Fixed-place ideals in commutative rings

Ali Rezaei Aliabad, Mehdi Badie

Abstract. Let I be a semi-prime ideal. Then $P_\circ \in \mathrm{Min}(I)$ is called irredundant with respect to I if $I \neq \bigcap_{P_\circ \neq P \in \mathrm{Min}(I)} P$. If I is the intersection of all irredundant ideals with respect to I, it is called a fixed-place ideal. If there are no irredundant ideals with respect to I, it is called an anti fixed-place ideal. We show that each semi-prime ideal has a unique representation as an intersection of a fixed-place ideal and an anti fixed-place ideal. We say the point $p \in \beta X$ is a fixed-place point if $O^p(X)$ is a fixed-place ideal. In this situation the fixed-place rank of p, denoted by $\text{FP-rank}_X(p)$, is defined as the cardinal of the set of all irredundant prime ideals with respect to $O^p(X)$. Let p be a fixed-place point, it is shown that $\text{FP-rank}_X(p) = \eta$ if and only if there is a family $\{Y_\alpha\}_{\alpha \in A}$ of cozero sets of X such that: 1- $|A| = \eta$, 2- $p \in \mathrm{cl}_{\beta X} Y_\alpha$ for each $\alpha \in A$, 3- $p \notin \mathrm{cl}_{\beta X}(Y_\alpha \cap Y_\beta)$ if $\alpha \neq \beta$ and 4- η is the greatest cardinal with the above properties. In this case p is an F-point with respect to Y_α for any $\alpha \in A$.

Keywords: ring of continuous functions, fixed-place, anti fixed-place, irredundant, semi-prime, annihilator, affiliated prime, fixed-place rank, Zariski topology

Classification: Primary 13Axx; Secondary 54C40

1. Introduction

In this article, any ring R is commutative with unity. A semi-prime ideal means an ideal which is an intersection of prime ideals. For each ideal I of R and each element a of R, we denote the ideals $\{x \in R : ax \in I\}$ by $(I : a)$. When $I = \{0\}$ we write instead $\mathrm{Ann}(a)$ and call this the annihilator of a. If $\mathrm{Ann}(a)$ is maximal in the set of all annihilators of nonzero elements of R, then $\mathrm{Ann}(a)$ is a prime ideal of R, and it is called an affiliated prime ideal. A prime ideal P is said to be a minimal prime ideal over an ideal I, if there are no prime ideals strictly contained in P that contain I. By $\mathrm{Min}(I)$ we mean the set of all minimal prime ideals over I; we use $\mathrm{Min}(R)$ instead of $\mathrm{Min}(\{0\})$. A ring is called reduced, if the ideal $\{0\}$ is a semi-prime ideal. $\mathrm{Zd}(R)$ stands for the set of all zero divisors of R. A prime ideal P is called a Bourbaki (resp. Zariski-Samuel) associated prime divisor of an ideal I if $(I : x) = P$ (resp., $(I : x)$ is P-primary) for some $x \in R$. We denote the set of Bourbaki (Zariski-Samuel) associated prime divisors of an ideal I by $\mathrm{B}(I)$ (resp., $[\text{Z-S}](I)$). It is clear that if I is a semi-prime ideal, then $\mathrm{B}(I) = [\text{Z-S}](I)$. A representation $I = \bigcap_{P \in \mathcal{P}} P$ of I as an intersection of prime ideals is called irredundant if no $P \in \mathcal{P}$ may be omitted.

For every $S \subseteq R$, by $h(S)$ and $h^c(S)$ we mean the sets $\{P \in \mathrm{Min}(R) : S \subseteq P\}$ and $\mathrm{Min}(R) \setminus h(S)$, respectively. Recall that the Zariski topology on $\mathrm{Min}(R)$ is the one generated by the closed base $\{h(a) : a \in R\}$.

We assume throughout the paper that any topological space X is Tychonoff, βX is the Stone-Čech compactification of X and $C(X)$ is the ring of all real valued continuous functions on X. For any $f \in C(X)$, we denote $f^{-1}\{0\}$ and $X \setminus f^{-1}\{0\}$ by $Z(f)$ and $Coz(f)$, respectively. Supposing $S \subseteq C(X)$, we define $Z(S) = \{Z(f) : f \in S\}$ and we use $Z(X)$ instead of $Z(C(X))$. For any $\mathcal{B} \subseteq Z(X)$, we define $Z^{-1}(\mathcal{B}) = \{f \in C(X) : Z(f) \in \mathcal{B}\}$. Suppose that $A \subseteq \beta X$, then by $M^A(X)$ and $O^A(X)$ we mean the sets $\{f \in C(X) : A \subseteq \mathrm{cl}_{\beta X} Z(f)\}$ and $\{f \in C(X) : A \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)\}$, respectively. If $A = \{p\}$ for some $p \in \beta X$, then for brevity, we use the notations $M^p(X)$ and $O^p(X)$. We denote $Z(O^p(X))$ by $\mathcal{O}^p(X)$. An element $p \in \beta X$ is called an F-point with respect to X if $O^p(X)$ is prime ideal. The set of all isolated points of a topological space X is denoted by $I(X)$. Clearly, $\overline{I(X)} = X$ if and only if X has a smallest dense subspace; exactly, this subspace is equal to $I(X)$. The reader is referred to [6] for other terms and notations.

In Section 2, we introduce irredundant families and irredundant prime ideals with respect to a semi-prime ideal. We use these notions to define central concepts of the article: fixed-place ideals, anti fixed-place ideals, and fixed-place rank for the fixed-place ideals. We give equivalence conditions for the above concepts. We prove that the intersection of two fixed-place ideals is fixed-place. In Section 3, we show that each semi-prime ideal has a unique intersection representation of a fixed-place ideal and an anti fixed-place ideal. In Section 4, we obtain some results from viewpoint of the Zariski topology. We show that I is a fixed-place (an anti fixed-place) ideal if and only if the set of isolated points of $\mathrm{Min}(I)$ is a dense subspace ($\mathrm{Min}(I)$ has no isolated point). We introduce fixed-place families in this section, and prove that $\mathcal{P} \subseteq Spec(R)$ is fixed-place if and only if $\mathcal{P}$ is discrete in the Zariski topology. In the final section, we study fixed-place ideals in $C(X)$. We show that the zero ideal of $C(X)$ is fixed-place (anti fixed-place) if and only if $I(X)$ is dense in X (X has no isolated point). We introduce fixed-place points and fixed-place rank of these points and we generalize Proposition 3.1 in [8] and Theorem 3.1 in [10].

2. Fixed-place ideal

By Theorem 2.1 and Proposition 4.11 of [12], each irredundant prime ideal of a semi-prime ideal I is of the form $(I : x)$, for some $x \in R$, and if a semi-prime ideal I is equal to an irredundant intersection of the family $\{P_\alpha\}_{\alpha \in A}$ of prime ideals, then $\{P_\alpha\}_{\alpha \in A}$ is the set of all irredundant prime ideals of I. We can summarize these facts as follows.

Theorem 2.1. *If I is a semi-prime ideal, then the following statements are equivalent.*

(a) *There is a family* $\mathcal{P}$ *of prime ideals such that* $I = \bigcap \mathcal{P}$ *is an irredundant intersection.*
(b) $I = \bigcap \mathcal{B}(I)$.
(c) $I = \bigcap [Z\text{-}S](I)$.
(d) $I = \bigcap \{(I : x) : x \in R \text{ and } (I : x) \text{ is a prime ideal } \}$.

In this situation, we have

$$\mathcal{P} = \mathcal{B}(I) = [Z\text{-}S](I) = \{(I : x) : x \in R \text{ and } (I : x) \text{ is a prime ideal } \}.$$

Definition 2.2. Suppose that I is a semi-prime ideal of a ring R and $\emptyset \neq \mathcal{P} \subseteq \mathrm{Min}(I)$. We say $\mathcal{P}$ is *irredundant* with respect to I if $I \neq \bigcap_{P \in \mathrm{Min}(I) \setminus \mathcal{P}} P$. If $\mathcal{P} = \{P\}$, then we say that P is irredundant with respect to I. If I is equal to the intersection of irredundant prime ideals of I, then we call I a *fixed-place ideal*, exactly, by Theorem 2.1, we have $I = \bigcap \mathcal{B}(I)$. In this situation the *fixed-place rank* of I is denoted by FP–rank(I), and it is defined by the cardinal of $\mathcal{B}(I)$. If $\mathcal{B}(I) = \emptyset$, i.e., I has no irredundant prime ideal, then we call I an *anti fixed-place ideal*.

The following proposition is an immediate consequence of Theorem 2.1.

Proposition 2.3. *If* I *is a semi-prime ideal of a ring* R *and* $\mathcal{A} \subseteq \mathcal{B}(I) \neq \emptyset$, *then* $J = \bigcap_{P \in \mathcal{A}} P$ *is a fixed-place ideal and* $\mathcal{B}(J) = \mathcal{A}$.

Proposition 2.4. *Let* I *be a semi-prime ideal of a ring* R *and* $\mathcal{P} \subseteq \mathrm{Min}(I)$. *If* $I = \bigcap_{P \in \mathcal{P}} P$, *then* $\mathcal{B}(I) \subseteq \mathcal{P}$.

Proof: If $P_0 \notin \mathcal{P}$, then $\mathcal{P} \subseteq \mathrm{Min}(I) \setminus \{P_0\}$. Thus

$$I \subseteq \bigcap_{P_0 \neq P \in \mathrm{Min}(I)} P \subseteq \bigcap_{P_0 \neq P \in \mathcal{P}} P = I \Rightarrow \bigcap_{P_0 \neq P \in \mathrm{Min}(I)} P = I \Rightarrow P_0 \notin \mathcal{B}(I).$$

Hence $\mathcal{B}(I) \subseteq \mathcal{P}$. □

Theorem 2.5. *Let* I *be a semi-prime ideal of a ring* R, $\mathcal{P} \subseteq \mathrm{Min}(I)$ *and* $\mathcal{Q} = \{P/I : P \in \mathcal{P}\}$. *The family* $\mathcal{P}$ *is irredundant with respect to* I *if and only if* $\mathcal{Q}$ *is irredundant with respect to the zero ideal of the ring* R/I.

Proof: We know that

$$I(a) \in \bigcap_{P/I \in \mathrm{Min}(R/I) \setminus \mathcal{Q}} \frac{P}{I} = \bigcap_{P \in \mathrm{Min}(I) \setminus \mathcal{P}} \frac{P}{I} \quad \Leftrightarrow \quad a \in \bigcap_{P \in \mathrm{Min}(I) \setminus \mathcal{P}} P.$$

Thus

$$\bigcap_{P/I \in \mathrm{Min}(R/I) \setminus \mathcal{Q}} \frac{P}{I} = \{0\} \quad \Leftrightarrow \quad \bigcap_{P \in \mathrm{Min}(I) \setminus \mathcal{P}} P = I.$$

Therefore, $\mathcal{P}$ is irredundant with respect to ideal I if and only if $\mathcal{Q}$ is irredundant with respect to the zero ideal of the ring R/I. □

By the above theorem, we can see that for studying the fixed-place ideals it is sufficient to focus on the zero ideal of the reduced rings. Thus, in the remainder of this section we assume that R is reduced. By this assumption, it is clear that if P is a minimal prime ideal of R, then P is irredundant with respect to the zero ideal if and only if P is an affiliated prime ideal.

Proposition 2.6. *Suppose that R is a ring and $\mathcal{P} \subseteq \operatorname{Min}(R)$.*

(a) *If $\mathcal{P} \cap \mathcal{B}(\{0\}) \neq \emptyset$, then $\mathcal{P}$ is irredundant with respect to $\{0\}$.*

(b) *If $J = \bigcap_{P\in\mathcal{B}(\{0\})} P$,*

$$\emptyset \neq \mathcal{S} = \{P \in \operatorname{Min}(R) : P \notin \mathcal{B}(\{0\}) \ \text{ and } \ P \not\supseteq J\} \ \text{ and}$$
$$\emptyset \neq \mathcal{T} = \{P \in \operatorname{Min}(R) : P \notin \mathcal{B}(\{0\}) \ \text{ and } \ P \supseteq J\}$$

then $\mathcal{S}$ is irredundant with respect to the zero ideal, where $\mathcal{S} \cap \mathcal{B}(\{0\}) = \emptyset$. Also, $\mathcal{T}$ is not irredundant with respect to the zero ideal.

Proof: The proof is straightforward. □

In this part, we study the irredundant family with respect to the zero ideal and give some equivalent conditions.

Proposition 2.7. *Let $\mathcal{P} \subseteq \operatorname{Min}(R)$. If $\mathcal{P}$ is an irredundant family with respect to the zero ideal R, then there exists $0 \neq a \in R$ such that $\bigcap \mathcal{P} \subseteq \operatorname{Ann}(a)$.*

Proof: Since $\mathcal{P}$ is an irredundant family with respect to the zero ideal of R, we have that

$$\bigcap_{P\in\operatorname{Min}(R)\setminus\mathcal{P}} P \neq \{0\}.$$

Say $0 \neq a \in \bigcap_{P\in\operatorname{Min}(R)\setminus\mathcal{P}} P$. For any $b \in \bigcap_{a\in\mathcal{P}} P$,

$$ab \in \bigcap_{P\in(\operatorname{Min}(R)\setminus\mathcal{P})\cup\mathcal{P}} P = \bigcap_{P\in\operatorname{Min}(R)} P = \{0\}.$$

Therefore, $\bigcap_{P\in\mathcal{P}} P \subseteq \operatorname{Ann}(a)$. □

Lemma 2.8. *For each element $a \in R$, we have $\operatorname{Min}(\operatorname{Ann}(a)) = h^c(a)$.*

Proof: We claim that $a \notin P$ for each $P \in \operatorname{Min}(\operatorname{Ann}(a))$. Suppose, on the contrary, that $a \in P$ for some $P \in \operatorname{Min}(\operatorname{Ann}(a))$. Then

$$\exists b \notin P,\ ab \in \operatorname{Ann}(a) \ \Rightarrow \ a^2b = 0 \ \Rightarrow \ ab = 0 \ \Rightarrow \ b \in \operatorname{Ann}(a) \ \Rightarrow \ b \in P,$$

which is impossible. Now, we prove that P is a minimal prime ideal for each $P \in \operatorname{Min}(\operatorname{Ann}(a))$. To see this,

$$\forall x \in P \quad \exists y \notin P \qquad xy \in \operatorname{Ann}(a) \quad \Rightarrow \quad xya = 0.$$

But $ya \notin P$, hence P is a minimal prime ideal and therefore $P \in h^c(a)$. Consequently, $\operatorname{Min}(\operatorname{Ann}(a)) \subseteq h^c(a)$. Now, we show that $h^c(a) \subseteq \operatorname{Min}(\operatorname{Ann}(a))$.

Suppose that $P \in h^c(a)$. It is sufficient to show that $\mathrm{Ann}(a) \subseteq P$. But it is evident, because $a\,\mathrm{Ann}(a) = \{0\} \subseteq P$. $\square$

Proposition 2.9. *Let a be a nonzero element of R and $\mathcal{P} \subseteq \mathrm{Min}(\mathrm{Ann}(a))$. The family $\mathcal{P}$ is irredundant with respect to the zero ideal of R if and only if $\mathcal{P}$ is irredundant with respect to* $\mathrm{Ann}(a)$.

Proof: $\Rightarrow$) Suppose that the assertion of the proposition is false, then

$$\mathrm{Ann}(a) = \bigcap_{P \in \mathrm{Min}(\mathrm{Ann}(a)) \setminus \mathcal{P}} P.$$

Thus

$$\{0\} = \bigcap_{P \in \mathrm{Min}(R)} P = \Big(\bigcap_{P \in \mathrm{Min}(\mathrm{Ann}(a))} P \Big) \cap \Big(\bigcap_{\substack{P \in \mathrm{Min}(R) \\ P \notin \mathrm{Min}(\mathrm{Ann}(a))}} P \Big)$$

$$= \Big(\bigcap_{P \in \mathrm{Min}(\mathrm{Ann}(a)) \setminus \mathcal{P}} P \Big) \cap \Big(\bigcap_{\substack{P \in \mathrm{Min}(R) \\ P \notin \mathrm{Min}(\mathrm{Ann}(a))}} P \Big) = \bigcap_{P \in \mathrm{Min}(R) \setminus \mathcal{P}} P.$$

Therefore, $\mathcal{P}$ is not irredundant with respect to the zero ideal of R and this is a contradiction.

$\Leftarrow$) Let $\mathcal{P} \subseteq \mathrm{Min}(\mathrm{Ann}(a)) = h^c(a)$ be irredundant with respect to $\mathrm{Ann}(a)$. We show that $\mathcal{P}$ is irredundant with respect to the zero ideal. On the contrary

$$\{0\} = \bigcap_{P \in \mathrm{Min}(R) \setminus \mathcal{P}} P = \Big(\bigcap_{P \in h^c(a) \setminus \mathcal{P}} P \Big) \cap \Big(\bigcap_{P \in h(a)} P \Big).$$

Since $\bigcap_{P \in h(a)} P \nsubseteq Q$ for each $Q \in h^c(a)$, it follows that

$$\forall Q \in h^c(a) \qquad \bigcap_{P \in h^c(a) \setminus \mathcal{P}} P \subseteq Q.$$

Hence

$$\bigcap_{P \in h^c(a) \setminus \mathcal{P}} P \subseteq \bigcap_{P \in h^c(a)} P = \mathrm{Ann}(a) \qquad \Rightarrow \qquad \bigcap_{P \in h^c(a) \setminus \mathcal{P}} P = \mathrm{Ann}(a),$$

which is a contradiction. $\square$

An immediate consequence of the above proposition is that $\mathrm{Min}(\mathrm{Ann}(a))$ is irredundant with respect to the zero ideal of R.

Proposition 2.10. *If the zero ideal in a ring R is a fixed-place ideal, then* $\mathrm{Zd}(R) = \bigcup_{P \in \mathcal{B}(\{0\})} P$.

Proof: Clearly, $\bigcup_{P \in \mathcal{B}(\{0\})} P \subseteq \mathrm{Zd}(R)$. We only need to show that $\mathrm{Zd}(R) \subseteq \bigcup_{P \in \mathcal{B}(\{0\})} P$. On the contrary, suppose that there exists a zero divisor a which is not in $\bigcup_{P \in \mathcal{B}(\{0\})} P$, then a nonzero element b exists such that $ab = 0$. Since for each $P \in \mathcal{B}(\{0\})$, $a \notin P$ and $ab = 0 \in P$, it follows that $0 \neq b \in \bigcap_{P\in\mathcal{B}(\{0\})} P$. This contradicts our assumption. $\square$

Now, we are going to show that the intersection of two fixed-place ideals is also a fixed-place ideal. To see this, we need the following lemma.

Lemma 2.11. *If I and J are two fixed-place ideals of R such that $I \nsubseteq P$ for each $P \in \mathcal{B}(J)$ and $J \nsubseteq P$ for each $P \in \mathcal{B}(I)$, then $I \cap J$ is a fixed-place ideal.*

Proof: Clearly, we have $I \cap J = \bigcap\{P \in Spec(R) : P \in \mathcal{B}(I) \cup \mathcal{B}(J)\}$. Now, by Theorem 2.1, it is enough to prove that $\mathcal{B}(I \cap J) = \mathcal{B}(I) \cup \mathcal{B}(J)$. To prove this, suppose that $P_\circ \in \mathcal{B}(I) \cup \mathcal{B}(J)$. Without loss of generality, we may assume that $P_\circ \in \mathcal{B}(I)$. On the contrary, let $I \cap J = \bigcap_{\substack{P\in\mathcal{B}(I)\cup\mathcal{B}(J)\\ P\neq P_0}} P$. Thus

$$\bigcap_{\substack{P\in\mathcal{B}(I)\cup\mathcal{B}(J)\\ P\neq P_\circ}} P \subseteq P_\circ \quad \Rightarrow \quad J \cap \Big(\bigcap_{\substack{P\in\mathcal{B}(I)\\ P\neq P_\circ}} P\Big) \subseteq P_\circ$$

$$\Rightarrow \quad \bigcap_{\substack{P\in\mathcal{B}(I)\\ P\neq P_\circ}} P \subseteq P_\circ \quad \Rightarrow \quad \bigcap_{\substack{P\in\mathcal{B}(I)\\ P\neq P_\circ}} P = I,$$

which is a contradiction. $\square$

Theorem 2.12. *If I and J are two fixed-place ideals of R, then $I \cap J$ is also a fixed-place ideal.*

Proof: Let $I' = \bigcap\{P \in \mathcal{B}(I) : P \nsupseteq J\}$ and $J' = \bigcap\{P \in \mathcal{B}(J) : P \nsupseteq I'\}$. Then, it is clear that $I \cap J = I' \cap J = I' \cap J'$, $\mathcal{B}(J') = \{P \in \mathcal{B}(J) : P \nsupseteq I\}$ and $\mathcal{B}(I') = \{P \in \mathcal{B}(I) : P \nsupseteq J\}$. Therefore

$$(1) \qquad \forall P \in \mathcal{B}(I') \quad J' \nsubseteq P, \quad \forall P \in \mathcal{B}(J') \quad I' \nsubseteq P.$$

On the other side, $I \subseteq I'$ and $J \subseteq J'$. Therefore, I' and J' are two fixed-place ideals such that they satisfy the condition of Lemma 2.11 and consequently $I' \cap J' = I \cap J$ is a fixed-place ideal. $\square$

In the following, we show that if I is not a fixed-place ideal, then there is no smallest fixed-place ideal containing I.

Proposition 2.13. *Let I be a semi-prime ideal of a ring R. If I is not fixed-place, then the set of all fixed-place ideals containing I has no minimal element.*

Proof: On the contrary, suppose that J is a minimal element of that set. Consequently, $J \cap P$ is a fixed-place ideal for each $P \in \mathrm{Min}(I)$. Thus $J = J \cap P$, hence $J \subseteq P$. This implies that $J = \bigcap_{P\in\mathrm{Min}(I)} P = I$. It follows that I is a fixed-place ideal, which is a contradiction. $\square$

3. Unique decomposition

Let $\mathscr{P}$ be the set of all subsets of $Spec(R)$. Throughout this section $\sim$ and $\leq$ denote the relations on $\mathscr{P}$ defined by

$$\mathcal{S} \leq \mathcal{T} \quad \Leftrightarrow \quad \forall P_1 \in \mathcal{S}, \quad \exists P_2 \in \mathcal{T} \qquad P_1 \subseteq P_2$$

and

$$\mathcal{S} \sim \mathcal{T} \quad \Leftrightarrow \quad \bigcap \mathcal{S} = \bigcap \mathcal{T}.$$

It is easy to check that $\mathscr{P}$ with relation $\leq$ is a complete lattice, in which $Spec(R)$ is the greatest element and $\emptyset$ is the smallest element. Furthermore, the relation $\sim$ is an equivalence relation. If we put $\mathfrak{P} = \{[\mathcal{S}] : \mathcal{S} \subseteq Spec(R)\}$ and $\mathfrak{I} = \{I : I$ is a semi-prime ideal of $R\}$, then the function $K : \mathfrak{P} \longrightarrow \mathfrak{I}$ (resp. $H : \mathfrak{I} \longrightarrow \mathfrak{P}$) is defined by $K([\mathcal{S}]) = \bigcap [\mathcal{S}]$ (resp. $H(I) = [\mathrm{Min}(I)]$). Throughout this section we use the above notation.

Lemma 3.1. *Let $\{\mathcal{S}_\alpha\}_{\alpha \in A}$ and $\{\mathcal{T}_\alpha\}_{\alpha \in A}$ be two families of subsets of $Spec(R)$. If $\mathcal{S}_\alpha \sim \mathcal{T}_\alpha$ for each $\alpha \in A$, then $[\bigcup_{\alpha \in A} \mathcal{S}_\alpha] = [\bigcup_{\alpha \in A} \mathcal{T}_\alpha]$.*

Proof: The proof is standard. □

Definition 3.2. We call two families $\mathcal{S}$ and $\mathcal{T}$ of $\mathscr{P}$ *essentially disjoint* if

$$\forall P \in \mathcal{S} \quad \bigcap \mathcal{T} \nsubseteq P, \qquad \forall P \in \mathcal{T} \quad \bigcap \mathcal{S} \nsubseteq P.$$

In this way, we call two classes $\mathscr{S}$ and $\mathscr{T}$ essentially disjoint, if there are two essentially disjoint families $\mathcal{S}$ and $\mathcal{T}$ in $\mathscr{S}$ and $\mathscr{T}$, respectively. Finally, we say two semi-prime ideals I and J are essentially disjoint if $[H(I)]$ and $[H(J)]$ are essentially disjoint.

Theorem 3.3. *If I is a semi-prime ideal of R, then we can write I as a unique intersection of a fixed-place ideal and an anti fixed-place ideal which are essentially disjoint.*

Proof: Let $J = \bigcap_{P \in \mathcal{B}(I)} P$, $\mathcal{A} = \{P \in \mathrm{Min}(I) : P \notin \mathcal{B}(I), P \nsupseteq J\}$, $K = \bigcap_{P \in \mathcal{A}} P$ and $\mathcal{B} = \{P \in \mathrm{Min}(I) : P \notin \mathcal{B}(I), P \supseteq J\}$. Then

$$I = \bigcap_{P \in \mathrm{Min}(I)} P = \Big(\bigcap_{P \in \mathcal{B}(I)} P \Big) \cap \Big(\bigcap_{P \in \mathcal{B}} P \Big) \cap \Big(\bigcap_{P \in \mathcal{A}} P \Big) = J \cap K.$$

By Corollary 2.3, J is fixed-place. We claim that $\mathcal{A} \subseteq \mathrm{Min}(K)$, because for each $P_\circ \in \mathcal{A}$, $K \subseteq P \subseteq P_\circ$, we have $I \subseteq K \subseteq P \subseteq P_\circ$. Since $P_\circ \in \mathrm{Min}(I)$, $P = P_\circ$ and therefore $P_\circ \in \mathrm{Min}(K)$. By Proposition 2.4, $\mathcal{B}(K) \subseteq \mathcal{A}$. For each $P_\circ \in \mathcal{A}$,

$$\begin{aligned} P_\circ \supseteq I = \bigcap_{P_\circ \neq P \in \mathrm{Min}(I)} P &= \Big(\bigcap_{P \in \mathcal{B}(I)} P \Big) \cap \Big(\bigcap_{P \in \mathcal{B}} P \Big) \cap \Big(\bigcap_{P_\circ \neq P \in \mathcal{A}} P \Big) \\ &= J \cap \Big(\bigcap_{P_\circ \neq P \in \mathcal{A}} P \Big). \end{aligned}$$

Since $J \nsubseteq P_\circ$, we can write

$$\bigcap_{P_\circ \neq P \in \mathcal{A}} P \subseteq P_\circ \quad \Rightarrow \quad K = \bigcap_{P \in \mathcal{A}} P = (\bigcap_{P_\circ \neq P \in \mathcal{A}} P) \cap P_\circ = \bigcap_{P_\circ \neq P \in \mathcal{A}} P.$$

This implies that $P_\circ \notin \mathcal{B}(K)$ and therefore $\mathcal{B}(K) = \emptyset$, thus K is an anti fixed-place ideal. Now, suppose that $I = J_1 \cap K_1$ where J_1 is a fixed-place ideal, K_1 is an anti fixed-place ideal and J_1 and K_1 are essentially disjoint. So there are some essentially disjoint families of prime ideals $\mathcal{P}$ and $\mathcal{Q}$ such that $J_1 = \bigcap \mathcal{P}$ and $K_1 = \bigcap \mathcal{Q}$. Since J_1 is fixed-place, we can assume that $\mathcal{P} = \mathcal{B}(J_1)$. Now, we prove that $J_1 = J$. To prove this, it is enough to show that $\mathcal{B}(J) = \mathcal{B}(J_1)$. If $P_\circ \in \mathcal{B}(J) = \mathcal{B}(I)$, then there exists $a \in R$ such that $P_\circ = (I : a)$, thus $P_\circ = (J_1 \cap K_1 : a) = (J_1 : a) \cap (K_1 : a)$. Since K_1 is anti fixed-place, by Theorem 2.1, we have $P_\circ \neq (K_1 : a)$. Hence $P_\circ = (J_1 : a)$ and this implies that $P_\circ \in \mathcal{B}(J_1)$. Therefore, $\mathcal{B}(J) \subseteq \mathcal{B}(J_1)$. Conversely, if $P_\circ \in \mathcal{B}(J_1)$, then $K_1 \nsubseteq P_\circ$ and so there exists $a \in K_1 \setminus P_\circ$. Hence, $(I : a) = (J_1 \cap K_1 : a) = (J_1 : a) = \bigcap_{a \notin P \in \mathcal{B}(J_1)} P$. Since $(J_1 : a)$ is fixed-place and $P_\circ \in \mathcal{B}(J_1 : a)$, Theorem 2.1 shows that there exists $b \in R$ such that $((J_1 : a) : b) = P_\circ$. Therefore, $(I : ab) = ((I : a) : b) = ((J_1 : a) : b) = P_\circ$. This implies that $P_\circ \in \mathcal{B}(I) = \mathcal{B}(J)$, thus $\mathcal{B}(J_1) \subseteq \mathcal{B}(J)$, and therefore $J_1 = J$. To complete the proof, we must show that $K = K_1$. Clearly, since $J \cap K = I = J_1 \cap K_1$, for each $Q \in \mathcal{Q}$, we have $J = J_1 \nsubseteq Q$ and so $K \subseteq Q$. It follows that $K \subseteq K_1$. By the same manner we can see $K_1 \subseteq K$. Thus, $K_1 = K$. □

Proposition 3.4. *The zero ideal in a reduced ring R is fixed-place if and only if* $\mathrm{Ann}(a)$ *is fixed-place, for each $a \in \mathrm{Zd}(R)$.*

Proof: $\Rightarrow$) Since the zero ideal is a fixed-place ideal, $\{0\} = \bigcap_{P \in \mathcal{B}(\{0\})} P$. Thus $\mathrm{Ann}(a) = (0 : a) = \left(\bigcap_{P \in \mathcal{B}(\{0\})} P : 0\right) = \bigcap_{a \notin P \in \mathcal{B}(\{0\})} P$ for each $a \in \mathrm{Zd}(R)$. By Corollary 2.3, it implies that $\mathrm{Ann}(a)$ is a fixed-place ideal.

$\Leftarrow$) By Theorem 3.3, $\{0\} = J \cap K$ where J is a fixed-place ideal, K is an anti fixed-place ideal and J and K are essentially disjoint. To obtain contradiction, suppose that $\{0\}$ is not fixed-place, then $J \neq \{0\}$, so there exists $0 \neq a \in J$. It is clear that $a \in \mathrm{Zd}(R)$. Consequently

$$\mathrm{Ann}(a) = (0 : a) = (J \cap K : a) = (K : a). \tag{1}$$

Since $\mathrm{Ann}(a)$ is a fixed-place ideal, it has an irredundant prime ideal P. Hence, by Theorem 2.1, there exists an element b in R such that $(\mathrm{Ann}(a) : b) = P$. We conclude from (1) that $(K : ab) = ((K : a) : b) = (\mathrm{Ann}(a) : b) = P$. Theorem 2.1 shows that $P \in \mathcal{B}(K)$, which contradicts our assumption. □

The following corollary is an immediate consequence of Theorem 2.5 and Proposition 3.4. This corollary will be needed to prove the next proposition.

Corollary 3.5. *Suppose that J is a semi-prime ideal of a ring R. The ideal J is fixed-place if and only if $(J : a)$ is a fixed-place ideal for each $a \in R$ where $J \subset (J : a)$.*

Proposition 3.6. *The zero ideal in a reduced ring R is an anti fixed-place ideal if and only if $\mathrm{Ann}(a)$ is an anti fixed-place ideal for each $a \in \mathrm{Zd}(R)$.*

PROOF: $\Rightarrow$) On the contrary, if for some $a \in \mathrm{Zd}(R)$, $\mathrm{Ann}(a)$ is not anti fixed-place, then there exist b in R and a prime ideal P such that $P = (\mathrm{Ann}(a) : b)$, by Theorem 2.1. In this way $P = (\mathrm{Ann}(a) : b) = (0 : ab) = \mathrm{Ann}(ab)$. So, by Theorem 2.1, $P \in \mathcal{B}(\{0\})$, which is impossible.

$\Leftarrow$) By Theorem 3.3, there is a fixed-place ideal J and an anti fixed-place ideal K such that $\{0\} = J \cap K$. Suppose that the zero ideal is not an anti fixed-place ideal, so $K \neq \{0\}$. Hence there exists $0 \neq a \in K$. It is clear that $a \in \mathrm{Zd}(R)$ and

$$\mathrm{Ann}(a) = (0 : a) = (J \cap K : a) = (J : a). \tag{2}$$

Since J is a fixed-place ideal, $(J : a)$ is also a fixed-place ideal, by Corollary 3.5. So, we can conclude from (2) that $\mathrm{Ann}(a)$ is a fixed-place ideal, which is a contradiction. □

4. Fixed-place ideal and Zariski topology

Throughout this section, for convenience, by Z we mean the set $\mathrm{Min}(R)$.

Theorem 4.1. *Let R be a reduced ring and $\mathcal{P} \subseteq Z$. The family $\mathcal{P}$ is irredundant with respect to the zero ideal of R if and only if $\mathrm{int}_Z \mathcal{P} \neq \emptyset$.*

PROOF: $\Rightarrow$) Since $\mathcal{P}$ is an irredundant family with respect to the zero ideal of R, we have

$$\bigcap_{P \in Z \setminus \mathcal{P}} P \neq \{0\}.$$

Hence, there is a nonzero element $a \in \bigcap_{P \in Z \setminus \mathcal{P}} P$. Obviously, $\emptyset \neq h^c(a) \subseteq \mathcal{P}$. Therefore, $\mathrm{int}_Z \mathcal{P} \neq \emptyset$.

$\Leftarrow$) Since $\mathrm{int}_Z \mathcal{P} \neq \emptyset$, there is an element $a \in R$ such that

$$\emptyset \neq h^c(a) \cap Z \subseteq \mathcal{P} \;\Rightarrow\; Z \setminus \mathcal{P} \subseteq h(a) \neq Z \;\Rightarrow\; 0 \neq a \in \bigcap_{P \in h(a)} P \subseteq \bigcap_{P \in Z \setminus \mathcal{P}} P.$$

Consequently, $\mathcal{P}$ is irredundant with respect to the zero ideal. □

An immediate conclusion of the above theorem is the following corollary.

Corollary 4.2. *Let P be a minimal prime ideal of a reduced ring R. The ideal P is an isolated point of Z if and only if $P \in \mathcal{B}(\{0\})$.*

Corollary 4.3. *The zero ideal of a reduced ring R is anti fixed-place if and only if Z has no isolated point.*

PROOF: It is evident, by Corollary 4.2. □

Theorem 4.4. *The zero ideal of a reduced ring R is a fixed-place ideal if and only if $\overline{I(Z)} = Z$.*

PROOF: $\Rightarrow$) Clearly, by Corollary 4.2, $\mathcal{B}(\{0\}) = I(Z)$. On the other side,

$$\mathrm{cl}_Z \mathcal{B}(\{0\}) = h\Big(\bigcap_{P\in\mathcal{B}(\{0\})} P\Big) = h(\{0\}) = Z.$$

Thus, $\overline{I(Z)} = Z$.

$\Leftarrow$) Using Corollary 4.2, since $\overline{I(Z)} = Z$, we have $Z = \mathrm{cl}_Z \mathcal{B}(\{0\})$. So, we can write

$$Z = \mathrm{cl}_Z \mathcal{B}(\{0\}) = h\Big(\bigcap_{P\in\mathcal{B}(\{0\})} P\Big)$$

$$\Rightarrow \qquad h\Big(\bigcap_{P\in\mathcal{B}(\{0\})} P\Big) = Z \qquad \Rightarrow \qquad \{0\} = \bigcap_{P\in\mathcal{B}(0)} P.$$

Therefore, the zero ideal is fixed-place. □

Definition 4.5. Let $\mathcal{P}$ be a family of prime ideals of a ring R. We say $\mathcal{P}$ is a *fixed-place family*, whenever $\bigcap_{P_\circ\neq P\in\mathcal{P}} P \nsubseteq P_\circ$ for each $P_\circ \in \mathcal{P}$. Obviously, if $\mathcal{P}$ is a fixed-place family, by Theorem 2.1, the ideal $I = \bigcap\mathcal{P}$ is fixed-place and $\mathcal{B}(I) = \mathcal{P}$.

Theorem 4.6. *Let $\mathcal{P}$ be a family of prime ideals of a ring R. The family $\mathcal{P}$ is fixed-place if and only if $\mathcal{P}$ is discrete as a subspace of Spec(R) with Zariski topology.*

PROOF: $\Rightarrow$) For convenience, assume that $V(S)$ denotes the set of all prime ideals containing $S \subseteq R$. Suppose that $P_\circ \in \mathcal{P}$ and $I = \bigcap\mathcal{P}$. It is enough to show that $P_\circ$ is isolated in $\mathrm{Min}(I)$ with Zariski topology. Set $J = \bigcap_{P_\circ\neq P\in\mathcal{P}} P$. Since $\mathcal{P}$ is a fixed-place family, we have $J \nsubseteq P_\circ$. Clearly, $V(J)\cap \mathrm{Min}(I) = \mathrm{Min}(I)\setminus\{P_\circ\}$ and hence $P_\circ$ is isolated in $\mathrm{Min}(I)$.

$\Leftarrow$) Consider $P_\circ \in \mathcal{P}$. Since $P_\circ$ is isolated in $\mathcal{P}$, there exists an ideal K of R such that $V(K)\cap\mathcal{P} = \mathcal{P}\setminus\{P_\circ\}$. It follows that $K \nsubseteq P_\circ$, $K \subseteq \bigcap_{P_\circ\neq P\in\mathcal{P}} P$ and hence $\bigcap_{P_\circ\neq P\in\mathcal{P}} P \nsubseteq P_\circ$. Therefore, $\mathcal{P}$ is a fixed-place family. □

5. Fixed-place ideal and $C(X)$

This section is divided in two parts. First we study the zero ideal of $C(X)$ and give two equivalent topological conditions. In the second part, we introduce the fixed-place point and the fixed-place rank of this point. Throughout this section $I(X)$ denotes the set of all isolated points of a topological space X.

Lemma 5.1. *Let A be a subset of a topological space X. The subset A is dense in X if and only if $M^A(X) = O^A(X) = \{0\}$.*

Proof: $\Rightarrow$) Suppose that $f \in M^A(X)$. Since $A \subseteq Z(f)$, A is dense in X and $Z(f)$ is closed, so $Z(f) = X$. Consequently, $M^A(X) = O^A(X) = \{0\}$.

$\Leftarrow$) If A is not dense in X, then there is a point p in X such that $p \notin \mathrm{cl}_X A$. Thus, there is a function f in $C(X)$ such that $p \notin Z(f)$ and $A \subseteq \mathrm{int}_X Z(f)$, hence $0 \neq f \in O^A(X)$. This clearly forces $M^A(X) \supseteq O^A(X) \neq 0$. □

Theorem 5.2. *Let $I(X)$ be the set of all isolated points of X, then $\mathcal{B}(\{0\}) = \{O^p(X) : p \in I(X)\}$.*

Proof: Define $A = \{p \in \beta X : \exists P \in \mathcal{B}(\{0\})\ \ O^p(X) \subseteq P\}$. We first prove that A is a subset of each dense subset of βX. Assume that D is a dense subset of βX. Fix $\mathcal{P} = \{P \in \mathrm{Min}(C(X)) : \exists p \in D\ \ O^p \subseteq P\}$. If $P_\circ$ is a minimal prime ideal and $P_\circ \notin \mathcal{P}$, then

$$\bigcap_{P_\circ \neq P \in \mathrm{Min}(C(X))} P \subseteq \bigcap_{P \in \mathcal{P}} P = \bigcap_{p \in D} \bigcap_{P \in \mathrm{Min}(O^p(X))} P = \bigcap_{p \in D} O^p(X) = O^D(X).$$

By Lemma 5.1, $O^D(X) = \{0\}$, thus $\bigcap_{P_\circ \neq P \in \mathrm{Min}(C(X))} P = \{0\}$. It shows that $P_\circ \notin \mathcal{B}(\{0\})$, and therefore $\mathcal{B}(\{0\}) \subseteq \mathcal{P}$. Now suppose $p \in A$, then there exists $P \in \mathcal{B}(\{0\})$ such that $O^p(X) \subseteq P$. Therefore $P \in \mathcal{P}$ and consequently there exists $p' \in D$ such that $O^{p'}(X) \subseteq P$. By [6, 2.11] and [6, 4I.4], every prime z-ideal contains a unique $O^x(X)$ for some $x \in \beta X$, hence $p = p' \in D$ and therefore $A \subseteq D$, which is desired. It is easily seen that the intersection of all dense subset of βX is equal to $I(X)$, thus $A \subseteq I(X)$, and therefore

$$\mathcal{B}(\{0\}) \subseteq \big\{O^p(X) : p \in I(X)\big\}. \tag{3}$$

Now, consider $p_\circ \in I(X)$. Then it is clear that $O^{p_\circ}(X)$ is prime and

$$\bigcap_{O^{p_\circ} \neq P \in \mathrm{Min}(C(X))} P \supseteq \bigcap_{p_\circ \neq p \in X} O^p(X) = O^{X \setminus \{p_\circ\}}(X) \neq 0.$$

Hence, $O^{p_\circ}(X) \in \mathcal{B}(\{0\})$. Thus

$$\mathcal{B}(\{0\}) \supseteq \big\{O^p : p \in I(X)\big\}. \tag{4}$$

From (3) and (4), we obtain $\mathcal{B}(\{0\}) = \{O^p(X) : p \in I(X)\}$. □

Corollary 5.3. *A space X has an isolated point if and only if the space $\mathrm{Min}(C(X))$ has an isolated point.*

Proof: Applying Corollary 4.2 and Theorem 5.2, it follows clearly. □

Corollary 5.4. *The zero ideal of $C(X)$ is anti fixed-place if and only if X has no isolated point.*

Proof: By Corollary 4.3 and Theorem 5.2, it is obvious. □

Theorem 5.5. *The zero ideal in the ring $C(X)$ is fixed-place if and only if $\overline{I(X)} = X$. Then $\mathrm{FP}\text{–}\mathrm{rank}(\{0\}) = |I(X)|$.*

PROOF: $\Rightarrow$) Since the zero ideal is a fixed-place ideal, Theorem 5.2 shows that

$$\{0\} = \bigcap_{P\in\mathcal{B}(\{0\})} P = \bigcap_{p\in I(X)} O^p(X) = O^{I(X)}(X).$$

We conclude from Lemma 5.1 that $I(X)$ is dense in X.

$\Leftarrow$) Since $\overline{I(X)} = X$, by Lemma 5.1, $O^{I(X)}(X) = \{0\}$. Now, using Theorem 5.2, we have

$$\bigcap_{P\in\mathcal{B}(\{0\})} P = \bigcap_{p\in I(X)} O_p = O^{I(X)} = \{0\}.$$

Therefore, the zero ideal of $C(X)$ is fixed-place. □

Corollary 5.6. *$I(X)$ is a dense subspace if and only if the set of isolated points of $\mathrm{Min}(C(X))$ is also a dense subspace.*

PROOF: It is evident, by Theorems 4.4 and 5.5. □

Example 5.7. Let X be an almost discrete space with the only non-isolated point p. The zero ideal of $C(X)$ is a fixed-place ideal and FP–rank$(\{0\}) = |X|$, by Theorem 5.5. Thus, for every cardinal number α there is a fixed-place ideal of fixed-place rank α.

The notion of rank of a point of a topological space was first introduced and studied in [8, 1.7], further in [10] and [11], and generalized in [1]. One can find in [9, 4.1] a basis of this concept as FMP-point. The following definition is based on similar definition in [3, 4.3] with a few differences. Actually, the root of this generalized definition may be found in [1].

Definition 5.8. Let X be a topological space and $p \in \beta X$. We call p a *fixed-place point* with respect to X if $O^p(X)$ is a fixed-place ideal and the fixed-place rank of p with respect to X, denoted by FP–rank$_X(p)$, is defined to be FP–rank$(O^p(X))$.

In [3, Theorem 4.4], it is shown that there is a point of fixed-place rank η for any given cardinal η. It is easy to see that p is a fixed-place point with respect to X if and only if it is a fixed-place point with respect to βX, for each $p \in \beta X$. Furthermore, if p is a fixed-place point with respect to X, then FP–rank$_X(p) =$ FP–rank$_{\beta X}(p)$.

The Proposition 3.1 of [8] states: "Let X be a compact space. A point $p \in X$ has rank $k(< \infty)$ if and only if there is a family of k pairwise disjoint cozero sets with p being in each of their closures, but no larger family of pairwise disjoint cozero sets with this feature." Also, in [10, Theorem 3.1] the same proposition is given for completely regular spaces. Of course, this proposition, with a few differences, was also shown in [2]. In the remainder of this section, we want to generalize this proposition for any $p \in \beta X$ and any cardinal number. To do this, we need some facts.

Suppose that Y is a subspace of X and $\mathbf{F}(X)$ and $\mathbf{F}(Y)$ are the set of all z-filters of X and Y, respectively. Let $\gamma : \mathbf{F}(Y) \to \mathbf{F}(X)$ be given by

$$\gamma(\mathscr{F}) = \{Z \in Z(X) : Z \cap Y \in \mathscr{F}\}.$$

Then $\psi = Z^{-1}\gamma Z$ is a map from all z-ideals of $C(Y)$ to the set of all z-ideals of $C(X)$. We consider $\Phi : \beta Y \to \beta X$ as extension of the identity map $Y \to X$. In the remainder of this section we use the above notation. See [4] for more information about the above maps.

Proposition 5.9. *Let Y be a subspace of X. The subspace*

$$Y' = \{p \in \beta X : \Phi^{-1}(p) \ \text{ is a singleton}\}$$

is the largest subspace of βX that can be considered homeomorphically as a subspace of βY by Φ.

Proof: The proof is standard. □

The above proposition leads us to the following definition.

Definition 5.10. Suppose that Y is a subspace of X and $p \in \beta X$. We say p is an F-point with respect to Y if $\Phi^{-1}(p)$ is a singleton and this point is an F-point with respect to Y.

Proposition 5.11. *Let X be a compact space, $p \in X$ and $Y = Coz(f)$, for some $f \in C(X)$. If $p \in \mathrm{cl}_X Y$, then ψ is a one-to-one correspondence map between $\bigcup_{\Phi(q)=p} \mathrm{Min}(O^q(Y))$ and $\{P \in \mathrm{Min}(O^p(X)) : f \notin P\}$.*

Proof: It follows from [4, Theorem 4.1 and Theorem 4.2]. □

Lemma 5.12. *Let X be a topological space, $f \in C(X)$ and $p \in \beta X$. A prime ideal P is irredundant with respect to $O^p(X)$ if and only if there is a cozero set Y in X such that p is an F-point with respect to Y and $P = \psi(O^q(Y))$, in which $\Phi(q) = p$.*

Proof: Suppose that $Y = Coz(f)$ for some $f \in C(X)$. By Proposition 5.11, there is the one-to-one correspondence ψ between $\bigcup_{\Phi(q)=p} \mathrm{Min}(O^q(Y))$ and $\{P \in \mathrm{Min}(O^p(X)) : f \notin P\}$. Thus

$$\begin{aligned}(O^p : f) &= \Big(\bigcap_{P \in \mathrm{Min}(O^p(X))} P : f\Big) = \bigcap_{P \in \mathrm{Min}(O^p(X))} (P : f) \\ &= \bigcap_{\substack{P \in \mathrm{Min}(O^p(X)) \\ f \notin P}} P = \bigcap_{\Phi(q)=p} \Big(\bigcap_{Q \in \mathrm{Min}(O^q(Y))} \psi(Q)\Big).\end{aligned} \tag{5}$$

Now, by Theorem 2.1, it follows that P is irredundant with respect to $O^p(X)$ if and only if there is $f \in C(X)$ such that $(O^p : f) = P$. By (5),

$$P = \bigcap_{\Phi(q)=p} \Big(\bigcap_{P \in \mathrm{Min}(O^q(Y))} \psi(Q)\Big).$$

Since the $\psi(Q)$'s in the above equality are minimal prime, this is equivalent to saying that $\Phi^{-1}(p)$ is a singleton and p is an F-point with respect to Y and consequently, this is equivalent to saying that $P = \psi(O^q(Y))$ where $\{q\} = \Phi^{-1}\{p\}$. $\square$

Lemma 5.13. *Let $p \in \beta X$ and $Y_1 = Coz(f_1)$, $Y_2 = Coz(f_2)$ for some $f_1, f_2 \in C(X)$. Then $f_1 f_2 \in O^p(X)$ if and only if $p \notin \mathrm{cl}_{\beta X}(Y_1 \cap Y_2)$.*

Proof: First we note that for each $f \in C^*(X)$

$$\mathrm{cl}_{\beta X} Coz(f^\beta) = \mathrm{cl}_{\beta X}\big(Coz(f^\beta) \cap X\big) = \mathrm{cl}_{\beta X} Coz(f).$$

Now, without loss of generality, we assume that $f_1, f_2 \in C^*(X)$. Thus

$$p \notin \mathrm{cl}_{\beta X}(Y_1 \cap Y_2) = \mathrm{cl}_{\beta X}\big(Coz(f_1) \cap Coz(f_2)\big)$$

$$= \mathrm{cl}_{\beta X} Coz(f_1 f_2) = \mathrm{cl}_{\beta X} Coz(f_1^\beta f_2^\beta)$$

$$\Leftrightarrow \; p \in \mathrm{int}_{\beta X} Z(f_1^\beta f_2^\beta) = \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f_1 f_2) \;\Leftrightarrow\; f_1 f_2 \in O^p(X).$$

$\square$

Proposition 5.14. *Let X be a topological space, $p \in \beta X$ and $\{Y_\alpha\}_{\alpha \in A}$ be a family of cozero sets of X. If*

(a) *p is F-point with respect to Y_α for each $\alpha \in A$,*
(b) *$p \notin \mathrm{cl}_{\beta X}(Y_\alpha \cap Y_\beta)$, if $\alpha \neq \beta$,*

then $|\mathcal{B}(O^p(X))| \geq |A|$.

Proof: Let $Y_\alpha = Coz(f_\alpha)$ for each $\alpha \in A$. By hypothesis, for each $\alpha \in A$ there is a unique $q_\alpha \in \beta Y_\alpha$ such that $\Phi_\alpha(q_\alpha) = p$. If we put $P_\alpha = \psi_\alpha(O^{q_\alpha}(Y_\alpha))$ for every $\alpha \in A$, then by Proposition 5.11, $\{P_\alpha\}_{\alpha \in A}$ is a family of irredundant ideals with respect to $O^p(X)$. It is sufficient to show that if $\alpha \neq \beta$, then $P_\alpha \neq P_\beta$. By Lemma 5.13, since $p \notin \mathrm{cl}_{\beta X}(Y_\alpha \cap Y_\beta)$, we have $f_\alpha f_\beta \in O^p(X) \subseteq P_\alpha$. It follows from Proposition 5.11 that $f_\alpha \notin P_\alpha$. Therefore, $f_\beta \in P_\alpha$ and hence $P_\alpha \neq P_\beta$. $\square$

Lemma 5.15. *Let X be a topological space and $p \in \beta X$ be a fixed-place point. If* FP–rank$_X(p) = \eta$ *and ζ is a cardinal such that $\zeta \leq \eta$, then there exist a family $\{Y_\alpha\}_{\alpha \in A}$ of distinct cozero sets such that*

(a) *$|A| = \zeta$;*
(b) *p is an F-point with respect to Y_α for every $\alpha \in A$;*
(c) *$p \notin \mathrm{cl}_{\beta X}(Y_\alpha \cap Y_\beta)$, if $\alpha \neq \beta$.*

Proof: Since FP–rank$_X(p) = \eta$, Theorem 2.1 shows that there exists a family $\{P_\alpha\}_{\alpha \in B}$ of minimal prime ideals such that $|B| = \eta$, $O^p(X) = \bigcap_{\alpha \in B} P_\beta$ and this intersection is irredundant. Suppose that $\{P_\alpha\}_{\alpha \in A}$ is a subfamily of $\{P_\alpha\}_{\alpha \in B}$ such that $|A| = \zeta$. By Lemma 5.12, for every $\alpha \in A$, there exists a cozero set $Y_\alpha = Coz(f_\alpha)$ such that p is an F-point with respect to Y_α, for each $\alpha \in A$. If $\alpha \neq \beta$, it is easy to check that either f_α or f_β must be in P_θ, for each $\theta \in B$. Thus $f_\alpha f_\beta \in P_\theta$, for each $\theta \in B$, and therefore $f_\alpha f_\beta \in O^p(X)$. It shows that $p \notin \mathrm{int}_{\beta X}(Y_\alpha \cap Y_\beta)$, by Lemma 5.13. $\square$

The following is another version of Lemma 5.14.

Lemma 5.16. *Let $p \in \beta X$ be a fixed-place point and $\{Y_\alpha\}_{\alpha \in A}$ be a family of cozero sets of X. If*

(a) *$p \in \mathrm{cl}_{\beta X} Y_\alpha$ for each $\alpha \in A$,*
(b) *$p \notin \mathrm{cl}_{\beta X}(Y_\alpha \cap Y_\beta)$ for $\alpha \neq \beta$,*

then $\mathrm{FP{-}rank}_X(p) \geq |A|$.

Proof: For each $\alpha \in A$, set $Y_\alpha = Coz(f_\alpha)$ with $f_\alpha \in C^*(X)$. Define

$$\mathcal{Q}_\alpha = \{Q \in \mathrm{Min}(O^q(Y_\alpha)) : \Phi_\alpha(q) = p\}.$$

We claim that $\psi_\alpha(\mathcal{Q}_\alpha) \cap \psi_\beta(\mathcal{Q}_\beta) = \emptyset$, for each $\alpha \neq \beta$. Let $\psi_\alpha(Q) \in \psi_\alpha(\mathcal{Q}_\alpha)$, then $f_\alpha \notin \psi_\alpha(Q)$. Since $p \notin \mathrm{cl}_{\beta X}(Y_\alpha \cap Y_\beta)$, by Lemma 5.13, obviously $f_\alpha f_\beta \in O^p(X)$. Thus $f_\beta \in \psi_\alpha(Q)$, this implies that $\psi_\alpha(Q) \notin \psi_\beta(\mathcal{Q}_\beta)$. To complete the proof, it is sufficient to show that $\mathcal{B}(O^p(X)) \cap \psi_\alpha(\mathcal{Q}_\alpha) \neq \emptyset$ for each $\alpha \in A$. We know that $f_\alpha \notin \psi_\alpha(Q)$, for each $Q \in \mathcal{Q}_\alpha$ and $f_\alpha \in \bigcap_{P \in \mathrm{Min}(O^p(X)) \setminus \psi_\alpha(\mathcal{Q}_\alpha)} P$. Thus

$$f_\alpha \in \bigcap_{P \in \mathrm{Min}(O^p(X)) \setminus \psi_\alpha(\mathcal{Q}_\alpha)} P \setminus O^p(X)$$

$$\Rightarrow \quad O^p(X) \neq \bigcap_{P \in \mathrm{Min}(O^p(X)) \setminus \psi_(\mathcal{Q}_\alpha)} P.$$

This implies that $\psi(\mathcal{Q}_\alpha)$ is irredundant with respect to $O^p(X)$. Since $O^p(X)$ is fixed-place, $\mathcal{B}(O^p(X)) \cap \psi_\alpha(\mathcal{Q}_\alpha) \neq \emptyset$. □

Now, we are ready to state the main theorem of this section.

Theorem 5.17. *Let $p \in \beta X$ be a fixed-place point. Then* $\mathrm{FP{-}rank}_X(p) = \eta$ *if and only if there exists a family $\{Y_\alpha\}_{\alpha \in A}$ of cozero sets of X such that*

(a) *$|A| = \eta$;*
(b) *$p \in \mathrm{cl}_{\beta X} Y_\alpha$, for each $\alpha \in A$;*
(c) *$p \notin \mathrm{cl}_{\beta X}(Y_\alpha \cap Y_\beta)$, if $\alpha \neq \beta$;*
(d) *η is the greatest cardinal with above properties.*

In this case, p is an F-point with respect to Y_α for each $\alpha \in A$.

Proof: $\Rightarrow$) By Lemma 5.15, there is a family $\{Y_\alpha\}_{\alpha \in A}$ of cozero sets of X with properties (a)–(c), and by Lemma 5.16, it follows that η is the greatest cardinal with properties (a)–(c).

$\Leftarrow$) Lemma 5.16 shows that $\mathrm{FP{-}rank}_X(p) \geq \eta$. Since η is the greatest cardinal with properties (a)–(c), by Lemma 5.15, $\mathrm{FP{-}rank}_X(p) = \eta$.

According to Lemma 5.12, p is an F-point with respect to Y_α, for each $\alpha \in A$. □

It is clear to see that if $f \in O^p(X)$, then there exists $g \notin M^p(X)$ such that $fg = 0$. Using this fact, we may obtain the following result.

Corollary 5.18. *A point $p \in \beta X$ has finite fixed-place rank n if and only if there is a collection of n pairwise disjoint cozero sets $\{Y_i\}_{i=1}^{n}$ such that p is in the closure of each Y_i, and there is no larger such collection. In this case, p is an F-point with respect to Y_i, for each $i = 1, \ldots, n$.*

References

[1] Aliabad A.R., *z°-ideals in $C(X)$*, Ph.D. Thesis, Chamran University of Ahvaz, Iran.
[2] Aliabad A.R., *Connections between $C(X)$ and $C(Y)$, where Y is a subspace of X*, Abstracts of International Conference on Applicable General Topology, August 12–18, 2001, Hacettepe University, Ankara, Turkey.
[3] Aliabad A.R., *Pasting topological spaces at one point*, Czechoslovak Math. J. **56** (131) (2006), 1193–1206.
[4] Aliabad A.R., Badie M., *Connection between $C(X)$ and $C(Y)$, where Y is subspace of X*, Bull. Iranian Math. Soc. **37** (2011), no. 4, 109–126.
[5] Engelking R., *General Topology*, PWN-Polish Scientific Publishing, Warsaw, 1977.
[6] Gillman L., Jerison M., *Rings of Continuous Functions*, Van Nostrand Reinhold, New York, 1960.
[7] Goodearl K.R., Warfield R.B., Jr. *Introduction to Noncommutative Noetherian Rings*, Cambridge University Press, Cambridge, 1989.
[8] Henriksen M., Larson S., Martinez J., Woods R.G., *Lattice-ordered algebras that are subdirect product of valuation domains*, Trans. Amer. Math. Soc. **345** (1994), 195–221.
[9] Henriksen M., Wilson R.G., *Almost discrete SV-space*, Topology Appl. **46** (1992), 89–97.
[10] Larson S., *f-Rings in which every maximal ideal contains finitely many minimal prime ideals*, Comm. Algebra **25** (1997), no. 12, 3859–3888.
[11] Larson S., *Constructing rings of continuous functions in which there are many maximal ideals with nontrivial rank*, Comm. Algebra **31** (2003), 2183–2206.
[12] Underwood D.H., *On some Uniqueness questions in primary representations of ideals*, Kyoto Math. J. **35** (1969), 69–94.
[13] Willard S., *General Topology*, Addison Wesley, Reading, Mass., 1970.

Mathematics Department, Chamran University, Ahvaz, Iran

E-mail: aliabady_r@scu.ac.ir

Basic Science Department, Jundi Shapur University of Technology, Dezful, Iran

E-mail: badie@jsu.ac.ir

(*Received* July 13, 2012, *revised* October 20, 2012)