Thinness and non-tangential limit associated to coupled PDE

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Abstract. In this paper, we study the reduit, the thinness and the non-tangential limit associated to a harmonic structure given by coupled partial differential equations. In particular, we obtain such results for biharmonic equation (i.e. $\triangle^2\varphi = 0$) and equations of $\triangle^2\varphi = \varphi$ type.

Keywords: thinness, non-tangential limit, Martin boundary, biharmonic functions, coupled partial differential equations

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1. Introduction

Let D be a domain in $\mathbb{R}^d$, $d \geq 1$ and let L_j; $j = 1, 2$, be two second order elliptic differential operators on D leading to harmonic spaces (D, H_{L_j}) with Green functions G_j. Moreover, we assume that every ball $B \subset \bar{B} \subset D$ is a L_j-regular set. Throughout this paper, we consider two positive Radon measures μ_1 and μ_2 such that $K_D^{\mu_j} = \int_D G_j(\cdot, y)\mu_j(dy)$ is a bounded continuous real function on D; $j = 1, 2$, and

$$\| K_D^{\mu_1} \|_\infty \cdot \| K_D^{\mu_2} \|_\infty < 1.$$

We consider the system:

$$\text{(S)} \qquad \begin{cases} L_1 u = -v \cdot \mu_1 \\ L_2 v = -u \cdot \mu_2. \end{cases}$$

Note that if U is a relatively compact open subset of D, $\mu_1 = \lambda^d$, where λ^d is the Lebesgue measure, $\mu_2 = 0$, and $L_1 = L_2 = \triangle$, then we obtain the classical biharmonic case on U. In the case where $\mu_1 = \mu_2 = \lambda^d$, and $\lambda^d(D) < \infty$, we obtain equations of $\triangle^2\varphi = \varphi$ type. In this work, we shall study the thinness notion and the non-tangential limit associated with the balayage space given by the system (S). Let us note that the notion of a balayage space defined by J. Bliedtner and W. Hansen in [6], [11] is more general than that of a P-harmonic space. It covers harmonic structures given by elliptic or parabolic partial differential equations, Riesz potentials, and biharmonic equations (which are a particular case of this work). In the biharmonic case, a similar study can be done using

couples of functions as presented in [2], [7], [12]. We are also grateful to the referee for his remarks and comments.

2. Notations and preliminaries

For $j = 1, 2$, let $X_j = D \times \{j\}$, and let $X = X_1 \cup X_2$, moreover, let i_j and π_j the mappings defined by:

$$i_j : D \longrightarrow X_j \quad \text{and} \quad \pi_j : X_j \longrightarrow D$$

$$x \longmapsto (x, j) \qquad\qquad (x, j) \longmapsto x.$$

Let $\mathcal{U}_0$ be the set of all balls B such that $B \subset \bar{B} \subset D$, $\mathcal{U}_j$ be the image of $\mathcal{U}_0$ by i_j; $j = 1, 2$ and $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$.

Definition 2.1. Let v be a measurable function on X. For $j, k \in \{1, 2\}$, $j \neq k$ and $U \in \mathcal{U}_j$, we define the kernel S_U, on X_j, by:

$$S_U v = (H^j_{\pi_j(U)}(v \ o\ i_j))\ o\ \pi_j + (K^{\mu_j}_{\pi_j(U)}(v \ o\ i_k))\ o\ \pi_j.$$

Where $H^j_{\pi_j(U)}$, $j = 1, 2$, denote harmonic kernels associated with (D, H_{L_j}) and

$$K^{\mu_j}_{\pi_j(U)}(w) = \int G^{\pi_j(U)}_j(\cdot, y) w(y) \mu_j(dy); \quad j = 1, 2.$$

Here w is a measurable function on D and $G^{\pi_j(U)}_j$ is the Green function associated with the operator L_j on $\pi_j(U)$. Let G_j, $j = 1, 2$, be the Green kernel associated with L_j on D. The family of kernels $(S_U)_{U \in \mathcal{U}}$ yields a balayage space on X as defined in [6], [11].

For all open subset V of X, let ${}^*\mathcal{H}(V)$ denote the set of all hyperharmonic functions on V:

$${}^*\mathcal{H}(V) := \{v \in \mathcal{B}(X) : v\,|_V \quad \text{is l.s.c and} \quad S_U v \leq v \quad \forall U \in \mathcal{U}(V)\}.$$

Here $\mathcal{U}(V) = \{U \in \mathcal{U} : \bar{U} \subset V\}$ and $\mathcal{B}(X)$ denotes the set of all Borel functions on X. Let $\mathcal{S}(V)$ be the set of all superharmonic functions on X, i.e.

$$\mathcal{S}(V) := \{s \in {}^*\mathcal{H}(V) : (S_U v)\,|_U \in C(U) \quad \forall U \in \mathcal{U}(V)\},$$

and $\mathcal{H}(V)$ be the set of all harmonic functions on X:

$$\mathcal{H}(V) := \{h \in \mathcal{S}(V) : S_U h = h \quad \forall U \in \mathcal{U}(V)\}.$$

We denote ${}^*\mathcal{H}^+(V)$ (resp. $\mathcal{S}^+(V)$, $\mathcal{H}^+(V)$) the set of all hyperharmonic (resp. superharmonic, harmonic) positive functions on V. We denote also, for $V \subset D$, ${}^*\mathcal{H}^+_j(V)$ (resp. $\mathcal{S}^+_j(V)$, $\mathcal{H}^+_j(V)$) the set of all L_j-hyperharmonic (resp. L_j-superharmonic, L_j-harmonic) positive functions on V.

Let φ be a positive hyperharmonic function on X and let φ_j be the function defined on D by:

$$\varphi_j := \begin{cases} \varphi \ o \ i_j - K_D^{\mu_j}(\varphi \ o \ i_k) & \text{if} \ \ K_D^{\mu_j}(\varphi \ o \ i_k) < \infty \\ +\infty & \text{otherwise} \end{cases}$$

where $j, k \in \{1, 2\}$ and $j \neq k$. We note that φ_j, $j = 1, 2$ are L_j-hyperharmonic on D (see [4, Corollary 2.2]).

3. Reduit and thinness

Let $A \subset X$ and let f be a positive numerical function on X. The reduit R_f^A of f relative to A in X is defined by:

$$R_f^A := \inf\{\varphi \in {}^*\mathcal{H}^+(X) : \varphi \geq f \ \text{ on } \ A\}.$$

Let $\widehat{R}_f^A$ be the lower semi-continuous regularization of R_f^A, i.e.

$$\widehat{R}_\varphi^A(x) := \liminf_{y \to x} R_\varphi^A(y), \quad x \in X.$$

We denote ${}^j R_g^A$ the reduit of a function g defined on D relative to a set A of D with respect to harmonic space (D, H_j), $j = 1, 2$ and ${}^j \hat{R}_g^A$ the l.s.c. regularization of ${}^j R_g^A$.

Proposition 3.1. *Let f be a positive numerical function on X and $A = (A_1 \times \{1\}) \cup (A_2 \times \{2\})$ with $A_j \subset D$, $j = 1, 2$. We have:*

$${}^j R_{f \circ i_j}^{A_j} \leq R_f^A \circ i_j, \quad j = 1, 2.$$

Proof: We consider the following sets:

$$B_1 = \{\varphi \circ i_1, \ \varphi \in {}^*\mathcal{H}^+(X), \ \varphi \geq f \ \text{ on } \ A\}$$

and

$$B_2 = \{g, \ g \in {}^*\mathcal{H}_1^+(D), \ g \geq f \circ i_1 \ \text{ on } \ A_1\}.$$

For showing ${}^1 R_{f \circ i_1}^{A_1} \leq R_f^A \circ i_1$, it suffices to prove that $B_1 \subset B_2$. Let $u \in B_1$, then there exists $\varphi \in {}^*\mathcal{H}^+(X)$ such that $u = \varphi \circ i_1$ and $\varphi \geq f$ on A. Since $\varphi \in {}^*\mathcal{H}^+(X)$, then $u \in {}^*\mathcal{H}_1^+(D)$ and $u = \varphi \circ i_1 \geq f \circ i_1$ on A_1. So $u \in B_2$, and ${}^1 R_{f \circ i_1}^{A_1} \leq R_f^A \circ i_1$. In the same way, we show that ${}^2 R_{f \circ i_2}^{A_2} \leq R_f^A \circ i_2$. □

Corollary 3.1. *Let f be a positive numerical function on X and $A \subset X$. We have:*

$${}^j \hat{R}_{f \circ i_j}^{A_j} \leq \hat{R}_f^A \circ i_j, \ j = 1, 2.$$

Here $A = (A_1 \times \{1\}) \cup (A_2 \times \{2\})$ and $A_j \subset D$; $j = 1, 2$.

Definition 3.1. (i) Let A be a subset of X. We say that A is thin at a point $x \in X$ if and only if there exist an open neighbourhood U of x in X and a positive hyperharmonic function v on U such that $\hat{R}_v^{A\cap U}(x) < v(x)$.

(ii) Let B be a subset of D. We say that B is L_j-thin at point $z \in D$, $j = 1, 2$, if and only if there exist an open neighbourhood U of z in D and a positive L_j-hyperharmonic function v on U such that ${}^j\hat{R}_v^{B\cap U}(z) < v(z)$.

Proposition 3.2. *Let $A = (A_1 \times \{1\}) \cup (A_2 \times \{2\})$ be a subset of X and $x = (x_0, j)$, $j = 1, 2$, where $x_0 \in D$. If A is thin at point x, then A_j is L_j-thin at point x_0.*

Proof: If A is thin at point $x = (x_0, 1)$ where $x_0 \in D$, then there exist an open neighbourhood U of x in X and a positive hyperharmonic function φ on U such that $\hat{R}_\varphi^{A\cap U}(x) < \varphi(x)$. Hence there exist an open neighbourhood U_1 of x_0, in D such that $(U_1 \times \{1\}) \subset U$. From Corollary 3.1,

$$ {}^1\hat{R}_{\varphi\circ i_1}^{A_1\cap U_1}(x_0) \leq (\hat{R}_\varphi^{A\cap U} \circ i_1)(x_0) < (\varphi \circ i_1)(x_0). $$

Since φ is a positive hyperharmonic function on U, then the function $\varphi \circ i_1$ is a positive L_1-hyperharmonic function on U_1. Therefore, A_1 is L_1-thin at point x_0. In the same way, we show that A_2 is L_2-thin at point x_0.

For $j, k \in \{1, 2\}$, $j \neq k$, we denote by $P_{j,k} := K_D^{\mu_j} K_D^{\mu_k}$ and $G_{P_{j,k}} := \sum_{n=0}^{+\infty} (P_{j,k})^n$ which coincides with $(I - P_{j,k})^{-1}$ on $\mathcal{B}_b(D)$. $\mathcal{B}_b(D)$ denotes the set of all bounded Borel measurable functions on D. We recall the following equalities:

$$ P_{j,k} G_{P_{j,k}} = G_{P_{j,k}} P_{j,k}, \tag{1} $$

$$ P_{j,k} G_{P_{j,k}} + I = G_{P_{j,k}}, \tag{2} $$

$$ G_{P_{j,k}}^2 - P_{j,k} G_{P_{j,k}}^2 = G_{P_{j,k}}, \tag{3} $$

$$ K_D^{\mu_j} G_{P_{k,j}} = G_{P_{j,k}} K_D^{\mu_j}. \tag{4} $$

□

Remark 3.1. (1) We note that if φ is a finite positive Borel measurable function on D such that $P_{j,k}\varphi$ is bounded, then $G_{P_{j,k}}\varphi < +\infty$.

(2) If s is a L_j-hyperharmonic positive function on D then $G_{P_{j,k}} s$ is L_j-hyperharmonic on D.

Let $J = (J_1 \times \{1\}) \cup (J_2 \times \{2\})$, $J' = ((J_1 \cap J_2) \times \{1\}) \cup ((J_1 \cap J_2) \times \{2\})$ with $J_j \subset D$, $j = 1, 2$, and $J_1 \cap J_2 \neq \emptyset$. Let t_j, $j = 1, 2$, be two positive L_j-hyperharmonic functions on D. We define two functions $v_{j,k}$, $j, k \in \{1, 2\}$, $j \neq k$ on X by:

$$ v_{j,k} := \begin{cases} (G_{P_{j,k}} t_j + P_{j,k} G_{P_{j,k}}^2 t_j) \ o \ \pi_j & \text{on } \ X_j \\ (K_D^{\mu_k} G_{P_{j,k}}^2 t_j) \ o \ \pi_k & \text{on } \ X_k. \end{cases} $$

From ([4, Corollary 2.2]), the functions $v_{j,k}$ are hyperharmonic on $(D \times \{1\}) \cup (D \times \{2\})$.

Remark 3.2. Note that, if $P_{j,k}G^2_{P_{j,k}}t_j < \infty$, we have $v_{j,k} \circ i_j = G^2_{P_{j,k}}t_j$ and

$$(v_{1,2} + v_{2,1}) \circ i_j - K_D^{\mu_j}(v_{1,2} + v_{2,1}) \circ i_k = P_{j,k}t_j,$$

$j, k \in \{1, 2\}$, $j \neq k$.

Proposition 3.3. *If $P_{j,k}G^2_{P_{j,k}}t_j < \infty$, we have*

$$R^{J'}_{v_{j,k}} \circ i_j \leq {}^jR^{J_j}_{G_{P_{j,k}}t_j} + P_{j,k}G^2_{P_{j,k}}t_j$$

and

$$R^{J'}_{v_{j,k}} \circ i_k \leq K_D^{\mu_k}G^2_{P_{j,k}}t_j$$

$j, k \in \{1, 2\}$, $j \neq k$.

Proof: (1) We give the proof for $j = 1$ and $k = 2$. Let s be a L_1-hyperharmonic function on D such that $s = G_{P_{1,2}}t_1$ on J_1 and $s \leq G_{P_{1,2}}t_1$. We consider on X the function

$$f := \begin{cases} (s + P_{1,2}G^2_{P_{1,2}}t_1)\ o\ \pi_1 & \text{on } X_1 \\ (K_D^{\mu_2}G^2_{P_{1,2}}t_1)\ o\ \pi_2 & \text{on } X_2. \end{cases}$$

So $f \circ i_1 = v_{1,2} \circ i_1$ on J_1 and $f \circ i_2 = v_{1,2} \circ i_2$. Hence $f = v_{1,2}$ on J' and $f \leq v_{1,2}$.

On one hand, we have

$$f \circ i_1 - K_D^{\mu_1}f \circ i_2 = s + P_{1,2}G^2_{P_{1,2}}t_1 - P_{1,2}G^2_{P_{1,2}}t_1 = s.$$

On the other hand, using the equalities (1), (2) and (3), we have

$$\begin{aligned} f \circ i_2 - K_D^{\mu_2}f \circ i_1 &= K_D^{\mu_2}G^2_{P_{1,2}}t_1 - K_D^{\mu_2}(s + P_{1,2}G^2_{P_{1,2}}t_1) \\ &= K_D^{\mu_2}G^2_{P_{1,2}}t_1 - K_D^{\mu_2}s - K_D^{\mu_2}P_{1,2}G^2_{P_{1,2}}t_1 \\ &= K_D^{\mu_2}(G^2_{P_{1,2}}t_1 - P_{1,2}G^2_{P_{1,2}}t_1) - K_D^{\mu_2}s \\ &= K_D^{\mu_2}G_{P_{1,2}}t_1 - K_D^{\mu_2}s \\ &= K_D^{\mu_2}(G_{P_{1,2}}t_1 - s). \end{aligned}$$

Hence $f \circ i_1 - K_D^{\mu_1}f \circ i_2$ and $f \circ i_2 - K_D^{\mu_2}f \circ i_1$ are respectively L_1 and L_2 hyperharmonic on D and therefore the function f is hyperharmonic on X ([4, Corollary 2.2]). So

$$R^{J'}_{v_{1,2}} \circ i_1 \leq {}^1R^{J_1}_{G_{P_{1,2}}t_1} + P_{1,2}G^2_{P_{1,2}}t_1$$

and

$$R^{J'}_{v_{1,2}} \circ i_2 \leq K_D^{\mu_2}G^2_{P_{1,2}}t_1.$$

□

The following theorem results from the previous proposition.

Theorem 3.1. *Let t_j, $j = 1, 2$, be two positive L_j-hyperharmonic functions on D such that $P_{j,k}G^2_{P_{j,k}} t_j < \infty$, $j, k \in \{1, 2\}$, $j \neq k$. Then*

$$\hat{R}^{J'}_{v_{1,2}+v_{2,1}} \circ i_j \leq {}^j\hat{R}^{J_j}_{G_{P_{j,k}} t_j} + P_{j,k}G^2_{P_{j,k}} t_j + K^{\mu_j}_D G^2_{P_{k,j}} t_k.$$

Remark 3.3. (1) In the biharmonic case, i.e. $\mu_1 = \lambda^d$, $\mu_2 = 0$, $L_j = \triangle$ for $j = 1, 2$ and $J = J_1 = J_2$, the result is given by A. Boukricha [7, Proposition 5.6].

(2) All the previous results are still valid if we substitute D by any L_j-regular subset V of D.

Proposition 3.4. *Let J_j, $j = 1, 2$ be two subsets of D such that $J_1 \cap J_2 \neq \emptyset$. Let $x_0 \in D$. If J_j are L_j-thin at point x_0 then the set $J' := ((J_1 \cap J_2) \times \{1\}) \cup ((J_1 \cap J_2) \times \{2\})$ is thin at points (x_0, j), $j = 1, 2$.*

Proof: Since J_j, $j \in \{1, 2\}$, is L_j-thin at point x_0, then there exist a L_j-regular open neighbourhood U_j of x_0 in D and a positive L_j-hyperharmonic function s_j on U_j such that

$${}^j\hat{R}^{J_j \cap U_j}_{s_j}(x_0) < s_j(x_0).$$

Letting $V := U_1 \cap U_2$, V is a regular open neighbourhood of x_0. Let φ be the positive hyperharmonic function on $W := (V \times \{1\}) \cup ((V) \times \{2\})$ defined on $V \times \{j\}$ by:

$$\varphi := (G_{P_{j,k}} s_j + K^{\mu_j}_V G^2_{P_{k,j}} s_k + P_{j,k}G^2_{P_{j,k}} s_j) \circ \pi_j.$$

We have, from Theorem 3.1,

$$\hat{R}^{J' \cap W}_{\varphi} \circ i_j \leq {}^j\hat{R}^{J_j}_{G_{P_{j,k}} s_j} + P_{j,k}G^2_{P_{j,k}} s_j + K^{\mu_j}_V G^2_{P_{k,j}} s_k.$$

Since

$$G_{P_{j,k}} s_j = s_j + P_{j,k}G_{P_{j,k}} s_j,$$

we have

$${}^j\hat{R}^{J_j \cap U_j}_{G_{P_{j,k}} s_j}(x_0) \leq {}^j\hat{R}^{J_j \cap U_j}_{s_j}(x_0) + {}^j\hat{R}^{J_j \cap U_j}_{P_{j,k}G_{P_{j,k}} s_j}(x_0).$$

Hence, from the hypothesis, we get

$${}^j\hat{R}^{J_j \cap U_j}_{G_{P_{j,k}} s_j}(x_0) < s_j(x_0) + P_{j,k}G_{P_{j,k}} s_j(x_0) = G_{P_{j,k}} s_j(x_0).$$

Therefore, we conclude

$$\hat{R}^{J' \cap W}_{\varphi}(x_0, 1) < \varphi(x_0, 1),$$

i.e. J' is thin at point $(x_0, 1)$. □

Note that our proof is direct. From Proposition 3.2 and Proposition 3.4 we have the following characterization of the thinness with respect to the system (S).

Theorem 3.2. *Let J_1 and J_2 be two subsets of D such that $J_1 \cap J_2 \neq \emptyset$. The following propositions are equivalent.*

(1) *J_1 is L_1-thin at point x_0 and J_2 is L_2-thin at point $x_0 \in D$.*
(2) *The set $J' := ((J_1 \cap J_2) \times \{1\}) \cup ((J_1 \cap J_2) \times \{2\})$ is thin at points (x_0, j), $j = 1, 2$.*

4. Minimal thinness

Let us fix $x_0 \in D$. For all $x, y \in D$ and $j \in \{1, 2\}$, we put:

$$g^j(x,y) := \begin{cases} \frac{G_j(x,y)}{G_j(x_0,y)}, & \text{if } \ x \neq x_0 \ \text{ or } \ y \neq x_0 \\ 1, & \text{if } \ x = y = x_0. \end{cases}$$

Let $\mathcal{A}_j = \{g^j(x, \cdot), \ x \in D\}$, and $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$.

As in [8], [9], we consider the Martin compactification $\widehat{D}$ of D associated with $\mathcal{A}$. The boundary $\partial_M D := \widehat{D} - D$ of D is called the Martin boundary of D associated with the system (S).

The function $g^j(x, \cdot)$, $j = 1, 2$, $x \in D$ can be extended, on $\widehat{D}$, to a continuous function denoted $g^j(x, \cdot)$, $j = 1, 2$, $x \in D$ as in [8]. Put $\tilde{\partial}_M D := \partial_M D \times \{1\} \cup \partial_M D \times \{2\}$. A couple of functions (u_1, u_2) defined on $\partial_M D$ can be identified with a function f on $\tilde{\partial}_M D$ such that $f \circ i_j = u_j$, where i_j, $j = 1, 2$ denote always the mappings of $\partial_M D$ into $\partial_M D \times \{j\}$ defined by: $i_j(z) = (z, j)$; $z \in \partial_M D$. We use also π, the mapping of $\tilde{\partial}_M D$ into $\partial_M D$ defined by: $\pi(Y) = \pi_j(Y)$, if $Y \in \partial_M D \times \{j\}$. Here $\pi_j(Y) = z$, if $Y = (z, j)$. We denote:

$$\partial_m^j D = \{y \in \partial_M D : g^j(\cdot, y) \ \text{ is } L_j\text{-minimal}\}.$$

We note that, for all $y \in \partial_M D$, the function $g^j(\cdot, y)$ is L_j-harmonic on D. In the following, we suppose that, for all $y \in \partial_M D$, the function $K_D^{\mu_j} g^k(\cdot, y)$ is finite and the function $P_{k,j} g^k(\cdot, y)$ is bounded for $j \neq k$, $j, k \in \{1, 2\}$. For all $Y \in \tilde{\partial}_M D$, we have $\pi(Y) \in \partial_M D$. Hence we can define on X, the following functions:

$$\Phi_Y := \begin{cases} (G_{P_{1,2}} g^1(\cdot, \pi(Y))) \ o \ \pi_1 & \text{on } \ X_1 \\ (K_D^{\mu_2} G_{P_{1,2}} g^1(\cdot, \pi(Y))) \ o \ \pi_2 & \text{on } \ X_2 \end{cases}$$

and

$$\Psi_Y := \begin{cases} (G_{P_{1,2}} K_D^{\mu_1} g^2(\cdot, \pi(Y))) \ o \ \pi_1 & \text{on } \ X_1 \\ (G_{P_{2,1}} g^2(\cdot, \pi(Y))) \ o \ \pi_2 & \text{on } \ X_2. \end{cases}$$

From [4, Theorem 3.1], Φ_Y and Ψ_Y are harmonic functions on X.

Definition 4.1. Let $Y \in \tilde{\partial}_M D$. We say that Y is a minimal point for $\tilde{\partial}_M D$ if Φ_Y is minimal or Ψ_Y is minimal.

Lemma 4.1. *$Y = (y, j)$, $j = 1, 2$ is a minimal point for $\tilde{\partial}_M D$, if and only if y is a minimal point for $\partial_M D$.*

Proof: Let $Y = (y, j)$ be a minimal point for $\tilde{\partial}_M D$, $j = 1, 2$, then, by the definition, Φ_Y is minimal or Ψ_Y is minimal. Suppose that Φ_Y is minimal. So, from [4, Proposition 4.2], the function $(\Phi_Y \circ i_1 - K_D^{\mu_1}(\Phi_Y \circ i_2))$ is L_1-minimal. Since

$$\Phi_Y \circ i_1 - K_D^{\mu_1}(\Phi_Y \circ i_2) = G_{P_{1,2}} g^1(\cdot, y) - P_{1,2} G_{P_{1,2}} g^1(\cdot, y),$$

then we have

$$\Phi_Y \circ i_1 - K_D^{\mu_1}(\Phi_Y \circ i_2) = g^1(\cdot, y). \tag{4.1}$$

Therefore, the function $g^1(\cdot, y)$ is minimal and we can deduce that the point y is a minimal point for $\partial_M D$. If we suppose that the function Ψ_Y is a minimal function, we show in an analogous way that the function $g^2(\cdot, y)$ is a minimal function, i.e. y is a minimal point for $\partial_M D$.

Conversely, let y be a minimal point for $\partial_M D$. Then $g^1(\cdot, y)$ is a L_1-minimal function or $g^2(\cdot, y)$ is a L_2-minimal function. If $g^1(\cdot, y)$ is minimal, then, by (4.1), the function $(\Phi_Y \circ i_1 - K_D^{\mu_1}(\Phi_Y \circ i_2))$ is L_1-minimal. Therefore, by [4, Proposition 4.2], Φ_Y is a minimal function. So Y is a minimal point for $\tilde{\partial}_M D$. Similarly, if we assume that the function $g^2(\cdot, y)$ is a minimal function we show that the function Ψ_Y is a minimal function, i.e. Y is a minimal point for $\tilde{\partial}_M D$. □

Definition 4.2. Let J be a subset of X and let Y be a minimal point for $\tilde{\partial}_M D$. We say that J has a minimal thinness at point Y if $\hat{R}^J_{\Phi_Y} \neq \Phi_Y$ or $\hat{R}^J_{\Psi_Y} \neq \Psi_Y$.

5. Non-tangential limit

In this section, we take $L_1 = L_2 = \triangle$ and D is the half space in R^d defined by:

$$D = \{(x', x_d) : x' \in R^{d-1} \text{ and } x_d > 0\}.$$

The Martin compactification of D can be identified with the closure of D and all Martin boundary points are minimal (see [1]). Let $x_0 = (0', 1)$ with $0' = (0, 0, \dots) \in R^{d-1}$. We recall that the Martin Kernel in this case is given by:

$$\begin{cases} M(x, y) = \frac{\|x_0 - y\|^d . x_d}{\|x - y\|^d}, & x \in D,\ y \in \partial D \\ M(x, \infty) = x_d, & x \in D. \end{cases}$$

For $a > 0$ and $y \in \partial D$, we define

$$\Gamma_{y,a} := \{(x', x_d) \in R^{d-1} \times R^{*+} : x_d > \|x' - y'\|\},\ y = (y', 0),\ y' \in R^{d-1}$$

and we define for $Y = (y, j) \in (\partial D \times \{1\}) \cup (\partial D \times \{2\})$,

$$\Omega_{Y,a} := (\Gamma_{y,a} \times \{1\}) \cup (\Gamma_{y,a} \times \{2\}).$$

We note that if h is a positive harmonic function on X, then the function $h_j = h\circ i_j - K_D^{\mu_j}(h\circ i_k)$ is harmonic on D [4, Theorem 2.1]. Moreover, $K_D^{\mu_j}(h\circ i_k) < \infty$ for $j,k = 1,2$, $j\neq k$.

Definition 5.1. (1) Let f be a function defined on X. We say that f has a fine minimal limit l at point $Y=(y,j)$ for $j=1,2$ and $y\in\partial D$, if there exist a subset J_1 of D having a L_1-minimal thinness at point y and a subset J_2 of D having a L_2-minimal thinness at point y such that

$$\lim_{x\longrightarrow Y,\ x\in X\setminus J} f(x) = l.$$

Here $J = (J_1\times\{1\})\cup(J_2\times\{2\})$.

(2) Let f be a function defined on X. We say that f has a non-tangential limit l at point $Y=(y,j)$ for $j=1,2$ and $y\in\partial D$ if

$$\forall\, a>0,\ \lim_{x\longrightarrow Y,\ x\in\Omega_{Y,a}} f(x) = l.$$

Remark 5.1. Let $Y=(y,j)$ for $j=1,2$ and $y\in\partial D$, then

$$\lim_{(z,j)\longrightarrow(y,j),\ (z,j)\in\Gamma_{y,a}\times\{j\}} f(z,j) = \lim_{z\longrightarrow y,\ z\in\Gamma_{y,a}} (f\circ i_j)(z).$$

Theorem 5.1. *Let $Y=(y,j)$ for $j=1,2$, $y\in\partial D$. Let u be a positive harmonic function on X and let h be a strictly positive harmonic function on X such that the function $\frac{u}{h}$ has a minimal fine limit l at point Y. Denote $h_j = h\circ i_j - K_D^{\mu_j}(h\circ i_k)$, $j,k=1,2$, $j\neq k$.*

If $h_1>0$ and $h_2>0$ then the function $\frac{u}{h}$ has a non-tangential limit at point Y.

Remark 5.2. If $h_j>0$, $h_k=0$ and $Y=(y,j)$ for $j,k\in\{1,2\}$, $j\neq k$, then

$$\lim_{z\longrightarrow y,\ z\in\Gamma_{y,a}} \frac{(u\circ i_j)(z)}{(h\circ i_j)(z)} = \lim_{x\longrightarrow Y,\ x\in(\Gamma_{y,a}\times\{1\})} \frac{u}{h}(x) = l.$$

Proof: Let $Y=(y,j)$ for $j=1,2$, $y\in\partial D$. We suppose that $h_1>0$ and $h_2>0$. Since the function $\frac{u}{h}$ has a minimal fine limit l at point Y, there exist a subset J_1 of D having a L_1-minimal thinness at point y and a subset J_2 of D having a L_2-minimal thinness at point y such that

$$\lim_{x\longrightarrow Y,\, x\in X\setminus J} \frac{u}{h}(x) = l.$$

Here $J=(J_1\times\{1\})\cup(J_2\times\{2\})$. Therefore $\lim_{z\longrightarrow y}\frac{(u\circ i_j)(z)}{(h\circ i_j)(z)} = l$ on $D\setminus J_j$. We have

$$\frac{(u\circ i_j)(z)}{(h\circ i_j)(z)} = \frac{u_j(z)+K_D^{\mu_j}(u\circ i_k)(z)}{h_j(z)+K_D^{\mu_j}(h\circ i_k)(z)},\ j\neq k.$$

Here $u_j = u \circ i_j - K_D^{\mu_j}(u \circ i_k)$; $j,k = 1,2,\ j \neq k$. Using [10, 18.1] or [1, Corollary 9.3.8], we have

$$\lim_{z \longrightarrow y \in \partial D} \frac{K_D^{\mu_j}(u \circ i_k)(z)}{h_j(z)} = \lim_{z \longrightarrow y \in \partial D} \frac{K_D^{\mu_j}(h \circ i_l)(z)}{h_j(z)} = 0; \mu_{h_j} - a.e. \text{ on } \partial_m^j D.$$

Here μ_{h_j} denotes the measure on $\partial_M^j D$ corresponding to h_j in the Martin representation. So, we get

$$\lim_{z \longrightarrow y} \frac{u_j(z)}{h_j(z)} = l \text{ on } D \setminus J_j.$$

Therefore, by Fatou Theorem (see [1, Theorem 9.7.4]) $\lim_{z \longrightarrow y} \frac{u_j(z)}{h_j(z)} = l$ on $\Gamma_{y,a}$. Since we have

$$\begin{aligned} \frac{u_j}{h_j} &= \frac{(u \circ i_j) - K_D^{\mu_j}(u \circ i_k)}{(h \circ i_j) - K_D^{\mu_j}(h \circ i_k)} \\ &= \frac{\frac{u \circ i_j}{h_j} - \frac{K_D^{\mu_j}(u \circ i_k)}{h_j}}{\frac{h \circ i_j}{h_j} - \frac{K_D^{\mu_j}(h \circ i_k)}{h_j}}, \end{aligned}$$

we conclude that $\lim_{z \longrightarrow y} \frac{u \circ i_j(z)}{h \circ i_j(z)} = l$ on $\Gamma_{y,a}$.

In the same way, we show the assertions in the previous remark. □

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