Spaces with large star cardinal number

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Abstract. In this paper, we prove the following statements:

(1) For any cardinal κ, there exists a Tychonoff star-Lindelöf space X such that $a(X) \geq \kappa$.

(2) There is a Tychonoff discretely star-Lindelöf space X such that $aa(X)$ does not exist.

(3) For any cardinal κ, there exists a Tychonoff pseudocompact σ-starcompact space X such that $\mathrm{st}\text{-}l(X) \geq \kappa$.

Keywords: star-Lindelöf number, the Aquaro number, the absolute Aquaro number, star-Lindelöf, centered-Lindelöf, discretely star-Lindelöf, absolutely discretely star-Lindelöf, σ-starcompact, pseudocompact

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1. Introduction

By a space, we mean a topological space. Recall from [6] that a space X is *starcompact* if for every open cover $\mathcal{U}$ of X, there exists a finite subset F of X such that $\mathrm{St}(F,\mathcal{U}) = X$, where $\mathrm{St}(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \emptyset\}$. It is well-known that starcompactness is equivalent to countably compactness for Hausdorff spaces (see [3], [6]).

A space X is *discretely absolutely star-Lindelöf* (see [12], [13]) if for every open cover $\mathcal{U}$ of X and every dense subset D of X, there exists a countable subset F of D such that F is discrete and closed in X and $\mathrm{St}(F,\mathcal{U}) = X$.

A space X is *star-Lindelöf* (see [1], [2], [3], [4], [6] under different names) (*discretely star-Lindelöf*) (see [11], [15]) if for every open cover $\mathcal{U}$ of X, there exists a countable subset (a countable discrete closed subset, respectively) F of X such that $\mathrm{St}(F,\mathcal{U}) = X$. It is clear that every separable space is star-Lindelöf as well as every space of countable extent (in particular, every countably compact space or every Lindelöf space).

A space X is *centered-Lindelöf* (see [1], [6]) if every open cover $\mathcal{U}$ of X has a σ-centered subcover. A family of sets is *centered* if every finite subfamily has non-empty intersection and a family is σ*-centered* if it can be represented as the union of countably many centered subfamilies.

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A space X is *σ-starcompact* (see [14]) if for every open cover $\mathcal{U}$ of X, there exists a σ-compact subset F of X such that $\mathrm{St}(F, \mathcal{U}) = X$.

From the above definitions, it is not difficult to see that every discretely absolutely star-Lindelöf space is discretely star-Lindelöf, every discretely star-Lindelöf space is star-Lindelöf, every star-Lindelöf space is centered-Lindelöf and every star-Lindelöf space is σ-starcompact.

As natural generalizations of star-Lindelöfness and discretely star-Lindelöfness, one can consider the following cardinal functions:

Definition 1.1 ([1], [6], [7])**.** The *star-Lindelöf number* of the space X is the cardinal number
$\mathrm{st}\text{-}l(X) = \min\{\kappa :$ for every open cover $\mathcal{U}$ of X, there exists a subset $F \subseteq X$ such that $|F| \leq \kappa$ and $\mathrm{St}(F, \mathcal{U}) = X\}$.

Definition 1.2 ([7])**.** The *Aquaro number* of the space X is the cardinal number
$a(X) = \min\{\kappa :$ for every open cover $\mathcal{U}$ of X, there exists a discrete closed subset $F \subseteq X$ such that $|F| \leq \kappa$ and $\mathrm{St}(F, \mathcal{U}) = X\}$.

As a natural generalization of discretely absolutely star-Lindelöfness, we can define the following cardinal function:

Definition 1.3. The *absolute Aquaro number* of the space X is the cardinal number
$aa(X) = \min\{\kappa :$ for every open cover $\mathcal{U}$ of X and for every dense subset D of X, there exists a discrete closed subset (in X) $F \subseteq D$ such that $|F| \leq \kappa$ and $\mathrm{St}(F, \mathcal{U}) = X\}$.

It is easily proved that the following inequalities hold for every space X:

$$\mathrm{st}\text{-}l(X) \leq a(X) \leq aa(X).$$

Bonanzinga-Matveev [1] and Matveev [6] asked if the $\mathrm{st}\text{-}l(X)$ of a Tychonoff centered-Lindelöf space X cannot be greater than $\mathfrak{c}$. The author [10] answered negatively the question by giving an example to show that for any cardinal κ there exists a Tychonoff centered-Lindelöf space X such that $\mathrm{st}\text{-}l(X) \geq \kappa$. In [14], the author constructed an example showing that there exists a Tychonoff σ-starcompact space that is not star-Lindelöf. However, the author's space is not pseudocompact and its star-Lindelöf number is not greater than $\mathfrak{c}$. It is natural for us to consider the following questions:

Question 1. Is it true that the Aquaro number of a Tychonoff star-Lindelöf space cannot be greater than $\mathfrak{c}$?

Question 2. Is it true that the absolute Aquaro number of a Tychonoff discretely star-Lindelöf space cannot be greater than $\mathfrak{c}$?

Question 3. Is it true that the star-Lindelöf number of a Tychonoff pseudocompact σ-starcompact space cannot be greater than $\mathfrak{c}$?

The purpose of this paper is to answer negatively the above three questions by showing the three statements stated in the abstract.

The cardinality of a set A is denoted by $|A|$. Let ω denote the first infinite cardinal and $\mathfrak{c}$ denote the cardinality of the continuum. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. For each ordinal α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$ and $(\alpha, \beta] = \{\gamma : \alpha < \gamma \leq \beta\}$. Other terms and symbols that we do not define will be used as in [5].

2. Spaces with large star cardinal number

In this section, we show the three statements stated in the abstract. All examples of this section are of the form

$$(X \times \alpha) \cup (Y \times \{\alpha\})$$

where X is a space, Y is a subspace of X and α is an ordinal. The first two examples use Matveev's space. We now sketch the construction of Matveev's space M defined in [8], [9]. Let κ be an infinite cardinal and $D = \{0, 1\}$ be the discrete space. For every $\alpha < \kappa$, let z_α be the point of D^κ defined by $z_\alpha(\alpha) = 1$ and $z_\alpha(\beta) = 0$ for $\beta \neq \alpha$. Put $Z = \{z_\alpha : \alpha < \kappa\}$. For a given ordinal τ, Matveev's space $M(\kappa, \tau)$ is the subspace

$$M(\kappa, \tau) = (D^\kappa \times \tau) \cup (Z \times \{\tau\})$$

of the product space $D^\kappa \times (\tau + 1)$. Then $M(\kappa, \tau)$ is Tychonoff and $Z \times \{\tau\}$ is a discrete closed set of $M(\kappa, \tau)$ with $|Z \times \{\tau\}| = \kappa$.

We need the following lemma:

Lemma 2.1 ([9], [10]). *Assume that there exists a family $\{V_\alpha : \alpha < \kappa\}$ of open sets in D^κ such that $z_\alpha \in V_\alpha$ for each $\alpha < \kappa$. Then there exists a countable set $S \subseteq D^\kappa$ such that $S \cap V_\alpha \neq \emptyset$ for each $\alpha < \kappa$ and $\mathrm{cl}_{D^\kappa} S \cap Z = \emptyset$.*

Theorem 2.2. *For any cardinal κ, there exists a Tychonoff star-Lindelöf space X such that $a(X) \geq \kappa$.*

Proof: Since for any cardinal κ there is a larger regular uncountable cardinal, we can assume that κ itself is a regular uncountable cardinal. Choose a regular uncountable cardinal τ such that $\tau > \kappa$ and let $X = M(\kappa, \tau)$.

First we show that X is star-Lindelöf. To this end, let $\mathcal{U}$ be an open cover of X. For every $\alpha < \kappa$, there exists an $U_\alpha \in \mathcal{U}$ such that $\langle z_\alpha, \tau\rangle \in U_\alpha$. Choose $\beta_\alpha < \tau$ and an open neighborhood V_α of z_α in D^κ such that

$$((V_\alpha \cap Z) \times \{\tau\}) \cup (V_\alpha \times (\beta_\alpha, \tau)) \subseteq U_\alpha.$$

By applying Lemma 2.1 to the family $\{V_\alpha : \alpha < \kappa\}$, then we can find a countable set $S \subseteq D^\kappa$ such that $S \cap V_\alpha \neq \emptyset$ for all $\alpha < \kappa$. Let $\beta' = \sup\{\beta_\alpha : \alpha < \kappa\}$. Then $\beta' < \tau$, since τ is regular and $\tau > \kappa$. Let $F_0 = S \times \{\beta'\}$. Then $Z \times \{\tau\} \subseteq \mathrm{St}(F_0, \mathcal{U})$, since $U_\alpha \cap F_0 \neq \emptyset$ for each $\alpha < \kappa$. On the other hand, since $D^\kappa \times \tau$ is countably compact, we can find a finite subset $F_1 \subseteq D^\kappa \times \tau$ such that $D^\kappa \times \tau \subseteq \mathrm{St}(F_1, \mathcal{U})$. If we put $F = F_0 \cup F_1$, then F is a countable subset of X such that $X = \mathrm{St}(F, \mathcal{U})$, which shows that X is star-Lindelöf.

Next we show that $a(X) \geq \kappa$. We can partition κ as $\kappa = \cup\{A_{n\gamma} : n \in \omega, \gamma < \kappa\}$ such that $|A_{n\gamma}| = n$ for each $n \in \omega$ and $\gamma < \kappa$, $A_{n\gamma} \cap A_{n'\gamma'} = \emptyset$ for $\langle n, \gamma\rangle \neq \langle n', \gamma'\rangle$. For each $\alpha < \kappa$, pick an open neighborhood U_α of $\langle z_\alpha, \tau\rangle$ such that $U_\alpha \cap (Z \times \{\tau\}) = \langle z_\alpha, \tau\rangle$, and $U_{\alpha_1} \cap U_{\alpha_2} = \emptyset$ if $\alpha_1, \alpha_2 \in A_{n\gamma}$ and $\alpha_1 \neq \alpha_2$ for each $n \in \omega$ and $\gamma < \kappa$.

Let us consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \kappa\} \cup \{D^\kappa \times \tau\}$$

of the space X. It remains to show that $\mathrm{St}(F, \mathcal{U}) \neq X$ for any discrete closed subset of X with $|F| < \kappa$. To show this, let F be any discrete closed subset of X with $|F| < \kappa$. Let

$$\alpha' = \sup\{\gamma : F \cap \{\langle z_\alpha, \tau\rangle : \alpha \in A_{n\gamma}\} \neq \emptyset \ \text{ for some } \ n \in \omega \ \text{ and some } \ \gamma < \kappa\}.$$

Then $\alpha' < \kappa$, since κ is regular and $|F| < \kappa$. Thus $F \cap \{\langle z_\alpha, \tau\rangle : \alpha \in A_{n\gamma}\} = \emptyset$ for each $n \in \omega$ and $\gamma > \alpha'$. On the other hand, since $D^\kappa \times \tau$ is countably compact, then $F \cap (D^\kappa \times \tau)$ is finite. Thus we choose $n_0 \in \omega$ and $\gamma_0 > \alpha'$ such that $\{\langle z_\alpha, \tau\rangle : \alpha \in A_{n_0\gamma_0}\} \cap F = \emptyset$. Therefore $\langle z_\alpha, \tau\rangle \notin \mathrm{St}(F, \mathcal{U})$ for each $\alpha \in A_{n_0\gamma_0}$, which shows $a(X) \geq \kappa$. □

For a Tychonoff space X, let βX denote the Čech-Stone compactification of the space X.

Theorem 2.3. *There is a Tychonoff discretely star-Lindelöf space X such that $aa(X)$ does not exist.*

Proof: The author [10] showed that $M(\omega_1, \omega)$ is discretely star-Lindelöf.

Let

$$X = (\beta M(\omega_1, \omega) \times \omega_1) \cup (M(\omega_1, \omega) \times \{\omega_1\})$$

be the subspace of the product space $\beta M(\omega_1, \omega) \times (\omega_1 + 1)$.

First we show that X is discretely star-Lindelöf. To this end, let $\mathcal{U}$ be an open cover of X. Since $\beta M(\omega_1, \omega) \times \omega_1$ is countably compact, we can find a finite subset $F_1 \subseteq \beta M(\omega_1, \omega) \times \omega_1$ such that

$$\beta M(\omega_1, \omega) \times \omega_1 \subseteq \mathrm{St}(F_1, \mathcal{U}).$$

On the other hand, $M(\omega_1, \omega) \times \{\omega_1\}$ is discretely star-Lindelöf, since it is homeomorphic to $M(\omega_1, \omega)$. Thus there exists a countable subset $F_2 \subseteq M(\omega_1, \omega) \times \{\omega_1\}$ such that F_2 is discrete closed in $M(\omega_1, \omega) \times \{\omega_1\}$ and

$$M(\omega_1, \omega) \times \{\omega_1\} \subseteq \mathrm{St}(F_2, \mathcal{U}).$$

Since $M(\omega_1, \omega) \times \{\omega_1\}$ is closed in X, then F_2 is closed in X. If we put $F = F_1 \cup F_2$, then F is a countable discrete closed subset of X such that $X = \mathrm{St}(F, \mathcal{U})$, which shows that X is discretely star-Lindelöf.

Next we show that $aa(X)$ does not exist. For each $\alpha < \omega_1$, let $U_\alpha = \{\langle z_\alpha, \omega\rangle\} \cup (D^{\omega_1} \times \omega)$. Since $Z \times \{\omega\}$ is relatively discrete the set U_α is an open neighborhood of $\langle z_\alpha, \omega\rangle$ such that $U_\alpha \cap (Z \times \{\omega\}) = \{\langle z_\alpha, \omega\rangle\}$.

Let us consider the open cover

$$\mathcal{U} = \{U_\alpha \times (\alpha, \omega_1] : \alpha < \omega_1\} \cup \{\beta M(\omega_1, \omega) \times \omega_1\}$$

of the space X and the dense subset $\beta M(\omega_1, \omega) \times \omega_1$ of the space X. It remains to show that $\mathrm{St}(F, \mathcal{U}) \neq X$ for any discrete closed subset F of $\beta M(\omega_1, \omega) \times \omega_1$. To show this, let F be any discrete closed subset of $\beta M(\omega_1, \omega) \times \omega_1$. Then F is finite subset of $\beta M(\omega_1, \omega) \times \omega_1$, since $\beta M(\omega_1, \omega) \times \omega_1$ is countably compact. Let $\alpha' = \sup\{\alpha : \alpha \in \pi(F)\}$, where $\pi : \beta M(\omega_1, \omega) \times \omega_1 \to \omega_1$ is the projection. Then $\alpha' < \omega_1$, since F is finite. If we pick $\beta > \alpha'$, then $\langle\langle z_\beta, \omega\rangle, \omega_1\rangle \notin \mathrm{St}(F, \mathcal{U})$, since $U_\beta \times (\beta, \omega_1]$ is the only element of $\mathcal{U}$ containing $\langle\langle z_\beta, \omega\rangle, \omega_1\rangle$ and $(U_\beta \times (\beta, \omega_1]) \cap F = \emptyset$, which shows that $aa(X)$ does not exist. □

Remark 2.1. The referee asked whether there is a Tychonoff star-Lindelöf space X such that $aa(X)$ does not exist. The author noticed that there is a Tychonoff countably compact (hence, starcompact, star-Lindelöf and discretely star-Lindelöf) space X such that $aa(X)$ does not exist. The construction of the example is very much simpler than the construction of the space X in Theorem 2.3. In fact, let $X = \omega_1 \times (\omega_1 + 1)$ be the product of ω_1 and $\omega_1 + 1$. Then X is Tychonoff countably compact space. Let us show that $aa(X)$ does not exist. For each $\alpha < \omega_1$, let $U_\alpha = [0, \alpha) \times (\alpha, \omega_1]$. Let us consider the open cover $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\} \cup \{D\}$ and the dense subspace D of X, where $D = \omega_1 \times \omega_1$. It remains to show that $\mathrm{St}(F, \mathcal{U}) \neq X$ for any discrete closed subset F of D. To show this, let F be any discrete closed subset of D. Then F is finite subset of D, since D is countably compact. Let $\alpha_0 = \sup\{\alpha : \alpha \in \pi(F)\}$, where $\pi : \omega_1 \times (\omega_1 + 1) \to \omega_1 + 1$ is the projection. Then $\alpha_0 < \omega_1$, since F is finite. If we pick $\alpha' > \alpha_0$, then $\langle \alpha', \omega_1\rangle \notin \mathrm{St}(F, \mathcal{U})$. Indeed, for every $U_\beta \in \mathcal{U}$, if $\langle \alpha', \omega_1\rangle \in U_\beta$, then $\beta > \alpha'$. Finally, for each $\beta > \alpha'$, $U_\beta \cap F = \emptyset$, which shows that $aa(X)$ does not exist.

Theorem 2.4. *For any cardinal κ, there exists a pseudocompact σ-starcompact Tychonoff space X such that* $\mathrm{st}\text{-}l(X) \geq \kappa$.

Proof: We may assume that κ is a regular uncountable cardinal, as we have done in Theorem 2.2. Let $D = \{d_\alpha : \alpha < \kappa\}$ be a discrete space of the cardinality κ and

$$Y = (\beta D \times \omega) \cup (D \times \{\omega\})$$

be the subspace of the product space $\beta D \times (\omega + 1)$. Then Y is σ-starcompact, since $\beta D \times \omega$ is a σ-compact dense subset of Y.

Let

$$X = (\beta Y \times \kappa) \cup (Y \times \{\kappa\})$$

be the subspace of the product space $\beta Y \times (\kappa + 1)$. Clearly, X is a Tychonoff space. Since κ has uncountable cofinality, then $\beta Y \times \kappa$ is a countably compact dense subset of X, hence X is pseudocompact.

First we show that X is σ-starcompact. To this end, let $\mathcal{U}$ be an open cover of X. Since $\beta Y \times \kappa$ is countably compact, there exists a finite subset F of $\beta Y \times \kappa$ such that

$$\beta Y \times \kappa \subseteq \mathrm{St}(F, \mathcal{U}).$$

On the other hand, $Y \times \{\kappa\}$ is σ-starcompact, since it is homeomorphic to Y. Thus

$$Y \times \{\kappa\} \subseteq \mathrm{St}((\beta D \times \omega) \times \{\kappa\}, \mathcal{U}),$$

since $(\beta D \times \omega) \times \{\kappa\}$ is a σ-compact dense subset of $Y \times \{\kappa\}$. Since $Y \times \{\kappa\}$ is closed in X, then $(\beta D \times \omega) \times \{\kappa\}$ is a σ-compact subset of X. If we put

$$E = F \cup ((\beta D \times \omega) \times \{\kappa\}).$$

Then E is a σ-compact subset of X such that $X = \mathrm{St}(E, \mathcal{U})$, which shows that X is σ-starcompact.

Next we show $\mathrm{st}\text{-}l(X) \geq \kappa$. For each $\alpha < \kappa$, let $U'_\alpha = \{d_\alpha\} \times [0, \omega]$, then U'_α is a compact subset of Y, hence U'_α is a clopen subset of Y and $U'_\alpha \cap U'_{\alpha'} = \emptyset$ for $\alpha \neq \alpha'$. For each $\alpha < \kappa$, let $U_\alpha = U'_\alpha \times (\kappa + 1)$, then U_α is an open subset of X and $U_\alpha \cap U_{\alpha'} = \emptyset$ for $\alpha \neq \alpha'$. For each $n \in \omega$, let $V'_n = \beta D \times \{n\}$, then V'_n is a compact subset of Y, hence V'_n is a clopen subset of Y and $V_n \cap V_m = \emptyset$ for $n \neq m$. For each $n \in \omega$, let $V_n = V'_n \times (\kappa + 1)$, then V_n is an open subset of X. Let us consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \kappa\} \cup \{V_n : n \in \omega\} \cup \{\beta Y \times [0, \kappa)\}$$

of X. It remains to show that $\mathrm{St}(F, \mathcal{U}) \neq X$ for any subset F of X with $|F| < \kappa$. To show this, let F be any subset of X with $|F| < \kappa$. Then there exists $\alpha_0 < \kappa$ such that $F \cap U_{\alpha_0} = \emptyset$, since κ is regular and $|F| < \kappa$. Hence $\langle\langle d_{\alpha_0}, \omega\rangle, \kappa\rangle \notin \mathrm{St}(F, \mathcal{U})$, since U_{α_0} is the only element of $\mathcal{U}$ containing $\langle\langle d_{\alpha_0}, \omega\rangle, \kappa\rangle$, which shows $\mathrm{st}\text{-}l(X) \geq \kappa$. $\square$

For normal spaces, it is well-known that countably compactness is equivalent with pseudocompactness, and countably compact space is starcompact. Thus we have the following result.

Theorem 2.5. *For any normal space X, the following conditions are equivalent:*

(1) *X is pseudocompact σ-starcompact;*
(2) *X is star-Lindelöf.*

Remark 2.2. The author does not know if there exists an example of a σ-starcompact normal space that is not star Lindelöf.

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References

[1] Bonanzinga M., *Star-Lindelöf and absolutely star-Lindelöf spaces*, Questions Answers Gen. Topology. **14** (1998), 79–104.
[2] Bonanzinga M., Matveev M.V., *Star-Lindelöfness versus centered-Lindelöfness*, Comment. Math. Univ. Carolin. **41** (2002), no. 1, 111–122.
[3] van Douwen E.K., Reed G.M., Roscoe A.W., Tree I.J., *Star covering properties*, Topology Appl. **39** (1991), 71–103.
[4] Dai M.-M., *A class of topological spaces containing Lindelöf spaces and separable spaces*, Chinese Ann. Math. Ser. **A 4** (1983), no. 5, 571–575.
[5] Engelking R., *General Topology. Revised and completed edition*, Heldermann Verlag, Berlin, 1989.
[6] Matveev M.V., *A survey on star covering properties*, Topology Atlas, preprint No. 330 (1998).
[7] Matveev M.V., *A survey on star covering properties II*, Topology Atlas, preprint No. 431 (2000).
[8] Matveev M.V., *How weak is weak extent?*, Topology Appl. **119** (2002), 229–232.
[9] Matveev M.V., *On space in countable web*, arxiv:math/0006199v1 (2000).
[10] Song Y.-K., *Spaces with large extent and large star-Lindelöf number*, Houston J. Math. **29** (2003), no. 2, 347–354.
[11] Song Y.-K., *Discretely star-Lindelöf spaces*, Tsukuba J. Math. **25** (2001), no. 2, 371–382.
[12] Song Y.-K., *Remarks on star-Lindelöf spaces*, Questions Answers Gen. Topology **20** (2002), 49–51.
[13] Song Y.-K., *Closed subsets of absolutely star-Lindelöf spaces II*, Comment. Math. Univ. Carolin. **44** (2003), no. 2, 329–334.
[14] Song Y.-K., *On σ-starcompact spaces*, Applied General. Topology **9** (2008), no. 2, 293–299.
[15] Yasui Y., Gao Z.-M., *Spaces in countable web*, Houston J. Math. **25** (1999), no. 2, 327–335.

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