Fixed points of periodic and firmly lipschitzian mappings in Banach spaces

Krzysztof Pupka

Abstract. W.A. Kirk in 1971 showed that if $T\colon C \to C$, where C is a closed and convex subset of a Banach space, is n-periodic and uniformly k-lipschitzian mapping with $k < k_0(n)$, then T has a fixed point. This result implies estimates of $k_0(n)$ for natural $n \geq 2$ for the general class of k-lipschitzian mappings. In these cases, $k_0(n)$ are less than or equal to 2. Using very simple method we extend this and later results for a certain subclass of the family of k-lipschitzian mappings. In the paper we show that $k_0(3) > 2$ in any Banach space. We also show that $\mathrm{Fix}(T)$ is a Hölder continuous retract of C.

Keywords: lipschitzian mapping, firmly lipschitzian mapping, n-periodic mapping, fixed point, retractions

Classification: 47H09, 47H10

1. Introduction

Let C be a nonempty closed convex subset of a Banach space E. A mapping $T : C \to C$ is called *k-lipschitzian* if for all x, y in C, $\|Tx - Ty\| \leq k\|x - y\|$. It is called *nonexpansive* if the same condition with $k = 1$ holds. In general, to assure the fixed point property for nonexpansive mappings some assumptions concerning the geometry of the spaces are added (see [9]). Another way is to put some additional restrictions on the mapping itself.

Recall that a mapping T is said to be n-periodic if $T^n = I$ (for $n = 2$, T is called *involution*). The first fixed point theorem for involutions are due to K. Goebel and E. Złotkiewicz [2], [5]. They investigated conditions under which k-lipschitzian involutions have a fixed point. K. Goebel [2] showed in 1970 that involutions have a fixed point if they are k-lipschitzian for $k < 2$ in a Banach space and for $k < \sqrt{5} \approx 2.2361$ in a Hilbert space. Moreover, in the same paper, he showed that if the space E satisfies $\varepsilon_0(E) < 1$, the same is true for k-lipschitzian involutions where k satisfies

$$\left(\frac{k}{2}\right)\left(1 - \delta_E\left(\frac{2}{k}\right)\right) < 1.$$

In 1971, W.A. Kirk [8] extend this result for all Banach spaces by proving that the same is true if T is n-periodic and such that $\|T^i x - T^i y\| \leq k\|x - y\|$ for

$x, y \in C$, $i = 1, 2, \dots, n-1$, where

$$\frac{1}{n^2}\left[(n-1)(n-2)k^2 + 2(n-1)k\right] < 1. \tag{1}$$

It follows from (1) that for $n = 3$, $k < 1.3452$; for $n = 4$, $k < 1.2078$; for $n = 5$, $k < 1.1280$; for $n = 6$, $k < 1.1147$.

If T is k-lipschitzian with $k > 1$, then $\|T^i x - T^i y\| \le k^{n-1}\|x - y\|$ for $x, y \in C$, $i = 1, 2, \dots, n-1$. Thus a k-lipschitzian mapping satisfying $T^n = I$ has fixed points if

$$\frac{1}{n^2}\left[(n-1)(n-2)k^{2(n-1)} + 2(n-1)k^{n-1}\right] < 1. \tag{2}$$

It follows from (2) that for $n = 3$, $k < 1.1598$; for $n = 4$, $k < 1.0649$; for $n = 5$, $k < 1.0351$; for $n = 6$, $k < 1.0219$.

In 1973, J. Linhart [11] slightly improved these results, namely he showed that a k-lipschitzian mapping $T\colon C \to C$ for which $T^n = I$ $(n > 1)$ has a fixed point if

$$\frac{1}{n}\sum_{j=n-1}^{2n-3} k^j < 1. \tag{3}$$

It follows from (3) that for $n = 3$, $k < 1.1745$; for $n = 4$, $k < 1.0741$; for $n = 5$, $k < 1.0412$; for $n = 6$, $k < 1.0262$.

In 2005, J. Górnicki and K. Pupka [7] obtained new improved evaluations of k for n-periodic $(n > 2)$ and k-lipschitzian mappings in a Banach space, namely for $n = 3$, $k < 1.3821$; for $n = 4$, $k < 1.2524$; for $n = 5$, $k < 1.1777$; for $n = 6$, $k < 1.1329$.

Recently in 2010, Victor Perez Garcia and Helga Fetter Nathansky [12] obtained better evaluation of k for n-periodic $(n > 2)$ and k-lipschitzian mappings in special case of a Hilbert space, namely for $n = 3$, $k < 1.5549$; for $n = 4$, $k < 1.3267$; for $n = 5$, $k < 1.2152$; for $n = 6$, $k < 1.1562$.

In the present paper, studying a simple iteration process, we extend Kirk's and Linhart's and later results for n-periodic mappings in a certain subclass of k-lipschitzian mappings, i.e., firmly k-lipschitzian mappings in general case of Banach space.

The notion of *firmly nonexpansive mapping* was introduced in 1973 by R.E. Bruck in [1]. The same class of mappings has been studied independently by K. Goebel and M. Koter in [4], where a different name is used, i.e., *regularly nonexpansive mappings*.

A mapping $T\colon C \to C$ is said to be *firmly k-lipschitzian* if for each $t \in [0, 1]$ and for any $x, y \in C$,

$$\|Tx - Ty\| \le \|k(1-t)(x-y) + t(Tx - Ty)\|. \tag{4}$$

Of course, each firmly k-lipschitzian mappings is k-lipschitzian.

In 1986, M. Koter [10] obtained theorems on the existence of a fixed point for the firmly k-lipschitzian and rotative mapping in a Banach space.

2. Firmly lipschitzian mappings

We will start with the following lemmas:

Lemma 1 ([6])**.** *Let C be a nonempty closed subset of a Banach space E and $T\colon C \to C$ be k-lipschitzian. Let $A, B \in \mathbb{R}$ and $0 \le A < 1$ and $0 < B$. If for arbitrary $x \in C$ there exists $u \in C$ such that*

$$\|Tu - u\| \le A\|Tx - x\|$$

and

$$\|u - x\| \le B\|Tx - x\|,$$

then T has a fixed point in C.

Lemma 2. *Let C be a nonempty subset of a Banach space E and a mapping $T\colon C \to C$ be firmly k-lipschitzian ($k > 1$) and n-periodic ($n > 2$), then for $x \in C$ we have*

$$\|T^{n-1}x - T^n x\| \le \left(\sum_{j=2}^{n-1} \left(\frac{k}{k+1}\right)^j k^{n-j} \frac{1 - k^{j-1}}{1 - k} \right) \|x - Tx\|.$$

Proof: Let $n > 2$. Note at the beginning that for a firmly k-lipschitzian mapping $T : C \to C$, putting $t = \frac{k}{k+1}$ in (4), we obtain

$$\|Tx - Ty\| \le \frac{k}{k+1}\|x - y + Tx - Ty\|. \tag{5}$$

Using the condition (5) two times, we obtain

$$\begin{aligned}
\|T^{n-1}x - T^n x\| &\le \frac{k}{k+1}\|T^{n-2}x - T^{n-1}x + T^{n-1}x - T^n x\| \\
&= \frac{k}{k+1}\|T^{n-2}x - T^n x\| \\
&\le \left(\frac{k}{k+1}\right)^2 \|T^{n-3}x - T^{n-1}x + T^{n-2}x - T^n x\| \\
&= \left(\frac{k}{k+1}\right)^2 \|T^{n-3}x - T^n x + T^{n-2}x - T^{n-1}x\| \\
&\le \left(\frac{k}{k+1}\right)^2 \Big(\|T^{n-3}x - T^n x\| + \|T^{n-2}x - T^{n-1}x\|\Big).
\end{aligned}$$

Repeating this estimate, we get

$$
\begin{aligned}
\|T^{n-1}x - T^n x\| &\le \left(\frac{k}{k+1}\right)^2 \left(\frac{k}{k+1}\|T^{n-4}x - T^{n-1}x + T^{n-3}x - T^n x\|\right.\\
&\qquad \left. + \|T^{n-2}x - T^{n-1}x\|\right)\\
&\le \left(\frac{k}{k+1}\right)^2 \left(\frac{k}{k+1}\|T^{n-4}x - T^n x\|\right.\\
&\qquad \left. + \frac{k}{k+1}\|T^{n-3}x - T^{n-1}x\| + \|T^{n-2}x - T^{n-1}x\|\right)\\
&\le \dots\\
&\le \left(\frac{k}{k+1}\right)^2 \left(\left(\frac{k}{k+1}\right)^{n-3} \|x - T^n x\|\right.\\
&\qquad + \left(\frac{k}{k+1}\right)^{n-3} \|Tx - T^{n-1}x\| + \dots\\
&\qquad \left. + \frac{k}{k+1}\|T^{n-3}x - T^{n-1}x\| + \|T^{n-2}x - T^{n-1}x\|\right).
\end{aligned}
$$

Note that mapping T is n-periodic, so we have

$$
\begin{aligned}
\|T^{n-1}x - T^n x\| \le \left(\frac{k}{k+1}\right)^2 &\left(\left(\frac{k}{k+1}\right)^{n-3} \|Tx - T^{n-1}x\| + \dots\right.\\
&\qquad \left. + \frac{k}{k+1}\|T^{n-3}x - T^{n-1}x\| + \|T^{n-2}x - T^{n-1}x\|\right).
\end{aligned}
$$

Finally, using the fact that mapping T is also k-lipschitzian, we have

$$
\begin{aligned}
\|T^{n-1}x - T^n x\| &\le \left(\frac{k}{k+1}\right)^2 \left(\sum_{j=2}^{n-1} \left(\frac{k}{k+1}\right)^{j-2} k^{n-j}\frac{1-k^{j-1}}{1-k}\right)\|x - Tx\|\\
&= \left(\sum_{j=2}^{n-1} \left(\frac{k}{k+1}\right)^{j} k^{n-j}\frac{1-k^{j-1}}{1-k}\right)\|x - Tx\|,
\end{aligned}
$$

which completes the proof. □

The following theorem can be proved using Lemma 2.

Theorem 1. *Let C be a nonempty closed and convex subset of a Banach space E and $T\colon C \to C$ be a firmly k-lipschitzian mapping $(k > 1)$ such that $T^n = I$*

$(n > 2)$. *If*

$$k < k_0(n) = \sup\left\{ s > 1 : \sum_{j=2}^{n-1} \left(\frac{s}{s+1}\right)^j s^{n-j}\frac{1-s^{j-1}}{1-s} = 1 \right\},$$

then T *has a fixed point in* C.

Proof: Let x be an arbitrary point in C and let $z = T^{n-1}x$. Then from Lemma 2, we get

$$\begin{aligned} \|z - Tz\| &= \|T^{n-1}x - T^n x\| \\ &\leq \left(\sum_{j=2}^{n-1} \left(\frac{k}{k+1}\right)^j k^{n-j}\frac{1-k^{j-1}}{1-k}\right)\|x - Tx\|. \end{aligned} \tag{6}$$

Moreover

$$\begin{aligned} \|z - x\| &= \|T^{n-1}x - x\| \\ &\leq \|T^{n-1}x - T^{n-2}x\| + \|T^{n-2}x - T^{n-3}x\| + \cdots + \|Tx - x\| \\ &\leq (k^{n-2} + k^{n-3} + \cdots + k + 1)\|Tx - x\|. \end{aligned} \tag{7}$$

Since

$$\sum_{j=2}^{n-1} \left(\frac{k}{k+1}\right)^j k^{n-j}\frac{1-k^{j-1}}{1-k} < 1$$

for $k < k_0(n)$, by inequality (6) and (7), Lemma 1 implies the existence of fixed points of T in C. □

Remark 1. Note that Theorem 1 implies

$$k_0(3) \geq \sqrt[3]{\frac{47}{54} - \frac{\sqrt{93}}{18}} + \sqrt[3]{\frac{47}{54} + \frac{\sqrt{93}}{18}} + \frac{1}{3} \approx 2.1479,$$

which is better than all estimates of $k_0(3)$ obtained in [8], [11], [7] for an arbitrary Banach space and better even than that obtained in [12] for a Hilbert space. It is worth noting that so far the estimates of $k_0(n)$ which are greater than 2 have been obtained only for $n = 2$ and in Hilbert space.

Remark 2. It follows from Theorem 1 that

$$k_0(4) \geq \sqrt{\frac{1}{8} + \frac{\sqrt{2}}{2}} + \frac{\sqrt{2}}{4} \approx 1.2657.$$

It is better estimate of $k_0(4)$ than obtained in [8], [11], [7] for a Banach space. For $n \geq 5$ Theorem 1 does not give better estimates than obtained in [7].

3. Hölder continuous retractions

In this section, we will show that, for a mapping T of a bounded, closed and convex set C, the limit of the iteration process discussed above, i.e.

$$\begin{aligned} x_0 &= x \in C \\ x_{m+1} &= T^{n-1}x_m, \qquad m = 0, 1, 2, \dots \end{aligned}$$

is a Hölder continuous retraction from C to $\mathrm{Fix}(T)$.

Let C be a nonempty, closed, convex and bounded subset of a Banach space E. Recall that a set $D \subset C$ is a *retract* of C if there is a continuous mapping $R : C \to D$ (*retraction*) with $\mathrm{Fix}(R) = D$. We say that a mapping $R : C \to C$ is *Hölder continuous* if there are constants $L \geq 0$ and $0 < \beta < 1$ such that for any $x, y \in C$ it holds:

$$\|Rx - Ry\| \leq L\|x - y\|^{\beta}. \tag{8}$$

An example of a real function (with $x \geq 0$) satisfying the Hölder condition but not satisfying the Lipschitz condition is a function $f(x) = x^{\beta}$.

The following lemma gives a condition for existence of a Hölder continuous retraction on the fixed point set.

Lemma 3 ([12]). *Let X be a complete metric space and $T : X \to X$ be a continuous mapping. Suppose there are $u : X \to X$, $0 < A < 1$ and $B > 0$, such that for every $x \in X$:*

(i) $d(Tu(x), u(x)) \leq A\, d(Tx, x)$,
(ii) $d(u(x), x) \leq B\, d(Tx, x)$.

Then we have that $\mathrm{Fix}(T) \neq \emptyset$.

If we define $R(x) = \lim_{n\to\infty} u^n(x)$ and u is a continuous mapping, then R is a retraction from X to $\mathrm{Fix}(T)$. *If additionally u satisfies the Lipschitz condition with constant $k > 1$ and* $\mathrm{diam}(X) < \infty$, *then R is a Hölder continuous retraction from X to* $\mathrm{Fix}(T)$.

Now, using Lemma 3, Theorem 1 and inequalities (6) and (7) we get the following conclusion.

Corollary 1. *Let $n > 2$ be natural and let C be a nonempty, closed, convex and bounded subset of a Banach space E. Let a mapping $T\colon C \to C$ be n-periodic and firmly k-lipschitzian with $1 < k < k_0(n)$. If we define mapping $F : C \to C$ such that $Fx = T^{n-1}x$, then the mapping $R\colon C \to C$ defined by*

$$R(x) = \lim_{p\to\infty} F^p(x)$$

is a Hölder continuous retraction from C to $\mathrm{Fix}(T)$.

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Department of Mathematics, Rzeszów University of Technology,
P.O. Box 85, 35-959 Rzeszów, Poland

E-mail: kpupka@prz.rzeszow.pl

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