,q,\alpha}^{-\frac{1}{\gamma}}\,.$$

(iii) *If $w$ satisfies* (1.13)*, then there is $\delta \in (0, |\Omega|)$ such that*

$$\sup_{u\in W_0^m L^{\frac{n}{m},q;w}(\Omega),\|\nabla^m u\|_{L^{\frac{n}{m},q;w}(\Omega)}\le 1} \frac{u^*(t)}{(1+\log(\frac{|\Omega|}{t}))^{\frac{1}{\gamma}}} < K_{n,m,q,\alpha}^{-\frac{1}{\gamma}} \quad \textit{for every} \;\; t \in (0,\delta)$$

*and*

$$\sup_{t\in(0,\delta),u\in W_0^m L^{\frac{n}{m},q;w}(\Omega),\|\nabla^m u\|_{L^{\frac{n}{m},q;w}(\Omega)}\le 1} \frac{u^*(t)}{(1+\log(\frac{|\Omega|}{t}))^{\frac{1}{\gamma}}} = K_{n,m,q,\alpha}^{-\frac{1}{\gamma}}\,.$$

(iv) *If $m = 1$, $q \in [n, \infty)$ and $w$ satisfies* (1.14)*, then for every $\delta \in (0, |\Omega|)$ we have*

$$\sup_{t\in(0,\delta),u\in W_0^1 L^{n,q;w}(\Omega),\|\nabla u\|_{L^{n,q;w}(\Omega)}\le 1} \frac{u^*(t)}{(1+\log(\frac{|\Omega|}{t}))^{\frac{1}{\gamma}}} > K_{n,1,q,\alpha}^{-\frac{1}{\gamma}}\,.$$

## 2. Preliminaries

By $B(x_0, R)$ we denote an open Euclidean ball in $\mathbb{R}^n$ centered at $x_0 \in \mathbb{R}^n$ with the radius $R > 0$. For the set $A \subset \mathbb{R}^n$, $|A|$ stands for the Lebesgue measure of $A$.

By $C$ we denote a generic positive constant. This constant may vary from expression to expression as usual.

In the rest of this section we recall some estimates and constructions from papers [9], [11] and [22] concerning the embeddings into exponential Orlicz spaces. These partial results are the main steps of our proofs (in fact, these estimates and constructions were also the most difficult parts of the proofs in papers [9], [11] and [22]).

Let us note that in the positive results, we can use the density of smooth functions and the equimeasurability of a function and its nonnegative radially symmetric rearrangement to pass to a.e. differentiable radially symmetric functions. Recall that the non-increasing radially symmetric rearrangement satisfies

$$u^{\star}(x) = u^*\Big(\frac{\omega_{n-1}}{n}|x|^n\Big) \qquad \text{for } \ x \in \Omega^{\star}\,, \tag{2.1}$$

where $\Omega^{\star}$ is a ball $B(0,R)$ such that $|B(0,R)| = |\Omega|$.

We would like to warn the reader that in papers [9], [11] and [22], there are considered the integrands $\exp(K|u|^\gamma)$, $\exp((K|u|)^\gamma)$ and $\exp((\frac{|u|}{K})^\gamma)$, respectively. Therefore several constants from these papers had to be modified so that the cited results from these papers were compatible with our notation (in this paper, we always consider the integrand $\exp(K|u|^\gamma)$).

**Orlicz-Sobolev case.** In the proof of [22, Theorem 1.1] it was shown that if the assumptions of Theorem 1.2(i) are satisfied, $u$ is a nonnegative radially symmetric function from the unit ball in $W_0L^\Phi(\Omega)$ and $g : [0,\infty) \mapsto [0,\infty)$ is such that $u(x) = g(|x|)$, then for every $\varepsilon > 0$ we can find $y_0 > 0$ such that

$$g(y) \le (1+\varepsilon)\frac{1}{\omega_{n-1}^{\frac{1}{n}} B^{\frac{n-1}{n}}} \log^{\frac{1}{\gamma}}\Big(\frac{1}{y}\Big) \qquad \text{for every } \ y \in (0, y_0).$$

Hence by (1.6) and (2.1) we have $t_0 > 0$ such that

$$\begin{aligned} u^*(t) &\le (1+\varepsilon)\frac{1}{\omega_{n-1}^{\frac{1}{n}} B^{\frac{n-1}{n}}} \log^{\frac{1}{\gamma}}\Big(\frac{1}{(\frac{n}{\omega_{n-1}}t)^{\frac{1}{n}}}\Big) \\ &= (1+\varepsilon) K_{n,\alpha}^{-\frac{1}{\gamma}} \log^{\frac{1}{\gamma}}\Big(\frac{1}{\frac{n}{\omega_{n-1}}t}\Big) \\ &\le (1+2\varepsilon) K_{n,\alpha}^{-\frac{1}{\gamma}} \log^{\frac{1}{\gamma}}\Big(\frac{1}{t}\Big) \qquad \text{for every } \ t \in (0, t_0). \end{aligned} \tag{2.2}$$

Similarly, under assumptions of Theorem 1.2(iii), the proof of [22, Theorem 4.2] gives us $a_1 \in (a, \min\{1, \frac{1}{\gamma}\})$ and $y_0 > 0$ such that

$$g(y) \le \frac{1}{\omega_{n-1}^{\frac{1}{n}} B^{\frac{n-1}{n}}} \log^{\frac{1}{\gamma}}\Big(\frac{1}{y}\Big)\Big(1 - \log^{-a_1}\Big(\frac{1}{y}\Big)\Big) \qquad \text{for every } \ y \in (0, y_0)$$

and thus we also have $t_0 > 0$ and $a_2 \in (a_1, \min\{1, \frac{1}{\gamma}\})$ such that

$$\begin{aligned} u^*(t) &\le \frac{1}{\omega_{n-1}^{\frac{1}{n}} B^{\frac{n-1}{n}}} \log^{\frac{1}{\gamma}}\Big(\frac{1}{(\frac{n}{\omega_{n-1}}t)^{\frac{1}{n}}}\Big)\Big(1 - \log^{-a_1}\Big(\frac{1}{(\frac{n}{\omega_{n-1}}t)^{\frac{1}{n}}}\Big)\Big) \\ &\le K_{n,\alpha}^{-\frac{1}{\gamma}} \log^{\frac{1}{\gamma}}\Big(\frac{1}{t}\Big)\Big(1 - \log^{-a_2}\Big(\frac{1}{t}\Big)\Big) \qquad \text{for every } \ t \in (0, t_0). \end{aligned} \tag{2.3}$$

The sharpness of our results is verified by constructing the following versions of Moser's sequence. First, let $x_0 \in \mathbb{R}^n$, $R > 0$ and $A > K_{n,\alpha}$. We define

$$w_k(x) = g_k(|x - x_0|)\,, \qquad \text{where}$$

$$g_k(y) = \begin{cases} 0 & \text{for } \ y \in [R, \infty) \\ (-\frac{2}{R}y + 2)A^{-\frac{1}{\gamma}} n^B \log^B(2) k^{\frac{1}{\gamma} - B} & \text{for } \ y \in [\frac{R}{2}, R] \\ A^{-\frac{1}{\gamma}} n^B \log^B(\frac{R}{y}) k^{\frac{1}{\gamma} - B} & \text{for } \ y \in [Re^{-\frac{k}{n}}, \frac{R}{2}] \\ A^{-\frac{1}{\gamma}} k^{\frac{1}{\gamma}} & \text{for } \ y \in [0, Re^{-\frac{k}{n}}]\,. \end{cases} \tag{2.4}$$

To obtain the sequence used in the borderline case (the case from Theorem 1.2(iv)), we set

$$\tilde{w}_k(x) = \tilde{g}_k(|x - x_0|)\,, \qquad \text{where} \tag{2.5}$$

$$\tilde{g}_k(y) = \begin{cases} 0 & \text{for } \ y \in [R, \infty) \\ (-\frac{2}{R}y + 2)K_{n,\alpha}^{-\frac{1}{\gamma}} n^B \log^B(2) k^{\frac{1}{\gamma} - B}\Big(1 + \frac{\log(k)}{k}\Big)^{\frac{1}{\gamma}} & \text{for } \ y \in [\frac{R}{2}, R] \\ K_{n,\alpha}^{-\frac{1}{\gamma}} n^B \log^B(\frac{R}{y}) k^{\frac{1}{\gamma} - B}\Big(1 + \frac{\log(k)}{k}\Big)^{\frac{1}{\gamma}} & \text{for } \ y \in [Re^{-\frac{k}{n}}, \frac{R}{2}] \\ K_{n,\alpha}^{-\frac{1}{\gamma}} k^{\frac{1}{\gamma}}\Big(1 + \frac{\log(k)}{k}\Big)^{\frac{1}{\gamma}} & \text{for } \ y \in [0, Re^{-\frac{k}{n}}]\,. \end{cases}$$

By the proof of [22, Theorem 1.2] we know that if the assumptions of Theorem 1.2(ii) are satisfied, then the members of the first sequence belong to the unit ball in $W_0L^\Phi(\Omega)$ for $k$ large enough. From the proof of [9, Example 5.1] we have that if the assumptions of Theorem 1.2(iv) are satisfied, then the members of the second sequence belong to the unit ball in $W_0L^\Phi(\Omega)$ for $k$ large enough.

**Lorentz-Sobolev case.** In the proof of [11, Theorem 1.4] it was shown that if the assumptions of Theorem 1.4(i) are satisfied and $u \in W_0^m L^{\frac{n}{m},q;w}(\Omega)$ with $\|\nabla^m u\|_{L^{\frac{n}{m},q;w}(\Omega)} \le 1$, then for given $\varepsilon > 0$ there is $t_0 > 0$ such that

$$u^*(t) \le (1+\varepsilon) K_{n,m,q,\alpha}^{-\frac{1}{\gamma}} \log^{\frac{1}{\gamma}}\Big(\frac{1}{t}\Big) \qquad \text{for every } \ t \in (0, t_0). \tag{2.6}$$

Under assumptions of Theorem 1.4(iii), the proof of [11, Proposition 5.1] gives us $\mu_1 \in (\mu, \min\{\frac{1}{\gamma}, \frac{1}{q'}\})$ and $t_0 > 0$ such that

$$u^*(t) \le K_{n,m,q,\alpha}^{-\frac{1}{\gamma}} \log^{\frac{1}{\gamma}}\Big(\frac{1}{t}\Big)\Big(1 - \log^{-\mu_1}\Big(\frac{1}{t}\Big)\Big) \qquad \text{for every } \ t \in (0, t_0). \tag{2.7}$$

The sharpness of the results in paper [11] was obtained by constructing similar sequences as in (2.4) and (2.5) (with different multiplicative constants and with some smoothing). By the proof of [11, Theorem 1.4(ii)] we have that under assumptions of Theorem 1.4(ii), for given $x_0 \in \mathbb{R}^n$, $R > 0$ and $A > K_{n,m,q,\alpha}$ there is a sequence of smooth functions from the unit ball in $W_0^m L^{\frac{n}{m},q;w}(\Omega)$ and supported on $B(x_0, R)$ satisfying

(2.8) $$w_k(x) = A^{-\frac{1}{\gamma}} k^{\frac{1}{\gamma}} \quad \text{for } x \in B(x_0, Re^{-\frac{k}{n}}) \text{ and } k \text{ sufficiently large.}$$

Furthermore, by the proof of [11, Proposition 5.2], under assumptions of Theorem 1.4(iv), for given $x_0 \in \mathbb{R}^n$ and $R > 0$, there are smooth functions from the unit ball in $W_0^1 L^{n,q;w}(\Omega)$ and supported on $B(x_0, R)$ satisfying

(2.9)
$$\tilde{w}_k(x) = K_{n,1,q,\alpha}^{-\frac{1}{\gamma}} k^{\frac{1}{\gamma}} \Big(1 + \frac{\log(k)}{k}\Big)^{\frac{1}{\gamma}} \quad \text{for } x \in B(x_0, Re^{-\frac{k}{n}}) \text{ and } k \text{ sufficiently large.}$$

## 3. Proofs of Theorem 1.2 and Theorem 1.4

Proof of Theorem 1.2(i): Fix $K > K_{n,\alpha}^{-\frac{1}{\gamma}}$. We can find $\varepsilon > 0$ so small that $K > (1 + 4\varepsilon) K_{n,\alpha}^{-\frac{1}{\gamma}}$. Let $t_0 > 0$ be such that (2.2) holds on $(0, t_0)$ with this $\varepsilon$. If $\delta \in (0, t_0)$ is sufficiently small, then we obtain from (2.2) for every $u \in W_0 L^{\Phi}(\Omega)$ such that $\|\nabla u\|_{L^{\Phi}(\Omega)} \leq 1$

$$\frac{u^*(t)}{(1 + \log(\frac{|\Omega|}{t}))^{\frac{1}{\gamma}}} \leq \frac{(1 + 2\varepsilon) K_{n,\alpha}^{-\frac{1}{\gamma}} \log^{\frac{1}{\gamma}}(\frac{1}{t})}{\log^{\frac{1}{\gamma}}(\frac{|\Omega|}{t})} \leq (1 + 3\varepsilon) K_{n,\alpha}^{-\frac{1}{\gamma}} < \frac{1 + 3\varepsilon}{1 + 4\varepsilon} K \ \text{ for } \ t \in (0, \delta)$$

and the result follows. □

Proof of Theorem 1.2(ii): Fix $K < K_{n,\alpha}^{-\frac{1}{\gamma}}$, $A \in (K_{n,\alpha}, K^{-\gamma})$, $\delta \in (0, |\Omega|)$ and $B(x_0, R) \subset \Omega$. Now, the sequence given by (2.4) satisfies

$$w_k^*\Big(\frac{\omega_{n-1}}{n} R^n e^{-k}\Big) = A^{-\frac{1}{\gamma}} k^{\frac{1}{\gamma}} \qquad \text{for every } \ k \in \mathbb{N}.$$

If $k_0 \in \mathbb{N}$ is sufficiently large, then $\frac{\omega_{n-1}}{n} R^n e^{-k} < \delta$ and $\|\nabla w_k\|_{L^{\Phi}(\Omega)} \leq 1$ for every $k > k_0$. Therefore we have

$$\begin{aligned} \sup_{t \in (0,\delta), u \in W_0 L^{\Phi}(\Omega), \|\nabla u\|_{L^{\Phi}(\Omega)} \leq 1} \frac{u^*(t)}{(1 + \log(\frac{|\Omega|}{t}))^{\frac{1}{\gamma}}} &\geq \sup_{k > k_0} \frac{w_k^*(\frac{\omega_{n-1}}{n} R^n e^{-k})}{(1 + \log(\frac{|\Omega|}{\frac{\omega_{n-1}}{n} R^n e^{-k}}))^{\frac{1}{\gamma}}} \\ &\geq \sup_{k > k_0} \frac{A^{-\frac{1}{\gamma}} k^{\frac{1}{\gamma}}}{(k + C)^{\frac{1}{\gamma}}} > K. \end{aligned}$$

Since $K < K_{n,\alpha}^{-\frac{1}{\gamma}}$ was arbitrary, the assertion follows. □

Proof of Theorem 1.2(iii): Let us prove the first assertion. If $u \in W_0 L^{\Phi}(\Omega)$, $\|\nabla u\|_{L^{\Phi}(\Omega)} \leq 1$, and $\delta$ is sufficiently small, then we can use (2.3) and $a_2 < 1$ to obtain

$$
\tag{3.1}
\begin{aligned}
\frac{u^*(t)}{(1+\log(\frac{|\Omega|}{t}))^{\frac{1}{\gamma}}} &\leq \frac{K_{n,\alpha}^{-\frac{1}{\gamma}}\log^{\frac{1}{\gamma}}(\frac{1}{t})(1-\log^{-a_2}(\frac{1}{t}))}{\log^{\frac{1}{\gamma}}(\frac{|\Omega|}{t})} \leq \frac{K_{n,\alpha}^{-\frac{1}{\gamma}}(1-\log^{-a_2}(\frac{1}{t}))}{(1-\frac{|\log(|\Omega|)|}{\log(\frac{1}{t})})^{\frac{1}{\gamma}}} \\
&\leq \frac{K_{n,\alpha}^{-\frac{1}{\gamma}}(1-\log^{-a_2}(\frac{1}{t}))}{(1-C\log^{-1}(\frac{1}{t}))} < K_{n,\alpha}^{-\frac{1}{\gamma}} \qquad \text{for every } t\in(0,\delta)
\end{aligned}
$$

and the first assertion follows. The second assertion plainly follows from the first one and from Theorem 1.2(ii). □

Proof of Theorem 1.2(iv): Fix $\delta\in(0,|\Omega|)$ and $B(x_0,R)\subset\Omega$. The sequence given by (2.5) satisfies

$$
\tilde{w}_k^*\Big(\frac{\omega_{n-1}}{n}R^n e^{-k}\Big) = K_{n,\alpha}^{-\frac{1}{\gamma}}k^{\frac{1}{\gamma}}\Big(1+\frac{\log(k)}{k}\Big)^{\frac{1}{\gamma}} \qquad \text{for every} \quad k\in\mathbb{N}\,.
$$

If $k_0\in\mathbb{N}$ is sufficiently large, then $\frac{\omega_{n-1}}{n}R^n e^{-k}<\delta$ and $\|\nabla\tilde{w}_k\|_{L^\Phi(\Omega)}\leq 1$ for every $k>k_0$. Therefore

$$
\begin{aligned}
\sup_{t\in(0,\delta),u\in W_0L^\Phi(\Omega),\|\nabla u\|_{L^\Phi(\Omega)}\leq 1}\frac{u^*(t)}{(1+\log(\frac{|\Omega|}{t}))^{\frac{1}{\gamma}}} &\geq \sup_{k>k_0}\frac{\tilde{w}_k^*(\frac{\omega_{n-1}}{n}R^ne^{-k})}{(1+\log(\frac{|\Omega|}{\frac{\omega_{n-1}}{n}R^ne^{-k}}))^{\frac{1}{\gamma}}} \\
&\geq \sup_{k>k_0}\frac{K_{n,\alpha}^{-\frac{1}{\gamma}}k^{\frac{1}{\gamma}}(1+\frac{\log(k)}{k})^{\frac{1}{\gamma}}}{(k+C)^{\frac{1}{\gamma}}} \\
&= \sup_{k>k_0}\frac{K_{n,\alpha}^{-\frac{1}{\gamma}}(k+\log(k))^{\frac{1}{\gamma}}}{(k+C)^{\frac{1}{\gamma}}} > K_{n,\alpha}^{-\frac{1}{\gamma}}\,.
\end{aligned}
$$

□

Proof of Theorem 1.4(i): Fix $K>K_{n,m,q,\alpha}^{-\frac{1}{\gamma}}$. Let $\varepsilon>0$ be so small that $K>(1+3\varepsilon)K_{n,m,q,\alpha}^{-\frac{1}{\gamma}}$. Let $t_0>0$ be such that (2.6) holds on $(0,t_0)$ with this $\varepsilon$. If $\delta\in(0,t_0)$ is sufficiently small, then we obtain from (2.6) for every $u\in W_0^mL^{\frac{n}{m},q;w}(\Omega)$ with $\|\nabla^m u\|_{L^{\frac{n}{m},q;w}(\Omega)}\leq 1$

$$
\begin{aligned}
\frac{u^*(t)}{(1+\log(\frac{|\Omega|}{t}))^{\frac{1}{\gamma}}} &\leq \frac{(1+\varepsilon)K_{n,m,q,\alpha}^{-\frac{1}{\gamma}}\log^{\frac{1}{\gamma}}(\frac{1}{t})}{\log^{\frac{1}{\gamma}}(\frac{|\Omega|}{t})} \\
&\leq (1+2\varepsilon)K_{m,n,q,\alpha}^{-\frac{1}{\gamma}} < \frac{1+2\varepsilon}{1+3\varepsilon}K \qquad \text{for} \quad t\in(0,\delta)
\end{aligned}
$$

and the result follows. □

Proof of Theorem 1.4(ii): Fix $K<K_{n,m,q,\alpha}^{-\frac{1}{\gamma}}$, $\delta\in(0,|\Omega|)$ and $A\in(K_{n,m,q,\alpha},K^{-\gamma})$. Let $\{w_k\}$ be the sequence from (2.8). If $k_0\in\mathbb{N}$ is large

enough, then $\frac{\omega_{n-1}}{n}R^n e^{-k} < \delta$ and $\|\nabla^m w_k\|_{L^{\frac{n}{m},q;w}(\Omega)} \leq 1$ for every $k > k_0$. Therefore

$$\sup_{t\in(0,\delta),u\in W_0^m L^{\frac{n}{m},q;w}(\Omega),\|\nabla^m u\|_{L^{\frac{n}{m},q;w}(\Omega)}\leq 1} \frac{u^*(t)}{(1+\log(\frac{|\Omega|}{t}))^{\frac{1}{\gamma}}}$$
$$\geq \sup_{k>k_0} \frac{w_k^*(\frac{\omega_{n-1}}{n}R^n e^{-k})}{(1+\log(\frac{|\Omega|}{\frac{\omega_{n-1}}{n}R^n e^{-k}}))^{\frac{1}{\gamma}}} \geq \sup_{k>k_0} \frac{A^{-\frac{1}{\gamma}} k^{\frac{1}{\gamma}}}{(k+C)^{\frac{1}{\gamma}}} > K.$$

Since $K < K_{n,m,q,\alpha}^{-\frac{1}{\gamma}}$ was arbitrary, the assertion follows. □

Proof of Theorem 1.4(iii): Let us prove the first assertion. If the function $u$ satisfies $u \in W_0^m L^{\frac{n}{m},q;w}(\Omega)$, $\|\nabla^m u\|_{L^{\frac{n}{m},q;w}(\Omega)} \leq 1$ and $\delta$ is sufficiently small, then we can use (2.7) and $\mu_1 < 1$ to obtain

$$\frac{u^*(t)}{(1+\log(\frac{|\Omega|}{t}))^{\frac{1}{\gamma}}} \leq \frac{K_{n,m,q,\alpha}^{-\frac{1}{\gamma}} \log^{\frac{1}{\gamma}}(\frac{1}{t})(1-\log^{-\mu_1}(\frac{1}{t}))}{\log^{\frac{1}{\gamma}}(\frac{|\Omega|}{t})} \leq \frac{K_{n,m,q,\alpha}^{-\frac{1}{\gamma}}(1-\log^{-\mu_1}(\frac{1}{t}))}{(1-\frac{|\log(|\Omega|)|}{\log(\frac{1}{t})})^{\frac{1}{\gamma}}}$$
$$\leq \frac{K_{n,m,q,\alpha}^{-\frac{1}{\gamma}}(1-\log^{-\mu_1}(\frac{1}{t}))}{(1-C\log^{-1}(\frac{1}{t}))} < K_{n,m,q,\alpha}^{-\frac{1}{\gamma}} \qquad \text{for } t \in (0,\delta)$$

and the first assertion follows. The second assertion plainly follows from the first one and from Theorem 1.4(ii). □

Proof of Theorem 1.4(iv): Let $\{\tilde{w}_k\}$ be the sequence from (2.9). If $k_0 \in \mathbb{N}$ is large enough, then $\frac{\omega_{n-1}}{n}R^n e^{-k} < \delta$ and $\|\nabla \tilde{w}_k\|_{L^{n,q;w}(\Omega)} \leq 1$ for every $k > k_0$. Therefore

$$\sup_{t\in(0,\delta),u\in W_0^1 L^{n,q;w}(\Omega),\|\nabla u\|_{L^{n,q;w}(\Omega)}\leq 1} \frac{u^*(t)}{(1+\log(\frac{|\Omega|}{t}))^{\frac{1}{\gamma}}}$$
$$\geq \sup_{k>k_0} \frac{\tilde{w}_k^*(\frac{\omega_{n-1}}{n}R^n e^{-k})}{(1+\log(\frac{|\Omega|}{\frac{\omega_{n-1}}{n}R^n e^{-k}}))^{\frac{1}{\gamma}}}$$
$$\geq \sup_{k>k_0} \frac{K_{n,1,q,\alpha}^{-\frac{1}{\gamma}} k^{\frac{1}{\gamma}}(1+\frac{\log(k)}{k})^{\frac{1}{\gamma}}}{(k+C)^{\frac{1}{\gamma}}}$$
$$\geq \sup_{k>k_0} \frac{K_{n,1,q,\alpha}^{-\frac{1}{\gamma}}(k+\log(k))^{\frac{1}{\gamma}}}{(k+C)^{\frac{1}{\gamma}}} > K_{n,1,q,\alpha}^{-\frac{1}{\gamma}}.$$

□

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