Notes on the Fučík spectrum and the mixed boundary value problem

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Abstract. The paper is devoted to the study of the properties of the Fučík spectrum. In the first part, we analyse the Fučík spectra of the problems with one second order ordinary differential equation with Dirichlet, Neumann and mixed boundary conditions and we present the explicit form of nontrivial solutions.

Then, we discuss the problem with two second order differential equations with mixed boundary conditions. We show the relation between the Dirichlet boundary value problem and mixed boundary value problem; using results of E. Massa and B. Ruf, we derive some properties of the Fučík spectrum of the mixed boundary value problem. Finally, we introduce a new proof of the closedness of the Fučík spectrum and a lemma about convergence of the corresponding nontrivial solutions.

Keywords: Fučík spectrum, system of ordinary differential equations of the second order, Dirichlet, Neumann and mixed boundary conditions

Classification: 34A34, 34B15, 47J10

Introduction

The aim of this paper is to summarize known results about Fučík spectra of a second order differential equation and for a system of two equations of this type and to study the problem with mixed boundary conditions.

We introduce the one equation problem in the first section and the descriptions of Fučík spectra of the problem with Dirichlet and Neumann boundary conditions in the second one. As we have not found the explicit form of eigenfunctions for Dirichlet and Neumann problem in the literature, we present it here. In the third section, we apply the idea of symmetric extension of solutions of mixed boundary value problem to prove that they are solutions of Dirichlet boundary value problem on a different domain.

The fourth section is devoted to a two equation problem. We use works of E. Massa and B. Ruf concerning the Fučík spectra of Dirichlet and Neumann problems. Applying the symmetrical extension, we obtain new results for the mixed boundary conditions which have not been studied yet. Most of them are consequences of the known results for Dirichlet and Neumann boundary value

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problem. In some cases (see Theorem 1, Remark 2, last section), we get stronger results.

In the last section, we present a more straightforward proof of the closedness of the Fučík spectrum of the mixed boundary value problem.

1. The Fučík spectrum

First, we define the main problem and its boundary conditions.

By $W^{1,2}(I)$, we denote the usual Sobolev space, i.e.

$$W^{1,2}(I) = \{u \in L^2(I);\ u' \in L^2(I)\}.$$

Definition 1. Let $I \subset \mathbb{R}$ be a bounded interval. Let μ, ν be given real numbers. Let the problem be given by the equation

$$-u'' = \mu u^+ - \nu u^- \quad \text{on } I, \tag{1}$$

where $u^\pm$ is defined by $u^+(t) = \max\{u(t); 0\}$ and by $u^-(t) = \max\{-u(t); 0\}$.

We call a function u the weak solution of the problem (1) with

1. Dirichlet boundary conditions if $u \in W_0^{1,2}(I)$ and

$$\int_I u'\varphi' = \mu \int_I u^+\varphi - \nu \int_I u^-\varphi \tag{2}$$

   holds for every function $\varphi \in W_0^{1,2}(I)$;
2. Neumann boundary conditions if $u \in W^{1,2}(I)$ and (2) holds for every function $\varphi \in W^{1,2}(I)$;
3. mixed boundary conditions if $u \in W^{1,2}(I)$, $u(a) = 0$ and (2) holds for every function

$$\varphi \in \{f \in W^{1,2}(I);\ f(a) = 0\}$$

   provided $I = (a, b)$.

To simplify, the problem with Dirichlet boundary conditions is called the Dirichlet problem and the others are shortened in the same way.

Now, we can proceed with the definition of the Fučík spectrum.

Definition 2. We define the Fučík spectrum of the problem (1) with boundary conditions 1, 2 or 3 to be the set of such couples $(\mu; \nu) \in \mathbb{R}^2$ that the problem (1) with parameters μ, ν and given boundary conditions has a nontrivial weak solution.

2. The Fučík spectra of the problem with one equation

Before pursuing two equation problems, we would like to summarize results concerning one equation problems.

According to the classical eigenvalue problem and some basic computations, we can restrict to nonnegative parameters μ, ν because for negative parameters the problem (1) has only the trivial solution.

In the 1970's, S. Fučík investigated the Dirichlet problem.

Lemma 1 ([3, Lemma 2.8]). *The problem* (1) *with Dirichlet boundary conditions on the interval* $[0,\pi]$ *has a nontrivial solution if and only if at least one of the following conditions holds:*

1. $\mu = 1$, ν *is arbitrary,*
2. μ *is arbitrary,* $\nu = 1$,
3. $\mu > 1$, $\nu > 1$ *and* $\frac{\sqrt{\mu}\sqrt{\nu}}{\sqrt{\mu}+\sqrt{\nu}} \in \mathbb{N}$,
4. $\mu > 1$, $\nu > 1$ *and* $\frac{\sqrt{\mu}(\sqrt{\nu}-1)}{\sqrt{\mu}+\sqrt{\nu}} \in \mathbb{N}$,
5. $\mu > 1$, $\nu > 1$ *and* $\frac{\sqrt{\nu}(\sqrt{\mu}-1)}{\sqrt{\mu}+\sqrt{\nu}} \in \mathbb{N}$.

Note 1. *We can explicitly write the form of solutions of the Dirichlet problem on* $[0,\pi]$ *with parameters* $(\mu;\nu)$, $\mu\nu \neq 0$. *Set*

$$c_k := k\pi\frac{\sqrt{\mu}+\sqrt{\nu}}{\sqrt{\mu\nu}}, \quad d_k := c_k + \frac{\pi}{\sqrt{\mu}}, \quad e_k := c_k + \frac{\pi}{\sqrt{\nu}}$$

for $k \in \mathbb{N}_0$. *Then the function*

$$u_D(t) := \begin{cases} c\sin\left(\sqrt{\mu}(t-c_k)\right) & \text{for } t \in [c_k, d_k] \cap [0,\pi], \\ -\frac{\sqrt{\mu}}{\sqrt{\nu}} c\sin\left(\sqrt{\nu}(t-d_k)\right) & \text{for } t \in [d_k, c_{k+1}] \cap [0,\pi], \end{cases}$$

for $c > 0$ *and the function*

$$\widetilde{u}_D(t) := \begin{cases} -d\sin\left(\sqrt{\nu}(t-c_k)\right) & \text{for } t \in [c_k, e_k] \cap [0,\pi], \\ \frac{\sqrt{\nu}}{\sqrt{\mu}} d\sin\left(\sqrt{\mu}(t-e_k)\right) & \text{for } t \in [e_k, c_{k+1}] \cap [0,\pi], \end{cases}$$

for $d > 0$ *are the weak solutions of the Dirichlet problem* (1). *Obviously,* u_D *and* $\widetilde{u}_D$ *are* C^2*-functions and thus the classical solutions.*

For the Neumann problem, it is easy to obtain the following lemma.

Lemma 2. *The point* $(\mu;\nu)$, $\mu,\nu \geq 0$ *belongs to the Fučík spectrum of the Neumann problem* (1) *on the interval* $[0,\pi]$ *if and only if it satisfies one of the following conditions:*

1. $\mu\nu = 0$,
2. $\mu > 0$, $\nu > 0$ *and* $\frac{2\sqrt{\mu}\sqrt{\nu}}{\sqrt{\mu}+\sqrt{\nu}} \in \mathbb{N}$.

Note 2. *We are able to write the explicit form of solutions for the Neumann problem on* $[0,\pi]$ *with parameters* $(\mu;\nu)$, $\mu\nu \neq 0$ *as well. Define*

$$c_k^N := k\frac{\pi}{2}\left(\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}}\right) = \frac{k\pi(\sqrt{\mu}+\sqrt{\nu})}{2\sqrt{\mu\nu}}$$

for $k \in \mathbb{N}_0$. Then the function

$$u_N(t) := \begin{cases} c\cos\left(\sqrt{\mu}(t-c_{2k}^N)\right) & \text{for } t \in [c_{2k}^N - \frac{\pi}{2\sqrt{\mu}}, c_{2k}^N + \frac{\pi}{2\sqrt{\mu}}] \cap [0,\pi], \\ -\frac{\sqrt{\mu}}{\sqrt{\nu}} c\cos\left(\sqrt{\nu}(t-c_{2k+1})\right) & \text{for } t \in [c_{2k+1}^N - \frac{\pi}{2\sqrt{\nu}}, c_{2k+1}^N + \frac{\pi}{2\sqrt{\nu}}] \cap [0,\pi], \end{cases}$$

for $c > 0$ and the function

$$\widetilde{u}_N(t) := \begin{cases} -d\cos\left(\sqrt{\nu}(t-c_{2k}^N)\right) & \text{for } t \in [c_{2k}^N - \frac{\pi}{2\sqrt{\nu}}, c_{2k}^N + \frac{\pi}{2\sqrt{\nu}}] \cap [0,\pi], \\ \frac{\sqrt{\nu}}{\sqrt{\mu}} d\cos\left(\sqrt{\mu}(t-c_{2k+1}^N)\right) & \text{for } t \in [c_{2k+1}^N - \frac{\pi}{2\sqrt{\mu}}, c_{2k+1}^N + \frac{\pi}{2\sqrt{\mu}}] \cap [0,\pi], \end{cases}$$

for $d > 0$ are the weak solutions of the Neumann problem (1). *Again, u_N and $\widetilde{u}_N$ are evidently C^2-functions and thus the classical solutions.*

Note 3. *As we can see, the Fučík spectra of problems with one equation can be described as a countable union of curves. These curves are disjoint in the Neumann case* (*except curves $\mu \equiv 0$ and $\nu \equiv 0$*). *In the Dirichlet case, they are almost disjoint — each of them intersects at most one of the other curves and it holds at exactly one point which is equal to an odd eigenvalue of the classical spectrum with Dirichlet boundary conditions on* $[0,\pi]$. *See Figures 1–3.*

3. The symmetrical extension

While solving the mixed problem on the interval $[0,\pi]$, we can observe its very close relation with the Dirichlet problem on $[0,2\pi]$.

Let u be a function belonging to $\mathcal{K} := \{u \in W^{1,2}((0,\pi));\ u(0)=0\}$. We denote by $\bar{u}$ a function defined on $[0,2\pi]$ in the following way:

$$\bar{u}(t) = \begin{cases} u(t) & t \in [0,\pi], \\ u(2\pi - t) & t \in [\pi, 2\pi]. \end{cases}$$

Using the fact that functions from $W^{1,2}(0,\pi)$ are absolutely continuous on $[0,\pi]$, we can prove that $\bar{u} \in W_0^{1,2}(0,2\pi)$.

Lemma 3. *Let $u \in \mathcal{K}$ be nontrivial. Then u is a solution of the problem* (1) *with mixed boundary conditions on the interval* $[0,\pi]$ *if and only if the function $\bar{u}$ is a solution of the Dirichlet problem on the interval* $[0,2\pi]$ *provided* $\bar{u}'(\pi) = 0$.

Note 4. *The assumption $\bar{u}'(\pi) = 0$ is unambiguous since we know that solutions of the Dirichlet problem are C^2-functions* (*see* [2, p. 317]).

Proof: If we have a solution u of the Dirichlet problem on $[0,2\pi]$ which satisfies $u'(\pi) = 0$, its restriction to $[0,\pi]$ is clearly a solution of the mixed boundary problem on $[0,\pi]$.

For the opposite direction, we use the definition of a weak solution. Let $\psi \in W_0^{1,2}(0,2\pi)$. We need to verify that the equation (2) with ψ holds. Set $\varphi_1(t) := \psi(t)$ for $t \in [0,\pi]$ and $\varphi_2(t) = \psi(2\pi - t)$ for $t \in [0,\pi]$. Then obviously $\varphi_1, \varphi_2 \in \mathcal{K}$.

Then, evidently, we have that

(3)
$$\int_0^\pi u'(t)\varphi_1'(t)\,\mathrm{d}t + \int_0^\pi u'(t)\varphi_2'(t)\,\mathrm{d}t = \int_0^\pi u'(t)\psi'(t)\,\mathrm{d}t + \int_\pi^{2\pi} u'(2\pi - t)\psi'(t)\,\mathrm{d}t = \int_0^{2\pi} \bar{u}'(t)\psi'(t)\,\mathrm{d}t.$$

At the same time, it holds that

$$\begin{aligned}\int_0^\pi u'(t)\varphi_1'(t)\,\mathrm{d}t + \int_0^\pi u'(t)\varphi_2'(t)\,\mathrm{d}t &= \mu\int_0^\pi u^+(t)\varphi_1(t)\,\mathrm{d}t \\ - \nu\int_0^\pi u^-(t)\varphi_1(t)\,\mathrm{d}t + \mu\int_0^\pi u^+(t)\varphi_2(t)\,\mathrm{d}t &- \nu\int_0^\pi u^-(t)\varphi_2(t)\,\mathrm{d}t \\ = \mu\int_0^{2\pi} \bar{u}^+(t)\psi(t)\,\mathrm{d}t - \nu\int_0^{2\pi} \bar{u}^-(t)\psi(t)\,\mathrm{d}t.\end{aligned} \tag{4}$$

The equations (3) and (4) show that $\bar{u}$ is a solution of the Dirichlet problem on $[0, 2\pi]$. □

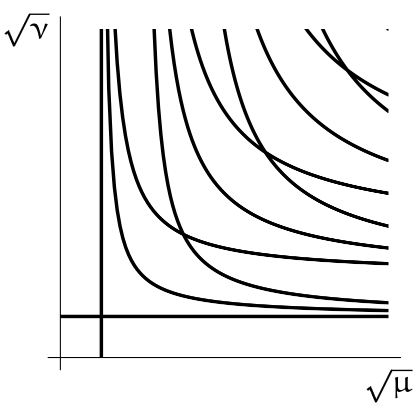

Figure 1. FS of the Dirichlet problem

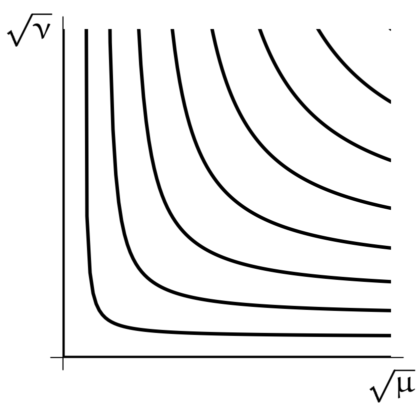

Figure 2. FS of the Neumann problem

Remark 1. It is natural to ask if we can extend a solution with respect to the point zero in any way and then search parameters for which there exists a nontrivial solution of the Neumann problem on $[-\pi, \pi]$ taking the zero value at the origin. The set of such parameters is very poor — the condition at origin holds only if the parameters are both equal to the same odd classical eigenvalue

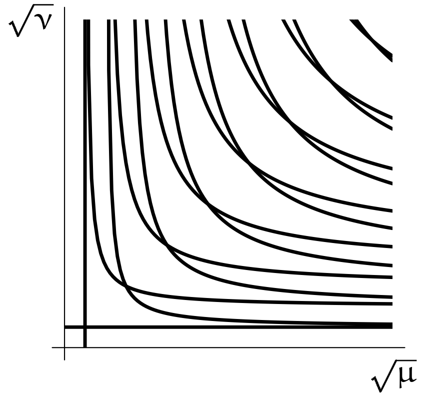

FIGURE 3. FS of the mixed problem

of the Neumann problem on $[-\pi, \pi]$, it means

$$\mu = \nu = \lambda_{2k-1}^N = \left(\frac{2k-1}{2}\right)^2, \ k \in \mathbb{N}.$$

In this case, a nontrivial solution equals to a classical eigenfunction of the Neumann problem (up to a multiple), exactly it is the solution $u(t) = c \sin \frac{2k-1}{2} t$ on $[-\pi, \pi]$ for $c \neq 0$.

By the above, the Fučík spectrum of the mixed problem is a subset of the Fučík spectrum of the Dirichlet problem on the double-long interval. We have to verify for which solutions of the Dirichlet problem the condition $\bar{u}'(\pi) = 0$ holds. This is not true only for parameters which satisfy condition similar to the claim 3 in Lemma 1. The result is summarized in the lemma below.

Lemma 4. *Let $(\mu; \nu) \in \mathbb{R}_+^2$. The point $(\mu; \nu)$ belongs to the Fučík spectrum of the problem on the interval $[0, \pi]$ with mixed boundary conditions if and only if it satisfies at least one of the following conditions:*

1. $\mu = \frac{1}{4}$, *ν is arbitrary,*
2. *μ is arbitrary,* $\nu = \frac{1}{4}$,
3. $\mu > \frac{1}{4}$, $\nu > \frac{1}{4}$ *and* $\frac{\sqrt{\nu}(2\sqrt{\mu}-1)}{\sqrt{\mu}+\sqrt{\nu}} \in \mathbb{N}$,
4. $\mu > \frac{1}{4}$, $\nu > \frac{1}{4}$ *and* $\frac{\sqrt{\mu}(2\sqrt{\nu}-1)}{\sqrt{\mu}+\sqrt{\nu}} \in \mathbb{N}$.

Note 5. *A solution of the mixed problem on $[0, \pi]$ extends to the solution of the Dirichlet one on $[0, 2\pi]$, so its explicit form for the corresponding parameters is given in Note 1.*

4. The Fučík spectrum of the two equation problem with mixed boundary conditions

This section is mainly the corollary of the symmetrical extension we mentioned above (but for the system of two equations) and results about the Dirichlet problem for such systems of Eugenio Massa and Bernhard Ruf which can be found in [4] and [5].

The problem that we want to study is given by the following equations:

$$\begin{aligned} -u''(t) &= \alpha v^+(t) - \beta v^-(t) \qquad \text{for } t \in [0,\pi], \\ -v''(t) &= \gamma u^+(t) - \delta u^-(t), \\ u(0) &= v(0) = 0, \\ u'(\pi) &= v'(\pi) = 0. \end{aligned} \tag{5}$$

Now, the test function space is $\mathcal{K} \times \mathcal{K} := \{(f;g) \in W^{1,2}((0,\pi)) \times W^{1,2}((0,\pi));$ $f(0) = 0 = g(0)\}$.

Definition 3. The set of points $(\alpha;\beta;\gamma;\delta) \in \mathbb{R}^4$ such that the problem (5) with parameters $\alpha,\beta,\gamma,\delta$ has a nontrivial weak solution is called the Fučík spectrum of the problem (5) and we denote it by Σ.

By using analogous arguments as in the previous section, it is easy to see that the couple $(u;v)$ is a solution of (5) if and only if the couple of symmetrically extended functions $(\bar{u};\bar{v})$ is a solution of the problem with the same equations on the interval $[0,2\pi]$ with the Dirichlet boundary conditions provided $u'(\pi) = v'(\pi) = 0$.

As a consequence, the Fučík spectrum of the mixed problem Σ is a subset of the Fučík spectrum of the Dirichlet problem on the interval $[0,2\pi]$ and thus Σ preserves all properties of the Fučík spectrum of the Dirichlet problem.

Due to known results ([4]; Lemma 2.1, Proposition 2.3), we can restrict to nonnegative parameters. If one of equations in (5) is identically zero, then the system has only the trivial solution. By basic computations, we get that if exactly one parameter vanishes in each of equations, we obtain a solution which does not change sign on the domain (it is either positive or negative) and we present it below. If a solution does change sign, then all parameters have to be strictly positive (strictly negative, respectively).

In this system, we can find some symmetries. The most important one is that if $(u;v)$ solves the problem with parameters $(\alpha;\beta;\gamma;\delta)$, then $(u;av)$ solves the problem with parameters $(\frac{\alpha}{a};\frac{\beta}{a};a\gamma;a\delta)$ for any $a > 0$. Therefore, we can find such a that $\frac{\alpha}{a} = a\gamma$.

The preceding ideas allow us to reduce the number of parameters. It is enough to study the set $\widehat{\Sigma}$ which is defined as a set of points $(\alpha;\beta;\delta) \in (\mathbb{R}_+)^3$ such that the problem (5) with parameters $\alpha,\beta,\alpha,\delta$ has a nontrivial weak solution. For more details see [4, Section 3].

We state now a lemma which is a slight modification of Lemma 3.2 in [5] because it is very useful below. The correctness of the modification of boundary conditions follows from the symmetrical extension.

Lemma 5 ([5, Lemma 3.2]). *Let $c, d \in L^\infty(0,\pi)$ with $c, d > 0$ a.e. and $(u;v)$ be a nontrivial solution of the boundary value problem on $(0,\pi)$*

$$\begin{aligned} -u''(t) &= c(t)v(t),\\ -v''(t) &= d(t)u(t),\\ u(0) = v(0) &= 0,\\ u'(\pi) = v'(\pi) &= 0. \end{aligned}$$

1. *For $x_0 \in [0,\pi]$, we have that*
   - *if $u(x_0) = 0$ or $v(x_0) = 0$, then $u'(x_0)v'(x_0) > 0$,*
   - *if $u'(x_0) = 0$ or $v'(x_0) = 0$, then $u(x_0)v(x_0) > 0$.*
2. *For $x_1, x_2 \in [0,\pi]$, $x_1 < x_2$, we have that*
   - *if $u'(x_1) = u'(x_2) = 0$ and $u'(x) \neq 0$ for $x \in (x_1, x_2)$, then $u(x_1)u(x_2) < 0$,*
   - *if $u(x_1) = u(x_2) = 0$ and $u(x) \neq 0$ for $x \in (x_1, x_2)$, then $u'(x_1)u'(x_2) < 0$.*

In the following text, we will assume that the nontrivial solution is normalized, it means $||u||_{L^2(0,\pi)} = ||v||_{L^2(0,\pi)} = 1$. By this condition, we avoid the problems with sequences of solutions in a form $\{(a_n u, a_n v)\}_{n=1}^\infty$ such that $a_n \to 0$ as n approaching the infinity.

Various properties of components of a solution are interconnected.

Lemma 6. *Let $(\alpha;\beta;\delta) \in \widehat{\Sigma}$ and $(u;v)$ be the corresponding nontrivial solution. Then the number of points $t \in (0,\pi)$ such that $u(t) = 0$ is finite and it is the same as the number of points $s \in (0,\pi)$ such that $v(s) = 0$.*

Proof: If u has no zero point in $(0,\pi)$, it is easy to figure out that $u(t) = \pm cv(t) = \pm d \sin\left(\frac{t}{2}\right)$ for some $c, d > 0$.

The claim was proved for the Dirichlet case provided u has a zero point in the interior of an interval (see [5, Proposition 3.4]). Thus $\bar{u}$ and $\bar{v}$ have the same number of zero points on $(0, 2\pi)$.

Suppose that $\bar{u}(\pi) = 0$. Then, by symmetry of the extension, $\bar{u}$ has an odd number of zeros and so does $\bar{v}$ by the claim for Dirichlet case. By symmetrical extension, necessarily $\bar{v}(\pi) = 0$. It means $u(\pi) = u'(\pi) = 0$ and $v(\pi) = v'(\pi) = 0$ and from the second equation in the system (5), $v''(\pi) = 0$ holds too. Using the uniqueness of the solution for differential equations with a Lipschitz right-hand side (see [1, Theorem 7.4]), we obtain that $v \equiv 0$, but it is a contradiction with assumptions of the claim. $\square$

E. Massa and B. Ruf [4, Proposition 3.2] proved that the Fučík spectrum is bounded away from zero in this sense:

Let $(\alpha;\beta;\delta)$ be a point of the Fučík spectrum $\widehat{\Sigma}$ and moreover, let $\alpha > 0$. Then $\alpha \geq \lambda_1^D$ and $\beta\delta \geq \lambda_1^D$. Here λ_1^D denotes the first eigenvalue of the classical spectrum problem with Dirichlet boundary conditions on $[0,2\pi]$ (which is the same as for the problem with mixed boundary conditions on $[0,\pi]$).

The relation between components of a solution is even stronger. Let $\alpha > \lambda_1^D$ and $\beta\delta > \lambda_1^D$. Then $u^+v^+ \not\equiv 0$ and $u^-v^- \not\equiv 0$. Moreover, at least one of multiples u^+v^- and u^-v^+ is not identically zero provided $\beta \neq \delta$.

Now, we can classify points which certainly do not belong to the Fučík spectrum. The similar claim was proven for the Dirichlet problem in [4, Proposition 4.1]. We do not give the proof here, because we use the methods adapted from mentioned article. However, we can get a stronger estimate on the set of points which do not belong to Fučík spectrum of the mixed problem. Namely, we have that

Theorem 1. *Let $\alpha_0, \beta_0, \delta_0$ be positive integers and let $a > 0$ be such that*

$$\left\{\frac{\alpha_0}{a}, \frac{\beta_0}{a}, a\alpha_0, a\delta_0\right\} \subset (\lambda_k, \lambda_{k+1})$$

where λ_k, λ_{k+1} denote the consecutive eigenvalues of the classical spectrum of the problem with mixed boundary value conditions on $[0,\pi]$. Then $(\alpha_0;\beta_0;\delta_0) \notin \widehat{\Sigma}$.

Note 6. *As we have written above, the first eigenvalue λ_1 of the classical spectrum problem with mixed boundary conditions on $[0,\pi]$ is the same as the first eigenvalue λ_1^D of the Dirichlet one on $[0,2\pi]$, it means $\lambda_1 = \lambda_1^D = \frac{1}{4}$. Generally, it holds that $\lambda_k = \lambda_{2k-1}^D = \left(\frac{2k-1}{2}\right)^2$ for all $k \in \mathbb{N}$, where λ_k (λ_k^D respectively) is the k-th eigenvalue of the classical spectrum of the mixed boundary value problem on $[0,\pi]$ (Dirichlet problem on $[0,2\pi]$ respectively).*

Due to results of E. Massa and B. Ruf about Dirichlet problem, we know that Fučík spectrum can be described as a countable union of the surfaces. These surfaces are globally given by graphs of C^1-functions — more precisely, the parameter α is a C^1-function of β and δ (see [5, Theorem 1.2]). The domain of α_+ is the set $\{(\beta;\delta) \in \mathbb{R}_+^2;\ \sqrt{\beta\delta} > \lambda_k^D\}$ and the domain of α_- is $\{(\beta;\delta) \in \mathbb{R}_+^2;\ \sqrt{\beta\delta} > \lambda_{k+1}^D\}$ (see [5, Proposition 4.9]).

Definition 4. Let $k \geq 2$ be an integer. Then we call the Fučík surfaces connected C^1-surfaces

$\Gamma_{k+} := \{(\alpha;\beta;\delta) \in \widehat{\Sigma};\ \alpha = \alpha_+(\beta,\delta);$ a corresponding solution of the problem (5) is positive in a right neighborhood of the origin$\}$

satisfying $(\lambda_k;\lambda_k;\lambda_k) \in \Gamma_{k+}$ and

$\Gamma_{k_-} := \{(\alpha;\beta;\delta) \in \widehat{\Sigma};\ \alpha = \alpha_-(\beta,\delta);$ a corresponding solution of the problem (5) is negative in a right neighborhood of the origin$\}$

satisfying $(\lambda_k; \lambda_k; \lambda_k) \in \Gamma_{k-}$. Moreover, we define

$$\Gamma_{1+} := \{(\alpha, \beta, \delta) \in \mathbb{R}^3_+;\ \alpha = \lambda_1\}$$

and

$$\Gamma_{1-} := \{(\alpha, \beta, \delta) \in \mathbb{R}^3_+;\ \beta\delta = \lambda_1\}.$$

It was proven ([5, Theorem 1.2]) that

$$\widehat{\Sigma} = \bigcup_{k=1}^{\infty} (\Gamma_{k+} \cup \Gamma_{k-}).$$

By [5, Proposition 4.10], the Fučík surfaces of different orders are disjoint:

$$(\Gamma_{k+} \cup \Gamma_{k-}) \cap (\Gamma_{l+} \cup \Gamma_{l-}) = \emptyset \ \text{ for all } \ k \neq l.$$

It was also proven in [5, Proposition 4.11], that in general, the surfaces Γ_{k+} and Γ_{k-} may coincide (and they do in case of Neumann problem and for even k in case of Dirichlet problem). But they do not coincide in the case of the mixed problem.

Remark 2. Because of the results of the first section, it is evident that there are two different unbounded curves which belong to the corresponding surfaces Γ_{k+} and Γ_{k-}. More precisely:

$$\gamma_{k+} := \left\{(\alpha; \beta; \beta) \in \mathbb{R}^3_+;\ \frac{\sqrt{\beta}(2\sqrt{\alpha} - 1)}{\sqrt{\alpha} + \sqrt{\beta}} = k\right\} \subset \Gamma_{k+},$$

and

$$\gamma_{k-} := \left\{(\alpha; \beta; \beta) \in \mathbb{R}^3_+;\ \frac{\sqrt{\alpha}(2\sqrt{\beta} - 1)}{\sqrt{\alpha} + \sqrt{\beta}} = k\right\} \subset \Gamma_{k-}.$$

Obviously, the intersection of these curves is the only point, exactly

$$\gamma_{k+} \cap \gamma_{k-} = \{(\alpha; \beta; \beta) \in \mathbb{R}^3_+;\ \alpha = \beta = \lambda_k\}.$$

Thus, the surfaces Γ_{k+} and Γ_{k-} do not coincide. The claim is evident for the surfaces Γ_{1+} and Γ_{1-}.

For further properties of the Fučík surfaces, we refer to [5].

5. The closedness of the Fučík spectrum

Although you can find a proof that the Fučík spectrum is a closed set in works of E. Massa and B. Ruf, we give here an alternative one which is more detailed and gives more than the simple closedness (see Lemma 7).

Theorem 2. *Let $\{A_n\}_{n=1}^{\infty}$ be a subset of $\widehat{\Sigma}$, $A_n = (\alpha_n; \beta_n; \delta_n)$. Let $A_n \to A = (\alpha; \beta; \delta) \in \mathbb{R}^3$. Then $A \in \widehat{\Sigma}$.*

PROOF: Denote by $\mathbf{u_n} := (u_n; v_n) \in W^{1,2}(0,\pi) \times W^{1,2}(0,\pi)$ a corresponding nontrivial solution of the problem (5) with parameters $\alpha_n, \beta_n, \delta_n$ such that

$$||u_n||_{L^2(0,\pi)} = ||v_n||_{L^2(0,\pi)} = 1.$$

We test the first equation in (5) with parameters $\alpha_n, \beta_n, \delta_n$ by the function u_n:

$$\begin{aligned} ||u_n'||^2_{L^2(0,\pi)} &= \left|\alpha_n \int_0^\pi v_n^+ u_n - \beta_n \int_0^\pi v_n^- u_n\right| \le (|\alpha_n| + |\beta_n|) \int_0^\pi |u_n v_n| \\ &\le ||u_n||_{L^2(0,\pi)} ||v_n||_{L^2(0,\pi)} (|\alpha_n| + |\beta_n|) = (|\alpha_n| + |\beta_n|) \end{aligned}$$

and analogically

$$||v_n'||^2_{L^2(0,\pi)} \le (|\alpha_n| + |\delta_n|).$$

The convergence of A_n implies that $\{||\mathbf{u_n}||_{W^{1,2}(0,\pi)\times W^{1,2}(0,\pi)}\}_{n=1}^\infty$ is bounded. Due to Eberlain-Šmuljan characterization of the reflexivity, we can choose a weakly convergent subsequence of $\{\mathbf{u_n}\}$. Because of the compact embedding of $W^{1,2}(0,\pi)$ into $L^2(0,\pi)$, we obtain even that

$$(u_n; v_n) \to (u; v) \text{ in } \left(L^2(0,\pi)\right)^2.$$

Obviously, $\sqrt{2} = ||\mathbf{u_n}||_{L^2(0,\pi)\times L^2(0,\pi)} \to ||\mathbf{u}||_{L^2(0,\pi)\times L^2(0,\pi)}$ and $\mathbf{u}$ is nontrivial.

By the compact embedding of $W^{1,2}(0,\pi)$ into $C([0,\pi])$, v_n converge to v uniformly. It is evident that $|v_n^\pm - v^\pm| \le |v_n - v|$. Therefore, $v_n^\pm \to v^\pm$ in $L^2(0,\pi)$.

Let $\varphi \in \mathcal{K}$. We can estimate

$$\begin{aligned} &\left|\alpha_n \int_0^\pi v_n^+ \varphi - \alpha \int_0^\pi v^+ \varphi\right| \\ &= \left|\alpha_n \int_0^\pi v_n^+ \varphi - \alpha_n \int_0^\pi v^+ \varphi + \alpha_n \int_0^\pi v^+ \varphi - \alpha \int_0^\pi v^+ \varphi\right| \\ &\le |\alpha_n| \left|\int_0^\pi v_n^+ \varphi - \int_0^\pi v^+ \varphi\right| + |\alpha_n - \alpha| \left|\int_0^\pi v^+ \varphi\right| \to 0. \end{aligned}$$

Since $\{\alpha_n\}$ is bounded and $\left|\int_0^\pi (v_n^+ - v^+)\varphi\right| \le \|v_n^+ - v^+\|_{L^2} \cdot \|\varphi\|_{L^2}$, the first term goes to zero. The convergence of $\alpha_n \to \alpha$ and $\left|\int_0^\pi v^+ \varphi\right| \le \|v^+\|_{L^2} \cdot \|\varphi\|_{L^2}$ imply the convergence of the second term.

Thus, it holds that

$$\alpha_n \int_0^\pi v_n^+ \varphi - \alpha \int_0^\pi v^+ \varphi \to 0. \tag{6}$$

The proof is analogous for other parameters.

As $\mathbf{u_n} \rightharpoonup \mathbf{u}$ in $W^{1,2}(0,\pi) \times W^{1,2}(0,\pi)$, we obtain for any $\varphi \in \mathcal{K}$ that

$$\int_0^\pi u_n'\varphi' \to \int_0^\pi u'\varphi', \tag{7}$$

$$\int_0^\pi v_n'\varphi' \to \int_0^\pi v'\varphi'. \tag{8}$$

Combining (6) and its analogies for other parameters and (7) and (8), it follows that $(u;v)$ is a nontrivial weak solution of the problem (5) with parameters α, β, δ. □

Lemma 7. *Let $\{A_n\}_{n=1}^\infty$ be a subset of $\widehat{\Sigma}$ satisfying $A_n \to A$ as $n \to \infty$. Let $(u_n; v_n)$ be the nontrivial solutions corresponding to A_n. Let u_n (v_n respectively) take the zero value on $(0,\pi)$ at the same number of points. Then the limit function u (v respectively) has the same number of zero values.*

Proof: From the proof of the previous theorem, we know that $(u;v)$ is a nontrivial solution of the equation (5) with parameters given by $A = \lim_{n\to\infty} A_n$.

Without loss of generality, we prove the claim for the limit function u. The proof for v is analogous. Denote the number of zero points of u_n by k.

First, we prove that u has only finitely many zero points. For contradiction, suppose that u takes the zero value in at least countably many points on $[0,\pi]$. Thus, by Bolzano-Weierstrass Theorem, there exists a limit point of the set $\{t \in [0,\pi]; u(t) = 0\}$. Let us denote it by z. Exactly, there exists a sequence $\{x_n\}_{n=1}^\infty$ satisfying $u(x_n) = 0$ for all $n \in \mathbb{N}$ and $x_n \to z$ as n approaching the infinity. But, since $u_n \in C^2([0,\pi])$ and u_n takes an extreme value on (x_n, x_{n+1}) (or (x_{n+1}, x_n) respectively), there also exists a sequence $\{y_n\}_{n=1}^\infty$ such that $y_n \in (x_n, x_{n+1})$ (or $y_n \in (x_{n+1}, x_n)$ respectively), $u'(y_n) = 0$ and obviously, $y_n \to z$ as n approaches the infinity. Since $u \in C^2([0,\pi])$, it holds that $u(z) = 0 = u'(z)$, but this is a contradiction with Lemma 5.

From now on, the zero points of u are denoted by x_i, $i = 0, 1, \ldots, N$, $0 = x_0 < x_1 < \cdots < x_N < \pi$.

Without loss of generality, suppose that u is positive in a right neighborhood of the origin. As u_n converge uniformly to u, using Lemma 5, for any $\varepsilon > 0$ sufficiently small, we can find n_ε such that u_n are strictly positive on $(x_{2i} + \varepsilon, x_{2i+1} - \varepsilon)$ and strictly negative on $(x_{2i+1} + \varepsilon, x_{2i+2} - \varepsilon)$ for every $n \geq n_\varepsilon$. It implies that $N \leq k$. Indeed, for n large enough, u_n has to take zero value at least once on $(x_i - \varepsilon, x_i + \varepsilon)$ since u_n are continuous on $[0,\pi]$. Moreover, u_n can take the zero value only on these intervals.

For contradiction, suppose now that $N < k$. First, assume that there exists a sequence $\{y_n\}_{n=1}^\infty \subset [0,\pi)$ such that $u_n(y_n) = 0$ and $y_n \to \pi$ as n approaching the infinity. But then $u(\pi) = u'(\pi) = 0$ and we get a contradiction with Lemma 5.

Secondly, if $N < k$, then necessarily there exist two sequences

$$\{y_n\}_{n=1}^\infty \subset [0,\pi) \ \text{ and } \ \{z_n\}_{n=1}^\infty \subset [0,\pi)$$

such that $y_n < z_n$, $u_n(y_n) = u_n(z_n) = 0$, $u_n \neq 0$ on (y_n, z_n) and

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = x \in \{x_i\}_{i=0}^{N}.$$

But simultaneously, for all $n \in \mathbb{N}$ there exists $\xi_n \in (y_n, z_n)$ such that $u_n'(\xi_n) = 0$. Necessarily, $\lim_{n\to\infty} \xi_n = x$. By using regularity of the problem, the solutions u_n, u belong to $W^{2,2}(0,\pi)$ and there exists a subsequence of $\{u_{n(k)}\}$ such that $u_{n(k)}' \rightrightarrows u'$. This fact and the continuity of u_n' imply that

$$|u'(x) - u_{n(k)}'(\xi_k)| \leq |u'(x) - u'(\xi_k)| + |u'(\xi_k) - u_{n(k)}'(\xi_k)| \to 0 \quad \text{as} \quad k \to \infty.$$

The conclusion is that $u(x) = u'(x) = 0$, which contradicts Lemma 5. $\square$

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