A model theory approach to structural limits

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Abstract. The goal of this paper is to unify two lines in a particular area of graph limits. First, we generalize and provide unified treatment of various graph limit concepts by means of a combination of model theory and analysis. Then, as an example, we generalize limits of bounded degree graphs from subgraph testing to finite model testing.

Keywords: graph, graph limits, model theory, first-order logic

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1. Introduction

Recently, graph sequences and graph limits are intensively studied, from diverse point of views: probability theory and statistics, property testing in computer science, flag algebras, logic, graphs homomorphisms, etc. Four standard notions of graph limits have inspired this work:

- the notion of *dense graph limit* [4], [15];
- the notion of *bounded degree graph limit* [3], [2];
- the notion of *elementary limit* e.g. [12], [13];
- the notion of *left limit* developed by the authors [20], [21].

Let us briefly introduce these notions. Our combinatorial terminology is standard and we refer to the standard books (such as [12], [17], [21], [23]) or original papers for more information.

The first approach consists in randomly picking a mapping from a test graph and to check whether this is a homomorphism. A sequence (G_n) of graphs will be said to be *L-convergent* if

$$t(F, G_n) = \frac{\hom(F, G_n)}{|G_n|^{|F|}}$$

converges for every fixed (connected) graph F.

The second one is used to define the convergence of a sequence of graphs with bounded degrees. A sequence (G_n) of graphs with bounded maximum degrees will be said to be *BS-convergent* if, for every integer r, the probability that the ball of radius r centered at a random vertex of G_n is isomorphic to a fixed rooted graph F converges for every F.

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The third one is a general notion of convergence based on the first-order properties satisfied by the elements of the sequence. A sequence $(G_i)_{i\in\mathbb{N}}$ is *elementarily convergent* if, for every sentence ϕ there exists an integer n_ϕ such that either all the G_i with $i > n_\phi$ satisfy ϕ or none of them do.

The fourth notion of convergence is based on testing existence of homomorphisms from fixed graphs: a sequence (G_n) is said to be *left-convergent* if, for every graph F, either all but a finite number of the graphs G_n contain a homomorphic image of F or only a finite number of G_n does. In other words, left-convergence is a weak notion of elementary convergence where we consider primitive positive sentences only.

These four notions proceed in different directions and, particularly, relate to either dense or sparse graphs. The sparse–dense dichotomy seems to be a key question in the area.

In this paper we provide a unifying approach to these limits. Our approach is a combination of a functional analytic and model theoretic approach and thus applies to more general structures (rather than graphs). Thus we use term *structural limits*.

The paper is organized as follows: In Section 2 we briefly introduce a general machinery based on the Boolean algebras and dualities, see [10] for standard background material. In Section 3 we apply this to Lindenbaum-Tarski algebras to get a representation of limits as measures (Theorem 1). In Section 4 we mention an alternative approach by means of ultraproducts (i.e. a non-standard approach) which yields another representation (of course ineffective) of limits (Proposition 4). In Section 5 we relate this to examples given in this section and particularly state results for bounded degree graphs, thus extending Benjamini-Schramm convergence [3] to the general setting of FO-convergence (Theorem 5). In the last section, we discuss the type of limit objects we would like to construct, and introduce some applications to the study of particular cases of first-order convergence which are going to appear elsewhere.

2. Boolean algebras, Stone representation, and measures

Recall that a *Boolean algebra* B is an algebra with two binary operations $\vee$ and $\wedge$, a unary operation $\neg$ and two elements 0 and 1, such that $(B, \vee, \wedge)$ is a distributive lattice with minimum 0 and maximum 1 which is complemented (in the sense that the complementation $\neg$ satisfies $a \vee \neg a = 1$ and $a \wedge \neg a = 0$).

The smallest Boolean algebra, denoted **2**, has elements 0 and 1. In this Boolean algebra it holds $0 \wedge a = 0$, $1 \wedge a = a$, $0 \vee a = a$, $1 \vee a = 1$, $\neg 0 = 1$, and $\neg 1 = 0$. Another example is the powerset 2^X of a set X which has a natural structure of Boolean algebra, with $0 = \emptyset, 1 = X$, $A \vee B = A \cup B$, $A \wedge B = A \cap B$ and $\neg A = X \setminus A$.

Key examples for us are the following:

Logical Example 1. The class of all first-order formulas on a language $\mathcal{L}$, considered up to logical equivalence, form a Boolean algebra with conjunction $\vee$, disjunction $\wedge$ and negation $\neg$ and constants "false" (0) and "true" (1). This Boolean algebra will be denoted $\mathrm{FO}(\mathcal{L})$.

Also, we denote by $\mathrm{FO}_0(\mathcal{L})$ the Boolean algebra of all first-order sentences (i.e. formulas without free variables) on a language $\mathcal{L}$, considered up to logical equivalence. Note $\mathrm{FO}_0(\mathcal{L})$ is a Boolean sub-algebra of $\mathrm{FO}(\mathcal{L})$.

Logical Example 2. Consider a logical theory T (with negation). The *Lindenbaum-Tarski* algebra L_T of T consists of the equivalence classes of sentences of T (here two sentences ϕ and ψ are equivalent if they are provably equivalent in T). The set of all the first-order formulas that are provably false from T forms an ideal $\mathcal{I}_T$ of the Boolean algebra $\mathrm{FO}_0(\mathcal{L})$ and L_T is nothing but the quotient algebra $\mathrm{FO}_0(\mathcal{L})/\mathcal{I}_T$.

With respect to a fixed Boolean algebra B, a *Boolean function* is a function obtained by a finite combination of the operations $\vee$, $\wedge$, and $\neg$.

Recall that a function $f : B \to B'$ is a *homomorphism* of Boolean algebras if $f(a \vee b) = f(a) \vee f(b)$, $f(a \wedge b) = f(a) \wedge f(b)$, $f(0) = 0$ and $f(1) = 1$. A filter of a Boolean algebra B is an upper set X (meaning that $x \in X$ and $y \geq x$ imply $y \in X$) that is a proper subset of B and that is closed under $\wedge$ operation ($\forall x, y \in X$ it holds $x \wedge y \in X$). It is characteristic for Boolean algebras that the maximal filters coincide with the *prime filters*, that is, the (proper) filters X such that $a \vee b \in X$ implies that either $a \in X$ or $b \in X$. One speaks of the maximal (i.e. prime filters) as of *ultrafilters* (they are also characterized by the fact that for each a either $a \in X$ or $\neg a \in X$). It is easily checked that the mapping $f \mapsto f^{-1}(1)$ is a bijection between the homomorphisms $B \to \mathbf{2}$ and the ultrafilters on B.

A *Stone space* is a compact Hausdorff space with a basis of clopen subsets. With a Boolean algebra B associate a topological space

$$S(B) = (\{x,\ x \text{ is a ultrafilter in } B\}, \tau),$$

where τ is the topology generated by all the $K_B(b) = \{x,\ b \in x\}$ (the subscript B will be omitted if obvious). Then $S(B)$ is a Stone space. By the well-known Stone Duality Theorem [24], the mappings $B \mapsto S(B)$ and $X \mapsto \Omega(X)$, where $\Omega(X)$ is the Boolean algebra of all clopen subsets of a Stone space X, constitute a one-one correspondence between the classes of all Boolean algebras and all Stone spaces.

In the language of category theory, Stone's representation theorem means that there is a duality between the category of Boolean algebras (with homomorphisms) and the category of Stone spaces (with continuous functions). The two contravariant functors defining this duality are denoted by S and Ω and defined as follows:

For every homomorphism $h : A \to B$ between two Boolean algebra, we define the map $S(h) : S(B) \to S(A)$ by $S(h)(g) = g \circ h$ (where points of $S(B)$ are identified with homomorphisms $g : B \to \mathbf{2}$). Then for every homomorphism $h : A \to B$, the map $S(h) : S(B) \to S(A)$ is a continuous function. Conversely, for every continuous function $f : X \to Y$ between two Stone spaces, define the map $\Omega(f) : \Omega(Y) \to \Omega(X)$ by $\Omega(f)(U) = f^{-1}(U)$ (where elements of $\Omega(X)$ are identified with clopen sets of X). Then for every continuous function $f : X \to Y$, the map $\Omega(f) : \Omega(Y) \to \Omega(X)$ is a homomorphism of Boolean algebras.

We denote by $K = \Omega \circ S$ one of the two natural isomorphisms defined by the duality. Hence, for a Boolean algebra B, $K(B)$ is the set algebra $\{K_B(b) : b \in B\}$, and this algebra is isomorphic to B.

Thus we have a natural notion for convergent sequence of elements of $S(B)$ (from Stone representation follows that this may be seen as the pointwise convergence).

Logical Example 3. Let $B = \mathrm{FO}_0(\mathcal{L})$ denote the Boolean Lindenbaum-Tarski algebra of all first-order sentences on a language $\mathcal{L}$ up to logical equivalence. Then the filters of B are the consistent theories of $\mathrm{FO}_0(\mathcal{L})$ and the ultrafilters of B are the *complete theories* of $\mathrm{FO}_0(\mathcal{L})$ (that is maximal consistent sets of sentences). It follows that the closed sets of $S(B)$ correspond to finite sets of consistent theories. According to Gödel's completeness theorem, every consistent theory has a model. It follows that the completeness theorem for first-order logic — which states that a set of first-order sentences has a model if and only if every finite subset of it has a model — amounts to say that $S(B)$ is compact. The points of $S(B)$ can also be identified with *elementary equivalence* classes of models. The notion of convergence of models induced by the topology of $S(B)$, called *elementary convergence*, has been extensively studied.

An ultrafilter on a Boolean algebra B can be considered as a finitely additive measure, for which every subset has either measure 0 or 1. Because of the equivalence of the notions of Boolean algebra and of set algebra, we define the *ba space* $\mathrm{ba}(B)$ of B as the space of all bounded additive functions $f : B \to \mathbb{R}$. Recall that a function $f : B \to \mathbb{R}$ is *additive* if for all $x, y \in B$ it holds

$$x \wedge y = 0 \quad \Longrightarrow \quad f(x \vee y) = f(x) + f(y).$$

The space $\mathrm{ba}(B)$ is a Banach space for the norm

$$\|f\| = \sup_{x \in B} f(x) - \inf_{x \in B} f(x).$$

(Recall that the ba space of an algebra of sets Σ is the Banach space consisting of all bounded and finitely additive measures on Σ with the total variation norm.)

Let h be a homomorphism $B \to \mathbf{2}$ and let $\iota : \mathbf{2} \to \mathbb{R}$ be defined by $\iota(0) = 0$ and $\iota(1) = 1$. Then $\iota \circ h \in \mathrm{ba}(B)$. Conversely, if $f \in \mathrm{ba}(B)$ is such that $f(B) = \{0, 1\}$ then $\iota^{-1} \circ f$ is a homomorphism $B \to \mathbf{2}$. This shows that $S(B)$ can be identified with a subset of $\mathrm{ba}(B)$.

One can also identify $\mathrm{ba}(B)$ with the space $\mathrm{ba}(K(B))$ of finitely additive measure defined on the set algebra $K(B)$. As vector spaces $\mathrm{ba}(B)$ is isomorphic to $\mathrm{ba}(K(B))$ and thus $\mathrm{ba}(B)$ is then clearly the (algebraic) dual of the normed vector space $V(B)$ (of so-called *simple functions*) generated by the indicator functions of the clopen sets (equipped with supremum norm). Indicator functions of clopen sets are denoted by $\mathbf{1}_{K(b)}$ (for some $b \in B$) and defined by

$$\mathbf{1}_{K(b)}(x) = \begin{cases} 1 & \text{if } x \in K(b) \\ 0 & \text{otherwise.} \end{cases}$$

The pairing of a function $f \in \mathrm{ba}(B)$ and a vector $X = \sum_{i=1}^n a_i \mathbf{1}_{K(b_i)}$ is defined by

$$[f, X] = \sum_{i=1}^n a_i f(b_i).$$

That $[f, X]$ does not depend on a particular choice of a decomposition of X follows from the additivity of f. We include a short proof for completeness: Assume $\sum_i \alpha_i \mathbf{1}_{K(b_i)} = \sum_i \beta_i \mathbf{1}_{K(b_i)}$. As for every $b, b' \in B$ it holds $f(b) = f(b \wedge b') + f(b \wedge \neg b')$ and $\mathbf{1}_{K(b)} = \mathbf{1}_{K(b\wedge b')} + \mathbf{1}_{K(b\wedge\neg b')}$ we can express the two sums as $\sum_j \alpha'_j \mathbf{1}_{K(b'_j)} = \sum_j \beta'_j \mathbf{1}_{K(b'_j)}$ (where $b'_i \wedge b'_j = 0$ for every $i \neq j$), with $\sum_i \alpha_i f(b_i) = \sum_j \alpha'_j f(b'_j)$ and $\sum_i \beta_i f(b_i) = \sum_j \beta'_j f(b'_j)$. As $b'_i \wedge b'_j = 0$ for every $i \neq j$, for $x \in K(b'_j)$ it holds $\alpha'_j = X(x) = \beta'_j$. Hence $\alpha'_j = \beta'_j$ for every j. Thus $\sum_i \alpha_i f(b_i) = \sum_i \beta_i f(b_i)$.

Note that $X \mapsto [f, X]$ is indeed continuous. Thus $\mathrm{ba}(B)$ can also be identified with the continuous dual of $V(B)$. We now show that the vector space $V(B)$ is dense in the space $C(S(B))$ of continuous functions from $S(B)$ to $\mathbb{R}$, hence that $\mathrm{ba}(B)$ can also be identified with the continuous dual of $C(S(B))$:

Lemma 1. *The vector space $V(B)$ is dense in $C(S(B))$ (with the uniform norm).*

Proof: Let $f \in C(S(B))$ and let $\epsilon > 0$. For $z \in f(S(B))$ let U_z be the preimage by f of the open ball $B_{\epsilon/2}(z)$ of $\mathbb{R}$ centered in z. As f is continuous, U_z is a open set of $S(B)$. As $\{K(b) : b \in B\}$ is a basis of the topology of $S(B)$, U_z can be expressed as a union $\bigcup_{b\in\mathcal{F}(U_z)} K(b)$. It follows that $\bigcup_{z\in f(S(B))} \bigcup_{b\in\mathcal{F}(U_z)} K(b)$ is a covering of $S(B)$ by open sets. As $S(B)$ is compact, there exists a finite subset $\mathcal{F}$ of $\bigcup_{z\in f(S(B))} \mathcal{F}(U_z)$ that covers $S(B)$. Moreover, as for every $b, b' \in B$ it holds $K(b) \cap K(b') = K(b \wedge b')$ and $K(b) \setminus K(b') = K(b \wedge \neg b')$ it follows that we can assume that there exists a finite family $\mathcal{F}'$ such that $S(B)$ is covered by open sets $K(b)$ (for $b \in \mathcal{F}'$) and such that for every $b \in \mathcal{F}'$ there exists $b' \in \mathcal{F}$ such that $K(b) \subseteq K(b')$. In particular, it follows that for every $b \in \mathcal{F}'$, $f(K(b))$ is included in an open ball of radius $\epsilon/2$ of $\mathbb{R}$. For each $b \in \mathcal{F}'$ choose a point $x_b \in S(B)$ such that $b \in x_b$. Now define

$$\hat{f} = \sum_{b\in\mathcal{F}'} f(x_b)\mathbf{1}_{K(b)}$$

Let $x \in S(B)$. Then there exists $b \in \mathcal{F}'$ such that $x \in K(b)$. Thus

$$|f(x) - \hat{f}(x)| = |f(x) - f(x_b)| < \epsilon.$$

Hence $\|f - \hat{f}\|_\infty < \epsilon$. □

It is difficult to exhibit a basis of $C(S(B))$ or $V(B)$. However, every meet subsemilattice of a Boolean algebra B generating B contains (via indicator functions) a basis of $V(B)$:

Lemma 2. *Let $X \subseteq B$ be closed by $\wedge$ and such that X* generates *B (meaning that every element of B can be obtained as a Boolean function of finitely many elements from X).*

Then $\{\mathbf{1}_b : \ b \in X\} \cup \{\mathbf{1}\}$ (where $\mathbf{1}$ is the constant function with value 1) includes a basis of the vector space $V(B)$.

Proof: Let $b \in B$. As X generates B there exist $b_1, \dots, b_k \in X$ and a Boolean function F such that $b = F(b_1, \dots, b_k)$. As $\mathbf{1}_{x\wedge y} = \mathbf{1}_x \mathbf{1}_y$ and $\mathbf{1}_{\neg x} = \mathbf{1} - \mathbf{1}_x$ there exists a polynomial P_F such that $\mathbf{1}_b = P_F(\mathbf{1}_{b_1}, \dots, \mathbf{1}_{b_k})$. For $I \subseteq [k]$, the monomial $\prod_{i\in I} \mathbf{1}_{b_i}$ rewrites as $\mathbf{1}_{b_I}$ where $b_I = \bigwedge_{i\in I} b_i$. It follows that $\mathbf{1}_b$ is a linear combination of the functions $\mathbf{1}_{b_I}$ ($I \subseteq [k]$) which belong to X if $I \neq \emptyset$ (as X is closed under $\wedge$ operation) and equal $\mathbf{1}$, otherwise. □

We are coming to the final transformation of our route: One can see that bounded additive real-value functions on a Boolean algebra B naturally define continuous linear forms on the vector space $V(B)$ hence, by density, on the Banach space $C(S(B))$ (of all continuous functions on $S(B)$ equipped with supremum norm). It follows (see e.g. [23]) from Riesz representation theorem that the topological dual of $C(S(B))$ is the space $\mathrm{rca}(S(B))$ of all regular countably additive measures on $S(B)$. Thus the equivalence of $\mathrm{ba}(B)$ and $\mathrm{rca}(S(B))$ follows. We summarize all of this as the following:

Proposition 1. *Let B be a Boolean algebra, let $\mathrm{ba}(B)$ be the Banach space of bounded additive real-valued functions equipped with the norm*

$$\|f\| = \sup_{b\in B} f(b) - \inf_{b\in B} f(b),$$

let $S(B)$ be the Stone space associated to B by Stone representation theorem, and let $\mathrm{rca}(S(B))$ be the Banach space of the regular countably additive measure on $S(B)$ equipped with the total variation norm.

Then the mapping $C_K : \mathrm{rca}(S(B)) \to \mathrm{ba}(B)$ defined by $C_K(\mu) = \mu \circ K$ is an isometric isomorphism. In other words, C_K is defined by

$$C_K(\mu)(b) = \mu(\{x \in S(B) : \ b \in x\})$$

(considering that the points of $S(B)$ are the ultrafilters on B).

Note also that, similarly, the restriction of C_K to the space $\mathrm{Pr}(S(B))$ of all (regular) probability measures on $S(B)$ is an isometric isomorphism of $\mathrm{Pr}(S(B))$ and the subset $\mathrm{ba}_1(B)$ of $\mathrm{ba}(B)$ of all positive additive functions f on B such that $f(1) = 1$.

A standard notion of convergence in $\mathrm{rca}(S(B))$ (as the continuous dual of $C(S(B))$) is the weak $*$-convergence: a sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures is convergent if, for every $f \in C(S(B))$ the sequence $\int f(x)\,\mathrm{d}\mu_n(x)$ is convergent. Thanks to the density of $V(B)$ this convergence translates as pointwise convergence in $\mathrm{ba}(B)$ as follows: a sequence $(g_n)_{n\in\mathbb{N}}$ of functions in $\mathrm{ba}(B)$ is convergent if, for every $b\in B$ the sequence $(g_n(b))_{n\in\mathbb{N}}$ is convergent. As $\mathrm{rca}(S(B))$ is complete, so is $\mathrm{rca}(B)$. Moreover, it is easily checked that $\mathrm{ba}_1(B)$ is closed in $\mathrm{ba}(B)$.

In a more concise way, we can write, for a sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $\mathrm{ba}(B)$ and for the corresponding sequence $(\mu_{f_n})_{n\in\mathbb{N}}$ of regular measures on $S(B)$:

$$\lim_{n\to\infty} f_n \text{ pointwise} \qquad \iff \qquad \mu_{f_n} \Rightarrow \mu_f.$$

The whole situation is summarized on Figure 1.

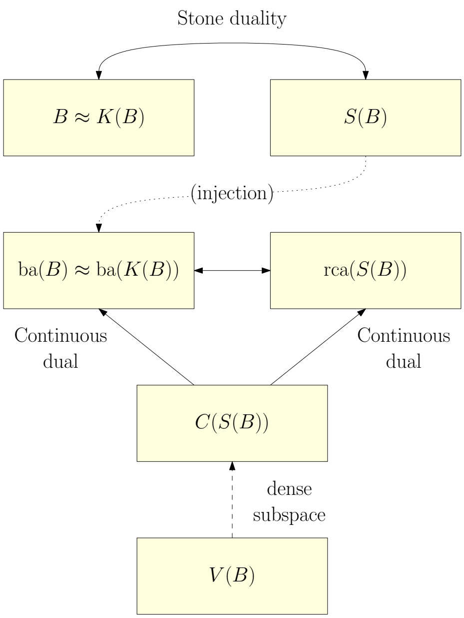

FIGURE 1. Several spaces defined from a Boolean algebra, and their inter-relations.

The above theory was not developed for its own sake but in order to demonstrate a natural approach to structural limits. The next example is a continuation of our main interpretation, which we started in Logical examples 2 and 3.

Logical Example 4. Let $B = \mathrm{FO}_0(\mathcal{L})$ denote the Boolean algebra of all first-order sentences on a language $\mathcal{L}$ up to logical equivalence. We already noted that the points of $S(B)$ are complete theories of $\mathrm{FO}_0(\mathcal{L})$, and that each complete theory has at least one model. Assume $\mathcal{L}$ is a finite language. Then for every $n\in\mathbb{N}$ there exists a sentence ϕ_n such that for every complete theory $T\in\mathrm{FO}_0(\mathcal{L})$ it holds $\phi_n\in T$ if and only if T has a unique model and this model has at most n elements. Let $U=\bigcup_{n\geq 1} K(\phi_n)$. Then U is open but not closed. The indicator function $\mathbf{1}_U$ is thus measurable but not continuous. This function has the nice property that for every complete theory $T \in S(B)$ it holds

$$\mathbf{1}_U(T) = \begin{cases} 1, & \text{if } T \text{ has a finite model;} \\ 0, & \text{otherwise.} \end{cases}$$

3. Limits via fragments and measures

We provide a unifying approach based on the previous section. We consider the special case of Boolean algebras induced by a fragment of the class $\mathrm{FO}(\mathcal{L})$ of the first-order formulas over a finite relational language $\mathcal{L}$. In this context, the language $\mathcal{L}$ will be described by its *signature*, that is the set of non-logical symbols (constant symbols, and relation symbols, along with the arities of the relation symbols). An $\mathrm{FO}(\mathcal{L})$-*structure* is then a set together with an interpretation of all relational and function symbols. Thus for example the signature of the language $\mathcal{L}^G$ of graphs is the symbol $\sim$ interpreted as the adjacency relation: $x \sim y$ if $\{x, y\}$ is an edge of the graph.

We now introduce our notion of convergence. Our approach is a combination of model theoretic and analytic approach.

Recall that a formula is obtained from atomic formulas by the use of the negation ($\neg$), logical connectives ($\vee$ and $\wedge$), and quantification ($\exists$ and $\forall$). A *sentence* (or closed formula) is a formula without free variables.

The *quantifier rank* $\mathrm{qrank}(\phi)$ of a formula ϕ is the maximum depth of a quantifier in ϕ. For instance, the quantifier rank of the formula

$$\exists x\ ((\exists y\ (x \sim y)) \vee (\forall y\ \forall z\ \neg(x \sim y) \wedge \neg(y \sim z)))$$

has quantifier rank 3.

The key to our approach is the following definition.

Definition 1. Let $\phi(x_1, \ldots, x_p)$ be a first-order formula with p free variables (in the language $\mathcal{L}$) and let G be an $\mathcal{L}$-structure. We denote

$$\langle \phi, G \rangle = \frac{|\{(v_1, \ldots, v_p) \in G^p :\ G \models \phi(v_1, \ldots, v_p)\}|}{|G|^p}. \tag{1}$$

In other words, $\langle \phi, G \rangle$ is the probability that ϕ is satisfied in G when the p free variables correspond to a random p-tuple of vertices of G. The value $\langle \phi, G \rangle$ is called the *density* of ϕ in G. Note that this definition is consistent in the sense that although any formula ϕ with p free variables can be considered as a formula with $q \geq p$ free variables with $q - p$ unused variables, we have

$$\frac{|\{(v_1, \ldots, v_q) :\ G \models \phi(v_1, \ldots, v_p)\}|}{|G|^q} = \frac{|\{(v_1, \ldots, v_p) :\ G \models \phi(v_1, \ldots, v_p)\}|}{|G|^p}.$$

It is immediate that for every formula ϕ it holds $\langle\neg\phi, G\rangle = 1-\langle\phi, G\rangle$. Moreover, if $\phi_1,\dots,\phi_n$ are formulas, then by de Moivre's formula, it holds

$$\langle\bigvee_{i=1}^n \phi_i, G\rangle = \sum_{k=1}^n(-1)^{k+1}\Bigg(\sum_{1\leq i_1<\dots<i_k\leq n}\langle\bigwedge_{j=1}^k \phi_{i_j}, G\rangle\Bigg).$$

In particular, if $\phi_1,\dots,\phi_k$ are *mutually exclusive* (meaning that ϕ_i and ϕ_j cannot hold simultaneously for $i\neq j$) then it holds

$$\langle\bigvee_{i=1}^k \phi_i, G\rangle = \sum_{i=1}^k\langle\phi_i, G\rangle.$$

In particular, for every fixed graph G, the mapping $\phi\mapsto\langle\phi, G\rangle$ is additive (i.e. $\langle\,\cdot\,, G\rangle\in \mathrm{ba}(\mathrm{FO}(\mathcal{L}))$):

$$\phi_1\wedge\phi_2 = 0 \quad\Longrightarrow\quad \langle\phi_1\vee\phi_2, G\rangle = \langle\phi_1, G\rangle + \langle\phi_2, G\rangle.$$

Thus we may apply the above theory to additive functions $\langle\,\cdot\,, G\rangle$ and to structural limits we shall define now.

Advancing this note that in the case of a sentence ϕ (that is a formula with no free variables, i.e. $p = 0$), the definition reduces to

$$\langle\phi, G\rangle = \begin{cases}1, & \text{if } G\models\phi;\\ 0, & \text{otherwise.}\end{cases}$$

Thus the definition of $\langle\phi, G\rangle$ will suit to the elementary convergence. Elementary convergence and all above graph limits are captured by the following definition:

Definition 2. Let X be a fragment of $\mathrm{FO}(\mathcal{L})$.

A sequence $(G_n)_{n\in\mathbb{N}}$ of $\mathcal{L}$-structures is *X-convergent* if for every $\phi\in X$, the sequence $(\langle\phi, G_n\rangle)_{n\in\mathbb{N}}$ converges.

For a Boolean sub-algebra X of $\mathrm{FO}(\mathcal{L})$, we define $\mathfrak{T}(X)$ as the space of all ultrafilters on X, which we call *complete X-theories*. The space $\mathfrak{T}(X)$ is endowed with the topology defined from its clopen sets, which are defined as the sets $K(\phi) = \{T\in\mathfrak{T}(X) : T\ni\phi\}$ for some $\phi\in X$. For the sake of simplicity, we denote by $\mathbf{1}_\phi$ (for $\phi\in X$) the indicator function of the clopen set $K(\phi)$ defined by ϕ. Hence, $\mathbf{1}_\phi(T) = 1$ if $\phi\in T$, and $\mathbf{1}_\phi(T) = 0$ otherwise.

It should be now clear that the above general approach yields the following:

Theorem 1. *Let X be a Boolean sub-algebra of* $\mathrm{FO}(\mathcal{L})$ *and let $\mathcal{G}$ be the class of all finite $\mathcal{L}$-structures.*

There exists an injective mapping $G \mapsto \mu_G$ from $\mathcal{G}$ to the space of probability measures on $\mathfrak{T}(X)$ such that for every $\phi \in X$ it holds

$$\langle \phi, G\rangle = \int \mathbf{1}_\phi(T)\,\mathrm{d}\mu_G(T).$$

A sequence $(G_n)_{n\in\mathbb{N}}$ of finite $\mathcal{L}$-structures is X-convergent if and only if the sequence $(\mu_{G_n})_{n\in\mathbb{N}}$ is weakly convergent. Moreover, if $\mu_{G_n} \Rightarrow \mu$ then for every $\phi \in X$ it holds

$$\lim_{n\to\infty}\langle \phi, G_n\rangle = \int \mathbf{1}_\phi(T)\,\mathrm{d}\mu(T).$$

In this paper, we shall be interested in specific fragments of $\mathrm{FO}(\mathcal{L})$:

- $\mathrm{FO}(\mathcal{L})$ itself;
- $\mathrm{FO}_p(\mathcal{L})$ (where $p \in \mathbb{N}$), which is the fragment consisting of all formulas with at most p free variables (in particular, $\mathrm{FO}_0(\mathcal{L})$ is the fragment of all first-order sentences);
- $\mathrm{QF}(\mathcal{L})$, which is the fragment of *quantifier-free formulas* (that is: propositional logic);
- $\mathrm{FO}^{\mathrm{local}}(\mathcal{L})$, which is the fragment of *local formulas*, defined as follows.

Let $r \in \mathbb{N}$. A formula $\phi(x_1,\dots,x_p)$ is *r-local* if, for every $\mathcal{L}$-structure G and every $v_1,\dots,v_p \in G^p$ it holds

$$G \models \phi(v_1,\dots,v_p) \qquad\Longleftrightarrow\qquad G[N_r(v_1,\dots,v_p)] \models \phi(v_1,\dots,v_p),$$

where $N_r(v_1,\dots,v_p)$ is the closed r-neighborhood of $x_1,\dots,x_p$ in the $\mathcal{L}$-structure G (that is the set of elements at distance at most r from at least one of $x_1,\dots,x_p$ in the Gaifman graph of G), and where $G[A]$ denotes the sub-$\mathcal{L}$-structure of G induced by A. A formula ϕ is *local* if it is r-local for some $r \in \mathbb{N}$; the fragment $\mathrm{FO}^{\mathrm{local}}(\mathcal{L})$ is the set of all local formulas (over the language $\mathcal{L}$). This fragment form an important fragment, particularly because of the following structure theorem.

Theorem 2 (Gaifman locality theorem [9])**.** *For every first-order formula $\phi(x_1,\dots,x_n)$ there exist integers t and r such that ϕ is equivalent to a Boolean combination of t-local formulas $\xi_j(x_{i_1},\dots,x_{i_s})$ and sentences of the form*

$$\exists y_1\dots\exists y_m\Big(\bigwedge_{1\le i<j\le m}\mathrm{dist}(y_i,y_j) > 2r \wedge \bigwedge_{1\le i\le m}\psi(y_i)\Big) \tag{2}$$

where ψ is r-local. Furthermore, if ϕ is a sentence, only sentences (2) *occur in the Boolean combination.*

From this theorem follows a general statement:

Proposition 2. *Let (G_n) be a sequence of graphs. Then (G_n) is* FO*-convergent if and only if it is both* $\mathrm{FO}^{\mathrm{local}}$*-convergent and elementarily-convergent.*

Proof: Assume $(G_n)_{n\in\mathbb{N}}$ is both $\mathrm{FO}^{\mathrm{local}}$-convergent and elementarily-convergent and let $\phi \in \mathrm{FO}$ be a first order formula with n free variables. According to Theorem 2, there exist integers t and r such that ϕ is equivalent to a Boolean combination of t-local formula $\xi(x_{i_1},\dots,x_{i_s})$ and of sentences. It follows that $\langle\phi, G\rangle$ can be expressed as a function of values of the form $\langle\xi, G\rangle$ where ξ is either a local formula or a sentence. Thus $(G_n)_{n\in\mathbb{N}}$ is FO-convergent. □

Notice that if ϕ_1 and ϕ_2 are local formulas, so are $\phi_1 \wedge \phi_2$, $\phi_1 \vee \phi_2$ and $\neg\phi_1$. It follows that $\mathrm{FO}^{\mathrm{local}}$ is a Boolean sub-algebra of FO. It is also clear that all the other fragments described above correspond to sub-algebras of FO. This means that there exist canonical injective Boolean-algebra homomorphisms from these fragments X to FO, that will correspond to surjective continuous functions (projections) from $S(\mathrm{FO})$ to $S(X)$ and it is not hard to see that they also correspond to surjective maps from ba(FO) to ba(X) and to surjective pushforwards from rca(S(FO) to rca($S(X)$).

Recall that a *theory* T is a set of sentences. (Here we shall only consider first-order theories, so a theory is a set of first-order sentences.) The theory T is *consistent* if one cannot deduce from T both a sentence ϕ and its negation. The theory T is *satisfiable* if it has a model. It follows from Gödel's completeness theorem that, in the context of first-order logic, a theory is consistent if and only if it is satisfiable. Also, according to the compactness theorem, a theory has a model if and only if every finite subset of it has a model. Moreover, according to the downward Löwenheim-Skolem theorem, there exists a countable model. A theory T is a *complete theory* if it is consistent and if, for every sentence $\phi \in \mathrm{FO}_0(\mathcal{L})$, either ϕ or $\neg\phi$ belongs to T. Hence every complete theory has a countable model. However, a complete theory which has an infinite model has infinitely many non-isomorphic models.

It is natural to ask whether one can consider fragments that are not Boolean sub-algebras of $\mathrm{FO}(\mathcal{L})$ and still have a description of the limit of a converging sequence as a probability measure on a nice measurable space. There is obviously a case where this is possible: when the convergence of $\langle\phi, G_n\rangle$ for every ϕ in a fragment X implies the convergence of $\langle\psi, G_n\rangle$ for every ψ in the minimum Boolean algebra containing X. We prove now that this is an instance of a more general phaenomenon:

Proposition 3. *Let X be a fragment of $\mathrm{FO}(\mathcal{L})$ closed under (finite) conjunction — that is:* a meet semilattice *of $\mathrm{FO}(\mathcal{L})$ — and let $\mathrm{BA}(X)$ be the Boolean algebra generated by X (that is the closure of X by $\vee, \wedge$ and $\neg$). Then X-convergence is equivalent to $\mathrm{BA}(X)$-convergence.*

Proof: Let $\Psi \in \mathrm{BA}(X)$. According to Lemma 2, there exist $\phi_1,\dots,\phi_k \in X$ and $\alpha_0,\alpha_1,\dots,\alpha_k \in \mathbb{R}$ such that

$$\mathbf{1}_\Psi = \alpha_0\mathbf{1} + \sum_{i=1}^{k}\alpha_i\mathbf{1}_{\phi_i}.$$

Let G be a graph, let $\Omega = S(\mathrm{BA}(X))$ and let $\mu_G \in \mathrm{rca}(\Omega)$ be the associated measure. Then

$$\langle \Psi, G\rangle = \int_\Omega \mathbf{1}_\Psi \,\mathrm{d}\mu_G = \int_\Omega \big(\alpha_0 \mathbf{1} + \sum_{i=1}^k \alpha_i \mathbf{1}_{\phi_i}\big)\,\mathrm{d}\mu_G = \alpha_0 + \sum_{i=1}^k \alpha_i \langle \phi_i, G\rangle.$$

Thus if $(G_n)_{n\in\mathbb{N}}$ is an X-convergent sequence, the sequence $(\langle \psi, G_n\rangle)_{n\in\mathbb{N}}$ converges for every $\psi \in \mathrm{BA}(X)$, that is $(G_n)_{n\in\mathbb{N}}$ is $\mathrm{BA}(X)$-convergent. □

Continuing to develop the general mechanism for the structural limits we consider fragments of FO quantified by the number of free variables.

We shall allow formulas with p free variables to be considered as a formula with $q > p$ variables, $q - p$ variables being unused. As the order of the free variables in the definition of the formula is primordial, it will be easier for us to consider sentences with p constants instead of formulas with p free variables. Formally, denote by $\mathcal{L}_p$ the language obtained from $\mathcal{L}$ by adding p (ordered) symbols of constants $c_1, \ldots, c_p$. There is a natural isomorphism of Boolean algebras $\nu_p : \mathrm{FO}_p(\mathcal{L}) \to \mathrm{FO}_0(\mathcal{L}_p)$, which replaces the occurrences of the p free variables $x_1, \ldots, x_p$ in a formula $\phi \in \mathrm{FO}_p$ by the corresponding symbols of constants $c_1, \ldots, c_p$, so that it holds, for every graph G, for every $\phi \in \mathrm{FO}_p$ and every $v_1, \ldots, v_p \in G$:

$$G \models \phi(v_1, \ldots, v_p) \quad\iff\quad (G, v_1, \ldots, v_p) \models \nu_p(\phi).$$

The Stone space associated to the Boolean algebra $\mathrm{FO}_0(\mathcal{L}_p)$ is the space $\mathfrak{T}(\mathcal{L}_p)$ of all complete theories in the language $\mathcal{L}_p$. Also, we denote by $\mathfrak{T}_\omega$ the Stone space representing the Boolean algebra $\mathfrak{T}(\mathrm{FO}_0(\mathcal{L}_\omega)) \approx \mathrm{FO}$. One of the specific properties of the spaces $\mathfrak{T}(\mathcal{L}_p)$ is that they are endowed with an ultrametric derived from the quantifier-rank:

$$\mathrm{dist}(T_1, T_2) = \begin{cases} 0 & \text{if } T_1 = T_2 \\ 2^{-\min\{\mathrm{qrank}(\theta):\ \theta \in T_1 \setminus T_2\}} & \text{otherwise.} \end{cases}$$

This ultrametric defines the same topology as the Stone representation theorem. As a compact metric space, $\mathfrak{T}(\mathcal{L}_p)$ is (with the Borel sets defined by the metric topology) a standard Borel space.

For each $p \geq 0$, there is a natural projection $\pi_p : \mathfrak{T}_{p+1} \to \mathfrak{T}_p$, which maps a complete theory $T \in \mathfrak{T}_{p+1}$ to the subset of T containing the sentences where only the p first constant symbols $c_1, \ldots, c_p$ are used. Of course we have to check that $\pi_p(T)$ is a complete theory in the language $\mathcal{L}_p$ but this is indeed so.

According to the ultrametrics defined above, the projections π_p are contractions (hence are continuous). Also, there is a natural isometric embedding $\eta_p : \mathfrak{T}_p \to \mathfrak{T}_{p+1}$ defined as follows: for $T \in \mathfrak{T}_p$, the theory $\eta_p(T)$ is the deductive closure of $T \cup \{c_p = c_{p+1}\}$. Notice that $\eta_p(T)$ is indeed complete: for every sentence $\phi \in \mathrm{FO}(\mathcal{L}_{p+1})$, let $\widetilde{\phi}$ be the sentence obtained from ϕ by replacing each symbol

c_{p+1} by c_p. It is clear that $c_p = c_{p+1} \vdash \phi \leftrightarrow \widetilde{\phi}$. As either $\widetilde{\phi}$ or $\neg\widetilde{\phi}$ belongs to T, either ϕ or $\neg\phi$ belongs to $\eta_p(T)$. Moreover, we deduce easily from the fact that $\widetilde{\phi}$ and ϕ have the same quantifier rank that η_p is an isometry. Finally, let us note that $\pi_p \circ \eta_p$ is the identity of $\mathfrak{T}_p$.

For these fragments we shall show a particular nice construction, well non-standard construction, of limiting measure.

4. A non-standard approach

The natural question that arises from the result of the previous section is whether one can always find a representation of the FO-limit of an FO-converging sequence by a "nice" measurable $\mathcal{L}$-structure.

It appears that a general notion of limit object for FO-convergence can be obtained by a non-standard approach. In this we follow closely Elek and Szegedy [7].

We first recall the ultraproduct construction. Let $(G_n)_{n\in\mathbb{N}}$ be a finite sequence of finite $\mathcal{L}$-structures and let U be a non-principal ultrafilter. Let $\widetilde{G} = \prod_{i\in\mathbb{N}} G_i$ and let $\sim$ be the equivalence relation on $\widetilde{V}$ defined by $(x_n) \sim (y_n)$ if $\{n : x_n = y_n\} \in U$. Then the *ultraproduct* of the $\mathcal{L}$-structures G_n is the quotient of $\widetilde{G}$ by $\sim$, and it is denoted $\prod_U G_i$. For each relational symbol R with arity p, the interpretation $R^{\widetilde{G}}$ of R in the ultraproduct is defined by

$$([v^1], \dots, [v^p]) \in R^{\widetilde{G}} \quad \iff \quad \{n : (v^1_n, \dots, v^p_n) \in R^{G_n}\} \in U.$$

The fundamental theorem of ultraproducts proved by Łoś makes ultraproducts particularly useful in model theory. We express it now in the particular case of $\mathcal{L}$-structures indexed by $\mathbb{N}$ but its general statement concerns structures indexed by a set I and the ultraproduct constructed by considering an ultrafilter U over I.

Theorem 3 ([14])**.** *For each formula $\phi(x_1, \dots, x_p)$ and each $v^1, \dots, v^p \in \prod_i G_i$ we have*

$$\prod_U G_i \models \phi([v^1], \dots, [v^p]) \quad \textit{iff} \quad \{i :\ G_i \models \phi(v^1_i, \dots, v^p_i)\} \in U.$$

Note that if (G_i) is elementary-convergent, then $\prod_U G_i$ is an elementary limit of the sequence: for every sentence ϕ, according to Theorem 3, we have

$$\prod_U G_i \models \phi \quad \iff \quad \{i :\ G_i \models \phi\} \in U.$$

A measure ν extending the normalised counting measures ν_i of G_i is then obtained via the Loeb measure construction. We denote by $\mathcal{P}(G_i)$ the Boolean algebra of the subsets of vertices of G_i, with the normalized measure $\nu_i(A) = \frac{|A|}{|G_i|}$. We define $\mathcal{P} = \prod_i \mathcal{P}(G_i)/I$, where I is the ideal of the elements $\{A_i\}_{i\in\mathbb{N}}$ such that $\{i :\ A_i = \emptyset\} \in U$. We have

$$[x] \in [A] \quad \text{iff} \quad \{i :\ x_i \in A_i\} \in U.$$

These sets form a Boolean algebra over $\prod_U G_i$. Recall that the ultralimit $\lim_U a_n$ defined for every $(a_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ is such that for every $\epsilon > 0$ we have

$$\{i:\ a_i \in [\lim_U a_n - \epsilon\,;\, \lim_U a_n + \epsilon]\} \in U.$$

Define

$$\nu([A]) = \lim_U \nu_i(A_i).$$

Then $\nu : \mathcal{P} \to \mathbb{R}$ is a finitely additive measure. Remark that, according to Hahn-Kolmogorov theorem, proving that ν extends to a countably additive measure amounts to prove that for every sequence $([A^n])$ of disjoint elements of $\mathcal{P}$ such that $\bigcup_n [A^n] \in \mathcal{P}$ it holds $\nu(\bigcup_n [A^n]) = \sum_n \nu([A^n])$.

A subset $N \subseteq \prod_U G_i$ is a *nullset* if for every $\epsilon > 0$ there exists $[A^\epsilon] \in \mathcal{P}$ such that $N \subseteq [A^\epsilon]$ and $\nu([A^\epsilon]) < \epsilon$. The set of nullsets is denoted by $\mathcal{N}$. A set $B \subseteq \prod_U G_i$ is *measurable* if there exists $\widetilde{B} \in \mathcal{P}$ such that $B \Delta \widetilde{B} \in \mathcal{N}$.

The following theorem is proved in [7]:

Theorem 4. *The measurable sets form a σ-algebra B_U and $\nu(B) = \nu(\widetilde{B})$ defines a probability measure on B_U.*

Notice that this construction extends to the case where to each G_i is associated a probability measure ν_i. Then the limit measure ν is non-atomic if and only if the following technical condition holds: for every $\epsilon > 0$ and for every $(A_n) \in \prod G_n$, if for U-almost all n it holds $\nu_n(A_n) \geq \epsilon$ then there exists $\delta > 0$ and $(B_n) \in \prod G_n$ such that for U-almost all n it holds $B_n \subseteq A_n$ and $\min(\nu_n(B_n), \nu_n(A_n \setminus B_n)) \geq \delta$. This obviously holds if ν_n is a normalized counting measure and $\lim_U |G_n| = \infty$.

Let $f_i : G_i \to [-d; d]$ be real functions, where $d > 0$. One can define $f : \prod_U G_i \to [-d; d]$ by

$$f([x]) = \lim_U f_i(x_i).$$

We say that f is the *ultralimit* of the functions $\{f_i\}_{i\in\mathbb{N}}$ and that f is an *ultralimit function*.

Let $\phi(x)$ be a first order formula with a single free variable, and let $f_i^\phi : G_i \to \{0, 1\}$ be defined by

$$f_i^\phi(x) = \begin{cases} 1 & \text{if } G_i \models \phi(x); \\ 0 & \text{otherwise} \end{cases}$$

and let $f^\phi : \prod_U G_i \to \{0, 1\}$ be defined similarly on the $\mathcal{L}$-structure $\prod_U G_i$. Then f^ϕ is the ultralimit of the functions $\{f_i^\phi\}$ according to Theorem 3.

The following lemma is proved in [7].

Lemma 3. *The ultralimit functions are measurable on $\prod_U G_i$ and*

$$\int_{\prod_U G_i} f \,\mathrm{d}\nu = \lim_U \frac{\sum_{x\in G_i} f_i(x)}{|G_i|}\,.$$

In particular, for every formula $\phi(x)$ with a single free variable, we have:

$$\nu\big(\{[x] : \prod_U G_i \models \phi([x])\}\big) = \lim_U \langle \phi, G_i \rangle.$$

Let $\psi(x,y)$ be a formula with two free variables. Define $f_i : G_i \to [0;1]$ by

$$f_i(x) = \frac{|\{y \in G_i :\ G_i \models \psi(x,y)\}|}{|G_i|}$$

and let

$$f([x]) = \mu\big(\{[y] : \prod_U G_i \models \psi([x],[y]\}\big).$$

Let us check that $f([x])$ is indeed the ultralimit of $f_i(x_i)$. Fix $[x]$. Let $g_i : G_i \to \{0,1\}$ be defined by

$$g_i(y) = \begin{cases} 1 & \text{if } G_i \models \psi(x_i, y) \\ 0 & \text{otherwise} \end{cases}$$

and let $g : \prod_U G_i \to \{0,1\}$ be defined similarly by

$$g([y]) = \begin{cases} 1 & \text{if } \prod_U G_i \models \psi([x],[y]) \\ 0 & \text{otherwise.} \end{cases}$$

According to Theorem 3 we have

$$\prod_U G_i \models \psi([x],[y]) \quad \iff \quad \{i :\ G_i \models \psi(x_i, y_i)\} \in U.$$

It follows that g is the ultralimit of the functions $\{g_i\}_{i\in\mathbb{N}}$. Thus, according to Lemma 3 we have

$$\nu\big(\{[y] : \prod_U G_I \models \psi([x],[y])\}\big) = \lim_U \frac{|\{y \in G_i : G_i \models \psi(x_i, y_i)\}|}{|G_i|},$$

that is:

$$f([x]) = \lim_U f_i(x_i).$$

Hence f is the ultralimit of the functions $\{f_i\}_{i\in\mathbb{N}}$ and, according to Lemma 3, we have

$$\iint 1_\psi([x],[y])\ \mathrm{d}\nu([x])\ \mathrm{d}\nu([y]) = \lim_U \langle \psi, G_i \rangle.$$

This property extends to any number of free variables. We formulate this as a summary of the results of this section.

Proposition 4. *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of finite $\mathcal{L}$-structures and let U be a non-principal ultrafilter on $\mathbb{N}$. Then there exists a measure ν on the ultraproduct $\widetilde{G} = \prod_U G_n$ such that for every first-order formula ϕ with p free variables it holds:*

$$\int_{\widetilde{G}^p} \cdots \int \mathbf{1}_\phi([x_1], \ldots, [x_p])\ \mathrm{d}\nu([x_1])\ \ldots\ \mathrm{d}\nu([x_p]) = \lim_U \langle \psi, G_i \rangle.$$

Moreover, the above integral is invariant by any permutation on the order of the integrations: for every permutation σ of $[p]$ it holds

$$\lim_U \langle \psi, G_i \rangle = \int_{\widetilde{G}^p} \cdots \int 1_\phi([x_1], \ldots, [x_p])\ \mathrm{d}\nu([x_{\sigma(1)}])\ \ldots\ \mathrm{d}\nu([x_{\sigma(p)}]).$$

However, the above constructed measure algebra is non-separable (see [7], [5] for discussion).

5. A particular case

Instead of restricting convergence to a fragment of $\mathrm{FO}(\mathcal{L})$, it is also interesting to consider restricted classes of structures. For instance, the class of graphs with maximum degree at most D (for some integer D) received much attention. Specifically, the notion of *local weak convergence* of bounded degree graphs was introduced in [3]:

A *rooted graph* is a pair (G, o), where $o \in V(G)$. An *isomorphism* of rooted graph $\phi : (G, o) \to (G', o')$ is an isomorphism of the underlying graphs which satisfies $\phi(o) = o'$. Let $D \in \mathbb{N}$. Let $\mathcal{G}_D$ denote the collection of all isomorphism classes of connected rooted graphs with maximal degree at most D. For simplicity's sake, we denote elements of $\mathcal{G}_D$ simply as graphs. For $(G, o) \in \mathcal{G}_D$ and $r \geq 0$ let $B_G(o, r)$ denote the subgraph of G spanned by the vertices at distance at most r from o. If $(G, o), (G', o') \in \mathcal{G}_D$ and r is the largest integer such that $(B_G(o, r), o)$ is rooted-graph isomorphic to $(B_{G'}(o', r), o')$, then set $\rho((G, o), (G', o')) = 1/r$, say. Also take $\rho((G, o), (G, o)) = 0$. Then ρ is metric on $\mathcal{G}_D$. Let $\mathfrak{M}_D$ denote the space of all probability measures on $\mathcal{G}_D$ that are measurable with respect to the Borel σ-field of ρ. Then $\mathfrak{M}_D$ is endowed with the topology of weak convergence, and is compact in this topology.

A sequence $(G_n)_{n\in\mathbb{N}}$ of finite connected graphs with maximum degree at most D is *BS-convergent* if, for every integer r and every rooted connected graph (F, o) with maximum degree at most D the following limit exists:

$$\lim_{n\to\infty} \frac{|\{v : B_{G_n}(v, r) \cong (F, o)\}|}{|G_n|}.$$

This notion of limits leads to the definition of a limit object as a probability measure on $\mathcal{G}_D$ [3].

However, as we shall see below, a nice representation of the limit structure can be given. To relate BS-convergence to X-convergence, we shall consider the fragment $\mathrm{FO}_1^{\mathrm{local}}$ of those formulas with at most 1 free variable that are local. Formally, let $\mathrm{FO}_1^{\mathrm{local}} = \mathrm{FO}^{\mathrm{local}} \cap \mathrm{FO}_1$.

Proposition 5. *Let (G_n) be a sequence of finite graphs with maximum degree d, with $\lim_{n\to\infty} |G_n| = \infty$.*

Then the following properties are equivalent:

1. *the sequence $(G_n)_{n\in\mathbb{N}}$ is BS-convergent;*
2. *the sequence $(G_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}_1^{\mathrm{local}}$-convergent;*
3. *the sequence $(G_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}^{\mathrm{local}}$-convergent.*

Proof: If $(G_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}^{\mathrm{local}}$-convergent, it is $\mathrm{FO}_1^{\mathrm{local}}$-convergent;

If $(G_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}_1^{\mathrm{local}}$-convergent then it is BS-convergent as for any finite rooted graph (F,o), testing whether the ball of radius r centered at a vertex x is isomorphic to (F,o) can be formulated by a local first order formula.

Assume $(G_n)_{n\in\mathbb{N}}$ is BS-convergent. As we consider graphs with maximum degree d, there are only finitely many isomorphism types for the balls of radius r centered at a vertex. It follows that any local formula $\xi(x)$ with a single variable can be expressed as the conjunction of a finite number of (mutually exclusive) formulas $\xi_{(F,o)}(x)$, which in turn correspond to subgraph testing. It follows that BS-convergence implies $\mathrm{FO}_1^{\mathrm{local}}$-convergence.

Assume $(G_n)_{n\in\mathbb{N}}$ is $\mathrm{FO}_1^{\mathrm{local}}$-convergent and let $\phi(x_1,\dots,x_p)$ be an r-local formula. Let $\mathcal{F}_\phi$ be the set of all p-tuples $((F_1,f_1),\dots,(F_p,f_p))$ of rooted connected graphs with maximum degree at most d and radius (from the root) at most r such that $\bigcup_i F_i \models \phi(f_1,\dots,f_p)$.

Then, for every graph G the sets

$$\{(v_1,\dots,v_p):\ G\models\phi(v_1,\dots,v_p)\}$$

and

$$\biguplus_{((F_1,f_1),\dots,(F_p,f_p))\in\mathcal{F}_\phi}\ \prod_{i=1}^{p}\{v:\ G\models\theta_{(F_i,f_i)}(v)\}$$

differ by at most $O(|G|^{p-1})$ elements. Indeed, according to the definition of an r-local formula, the p-tuples $(x_1,\dots,x_p)$ belonging to exactly one of these sets are such that there exists $1\le i<j\le p$ such that $\mathrm{dist}(x_i,x_j)\le 2r$.

It follows that

$$\langle\phi,G\rangle = \Big(\sum_{((F_i,f_i))_{1\le i\le p}\in\mathcal{F}_\phi}\ \prod_{i=1}^{p}\langle\theta_{(F_i,f_i)},G\rangle\Big) + O(|G|^{-1}).$$

It follows that $\mathrm{FO}_1^{\mathrm{local}}$-convergence (hence BS-convergence) implies full $\mathrm{FO}^{\mathrm{local}}$-convergence. □

According to this proposition, the BS-limit of a sequence of graphs with maximum degree at most D corresponds to a probability measure on $S(\mathrm{FO}_1^{\mathrm{local}}(\mathcal{L}))$ (where $\mathcal{L}$ is the language of graphs) whose support is included in the clopen set $K(\zeta_D)$, where ζ_D is the sentence expressing that the maximum degree is at most D. As above, the Boolean algebra $\mathrm{FO}_1^{\mathrm{local}}(\mathcal{L})$ is isomorphic to the Boolean algebra defined by the fragment $X \subset \mathrm{FO}_0(\mathcal{L}_1)$ of sentences in the language of rooted graphs that are local with respect to the root. According to this locality, for any two countable rooted graphs (G_1, r_1) and (G_2, r_2), the trace of the complete theories of (G_1, r_1) and (G_2, r_2) on X are the same if and only if the (rooted) connected component (G_1', r_1) of (G_1, r_1) containing the root r_1 is elementary equivalent to the (rooted) connected component (G_2', r_2) of (G_2, r_2) containing the root r_2. As isomorphism and elementary equivalence are equivalent for countable connected graphs with bounded degrees it is easily checked that $K_X(\zeta_D)$ is homeomorphic to $\mathcal{G}_D$. Hence our setting leads essentially to the same limit object as [3] for BS-convergent sequences.

We now consider how full FO-convergence differs to BS-convergence for sequence of graphs with maximum degree at most D. This shows a remarkable stability of BS-convergence.

Corollary 1. *A sequence (G_n) of finite graphs with maximum degree at most d such that $\lim_{n\to\infty} |G_n| = \infty$ is* FO*-convergent if and only if it is both BS-convergent and elementarily convergent.*

Proof: This is a direct consequence of Propositions 2 and 5. □

Explicit limit objects are known for sequence of bounded degree graphs, both for BS-convergence (graphing) and for elementary convergence (countable graphs). It is natural to ask whether a nice limit object could exist for full FO-convergence. We shall now answer this question by the positive.

Let V be a standard Borel space with a measure μ. Suppose that $T_1, T_2, \dots, T_k$ are measure preserving Borel involutions of X. Then the system

$$\mathbf{G} = (V, T_1, T_2, \dots, T_k, \mu)$$

is called a *measurable graphing* [1]. Here x is adjacent to y, if $x \neq y$ and $T_j(x) = y$ for some $1 \leq j \leq k$. Now if V is a compact metric space with a Borel measure μ and $T_1, T_2, \dots, T_k$ are continuous measure preserving involutions of V, then $\mathbf{G} = (V, T_1, T_2, \dots, T_k, \mu)$ is a *topological graphing*. It is a consequence of [3] and [8] that every local weak limit of finite connected graphs with maximum degree at most D can be represented as a measurable graphing. Elek [6] further proved the representation can be required to be a topological graphing.

For an integer r, a graphing $\mathbf{G} = (V, T_1, \dots, T_k, \mu)$ and a finite rooted graph (F, o) we define the set

$$D_r(\mathbf{G}, (F, o)) = \{x \in \mathbf{G}, B_r(\mathbf{G}, x) \simeq (F, o)\}.$$

We shall make use of the following lemma which reduces a graphing to its essential support.

Lemma 4 (Cleaning Lemma)**.** *Let $\mathbf{G} = (V, T_1, \ldots, T_d, \mu)$ be a graphing.*

Then there exists a subset $X \subset V$ with 0 measure such that X is globally invariant by each of the T_i and $\mathbf{G}' = (V - X, T_1, \ldots, T_d, \mu)$ is a graphing such that for every finite rooted graph (F, o) and integer r it holds

$$\mu(D_r(\mathbf{G}', (F, o))) = \mu(D_r(\mathbf{G}, (F, o)))$$

(*which means that $\mathbf{G}'$ is equivalent to $\mathbf{G}$*) *and*

$$D_r(\mathbf{G}', (F, o)) \neq \emptyset \quad \iff \quad \mu(D_r(\mathbf{G}', (F, o))) > 0.$$

Proof: For a fixed r, define $\mathcal{F}_r$ as the set of all (isomorphism types of) finite rooted graphs (F, o) with radius at most r such that $\mu(D_r(\mathbf{G}, (F, o))) = 0$. Define

$$X = \bigcup_{r \in \mathbb{N}} \bigcup_{(F,o) \in \mathcal{F}_r} D_r(\mathbf{G}, (F, o)).$$

Then $\mu(X) = 0$, as it is a countable union of 0-measure sets.

We shall now prove that X is a union of connected components of $\mathbf{G}$, that is that X is globally invariant by each of the T_i. Namely, if $x \in X$ and y is adjacent to x, then $y \in X$. Indeed: if $x \in X$ then there exists an integer r such that $\mu(D(\mathbf{G}, B_r(\mathbf{G}, x))) = 0$. But it is easily checked that

$$\mu(D(\mathbf{G}, B_{r+1}(\mathbf{G}, y))) \leq d \cdot \mu(D(\mathbf{G}, B_r(\mathbf{G}, x))).$$

Hence $y \in X$. It follows that for every $1 \leq i \leq d$ we have $T_i(X) = X$. So we can define the graphing $\mathbf{G}' = (V - X, T_1, \ldots, T_d, \mu)$.

Let (F, o) be a rooted finite graph. Assume there exists $x \in \mathbf{G}'$ such that $B_r(\mathbf{G}', r) \simeq (F, o)$. As X is a union of connected components, we also have $B_r(\mathbf{G}, r) \simeq (F, o)$ and $x \notin X$.

It follows that $\mu(D(\mathbf{G}, (F, o))) > 0$ hence $\mu(D_r(\mathbf{G}', (F, o))) > 0$. □

The cleaning lemma allows us a clean description of FO-limits in the bounded degree case:

Theorem 5. *Let $(G_n)_{n \in \mathbb{N}}$ be a* FO*-convergent sequence of finite graphs with maximum degree d, with $\lim_{n \to \infty} |G_n| = \infty$. Then there exists a graphing $\mathbf{G}$ and a countable graph $\hat{G}$ such that*

- *$\mathbf{G}$ is a BS-limit of the sequence,*
- *$\hat{G}$ is an elementary limit of the sequence,*
- *$\mathbf{G} \cup \hat{G}$ is an* FO*-limit of the sequence.*

Proof: Let $\mathbf{G}$ be a BS-limit, which has been "cleaned" using the previous lemma, and let $\hat{G}$ be an elementary limit of G. It is clear that $\mathbf{G} \cup \hat{G}$ is also a BS-limit of the sequence, so the lemma amounts in proving that $\mathbf{G} \cup \hat{G}$ is elementarily equivalent to $\hat{G}$.

According to Hanf's theorem [11], it is sufficient to prove that for every integers r, t and for every rooted finite graph (F, o) (with maximum degree d) the following equality holds:

$$\min(t, |D_r(\mathbf{G} \cup \hat{G}, (F, o))|) = \min(t, |D_r(\hat{G}, (F, o))|).$$

Assume for contradiction that this is not the case. Then $|D_r(\hat{G}, (F, o))| < t$ and $D_r(\mathbf{G}, (F, o))$ is not empty. However, as $\mathbf{G}$ is clean, this implies $\mu(D_r(\mathbf{G}, (F, o))) = \alpha > 0$. It follows that for every sufficiently large n it holds $|D_r(G_n, (F, o))| > \alpha/2\,|G_n| > t$. Hence $|D_r(\hat{G}, (F, o))| > t$, contradicting our hypothesis.

Note that the reduction of the satisfaction problem of a general first-order formula ϕ with p free variables to a case analysis based on the isomorphism type of a bounded neighborhood of the free variables shows that every first-order definable subset of $(\mathbf{G} \cup \hat{G})^p$ is indeed measurable (we extend μ to $\mathbf{G} \cup \hat{G}$ in the obvious way, considering $\hat{G}$ as zero measure). $\square$

The cleaning lemma sometimes applies in a non-trivial way:

Example 5. Consider the graph G_n obtained from a De Bruijn sequence (see e.g. [17]) of length 2^n as shown in Figure 2.

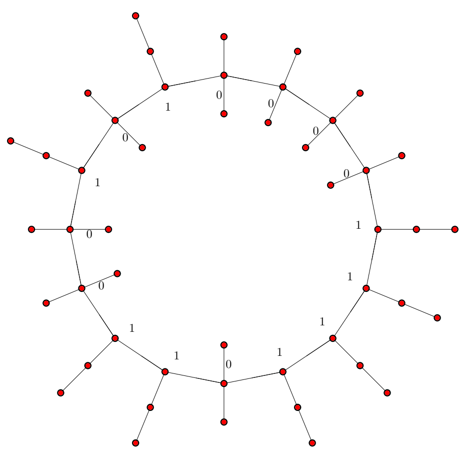

FIGURE 2. The graph G_n is constructed from a De Bruijn sequence of length 2^n.

It is easy to define a graphing $\mathbf{G}$, which is the limit of the sequence $(G_n)_{n\in\mathbb{N}}$: as vertex set, we consider the rectangle $[0; 1) \times [0; 3)$. We define a measure preserving function f and two measure preserving involutions T_1, T_2 as follows:

$$f(x,y) = \begin{cases} (2x, y/2) & \text{if } x < 1/2 \text{ and } y < 1 \\ (2x-1, (y+1)/2) & \text{if } 1/2 \leq x \text{ and } y < 1 \\ (x,y) & \text{otherwise} \end{cases}$$

$$T_1(x,y) = \begin{cases} (x, y+1) & \text{if } y < 1 \\ (x, y-1) & \text{if } 1 \leq y < 2 \\ (x,y) & \text{otherwise} \end{cases}$$

$$T_2(x,y) = \begin{cases} (x, y+1) & \text{if } x < 1/2 \text{ and } 1 \leq y < 2 \\ (x, y+2) & \text{if } 1/2 \leq x \text{ and } y < 1 \\ (x, y-1) & \text{if } x < 1/2 \text{ and } 2 \leq y \\ (x, y-2) & \text{if } 1/2 \leq x \text{ and } 2 \leq y \\ (x,y) & \text{otherwise} \end{cases}$$

Then the edges of $\mathbf{G}$ are the pairs $\{(x,y),(x',y')\}$ such that $(x,y) \neq (x',y')$ and either $(x',y') = f(x,y)$, or $(x,y) = f(x',y')$, or $(x',y') = T_1(x,y)$, or $(x',y') = T_2(x,y)$.

If one considers a random root (x,y) in $\mathbf{G}$, then the connected component of (x,y) will almost surely be a rooted line with some decoration, as expected from what is seen from a random root in a sufficiently large G_n. However, special behaviour may happen when x and y are rational. Namely, it is possible that the connected component of (x,y) becomes finite. For instance, if $x = 1/(2^n - 1)$ and $y = 2^{n-1}x$ then the orbit of (x,y) under the action of f has length n thus the connected component of (x,y) in $\mathbf{G}$ has order $3n$. Of course, such finite connected components do not appear in G_n. Hence, in order to clean $\mathbf{G}$, infinitely many components have to be removed.

6. Conclusion and further works

In a forthcoming paper [22], we apply the theory developed here to the context of classes of graphs with bounded diameter connected components, and in particular to classes with bounded tree-depth [19]. Specifically, we prove that if a uniform bound is fixed on the diameter of the connected components, FO-convergence may be considered component-wise (up to some residue for which FO_1-convergence is sufficient).

The prototype of convenient limit objects for sequences of finite graphs is a quadruple $\mathbb{G} = (V, E, \Sigma, \mu)$, where (V, E) is a graph, (V, Σ, μ) is a standard probability space, and E is a measurable subset of V^2. In such a context, modulo the axiom of projective determinacy (which would follow from the existence of infinitely many Woodin cardinals [16]), every first-order definable subset of V^p is measurable in $(V^p, \Sigma^{\otimes p})$ [18]. Then, for every first-order formula ϕ with p free variables, it is natural to define

$$\langle \psi, \mathbb{G} \rangle = \int_{V^p} 1_\phi \, \mathrm{d}\mu^{\otimes p}.$$

In this setting, $\mathbb{G} = (V, E, \Sigma, \mu)$ is a limit — we do not pretend to have uniqueness — of an FO-convergent sequence $(G_n)_{n \in \mathbb{N}}$ of finite graphs if for every first-order formula ψ it holds

$$\langle \psi, \mathbb{G} \rangle = \lim_{n \to \infty} \langle \psi, G_n \rangle.$$

We obtain in [22] an explicit construction of such limits for FO-convergent sequences of finite graphs bound to a class of graphs with bounded tree-depth. It is also there where we develop in a greater detail the general theory explained in the Sections 2 and 3. Notice that in some special cases, one does not need a standard probability space and a Borel measurable space is sufficient. This is for instance the case when we consider limits of finite connected graphs with bounded degrees (as we can use a quantifier elimination scheme to prove that definable sets are measurable) or quantifier-free convergence of graphs (definable sets form indeed a sub-algebra of the σ-algebra).

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References

[1] Adams S., *Trees and amenable equivalence relations*, Ergodic Theory Dynam. Systems **10** (1990), 1–14.

[2] Aldous D., Lyons R., *Processes on unimodular random networks*, arXiv:math/0603062 (2006).

[3] Benjamini I., Schramm O., *Recurrence of distibutional limits of finite planar graphs*, Electron. J. Probab. **6** (2001), no. 23, 13pp.

[4] Borgs C., Chayes J., Lovász L., Sós V., Szegedy B., Vesztergombi K., *Graph limits and parameter testing*, in Proc. 38th Annual ACM Symp. Principles of Dist. Comp., pp. 51–59, 2005.

[5] Conley C., Kechris A., Tucker-Drob R., *Ultraproducts of measure preserving actions and graph combinatorics*, Ergodic Theory and Dynamical Systems (2012), DOI 10.1017/S0143385711001143.

[6] Elek G., *Note on limits of finite graphs*, Combinatorica **27** (2007), 503–507, DOI 10.1007/s00493-007-2214-8.

[7] Elek G., Szegedy B., *Limits of hypergraphs, removal and regularity lemmas. A non-standard approach*, arXiv:0705.2179v1 [math.CO] (2007).

[8] Gaboriau D., *Invariant percolation and harmonic Dirichlet functions*, Geom. Funct. Anal. **15** (2005), 1004–1051, DOI 10.1007/s00039-005-0539-2.

[9] Gaifman H., *On local and non-local properties*, in Proceedings of the Herbrand Symposium, Logic Colloquium '81, 1982.

[10] Halmos P., Givant S., *Logic as Algebra*, Dolciani Mathematical Expositions, 21, The Mathematical Association of America, Washington DC, 1998.

[11] Hanf W., *Model-theoretic methods in the study of elementary logic*, in Theory of Models, J. Addison, L. Henkin, A. Tarski (eds.), pp. 132–145, North-Holland, Amsterdam, 1965.

[12] Hodges W., *A Shorter Model Theory*, Cambridge University Press, Cambridge, 1997.

[13] Lascar D., *La théorie des modèles en peu de maux*, Cassini, 2009.
[14] Łoś J., *Quelques remarques, théorèmes et problèmes sur les classes définissables d'algèbres*, in Mathematical Interpretation of Formal Systems, Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam, 1955.
[15] Lovász L., Szegedy B., *Limits of dense graph sequences*, J. Combin. Theory Ser. B **96** (2006), 933–957.
[16] Martin D., Steel J., *A proof of projective determinacy*, J. Amer. Math. Soc. **2** (1989), no. 1, 71–125.
[17] Matoušek J., Nešetřil J., *Invitation to Discrete Mathematics*, Oxford University Press, New York, 1998 (second edition 2009)
[18] Mycielski J., Świerczkowski S., *On the Lebesgue measurability and the axiom of determinateness*, Fund. Math. **54** (1964), 67–71.
[19] Nešetřil J., Ossona de Mendez P., *Tree depth, subgraph coloring and homomorphism bounds*, European J. Combin. **27** (2006), no. 6, 1022–1041, DOI 10.1016/j.ejc.2005.01.010.
[20] Nešetřil J., Ossona de Mendez P., *From sparse graphs to nowhere dense structures: Decompositions, independence, dualities and limits*, in European Congress of Mathematics, pp. 135–165, European Mathematical Society, Zürich, 2010, DOI 10.4171/077-1/7.
[21] Nešetřil J., Ossona de Mendez P., *Sparsity. Graphs, Structures, and Algorithms*, Algorithms and Combinatorics, 28, Springer, Heidelberg, 2012, 465 pp.
[22] Nešetřil J., Ossona de Mendez P., *Graph limits: a unified approach with application to the study of limits of graphs with bounded diameter components*, manuscript, 2012.
[23] Rudin W., *Functional Analysis*, Mc-Graw Hill, New York, 1973.
[24] Stone M., *The theory of representations of Boolean algebras*, Trans. Amer. Math. Soc. **40** (1936), 37–111.

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