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Diagonals and discrete subsets of squares

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Abstract. In 2008 Juhász and Szentmiklóssy established that for every compact space X there exists a discrete $D \subset X \times X$ with $|D| = d(X)$. We generalize this result in two directions: the first one is to prove that the same holds for any Lindelöf Σ-space X and hence X^ω is d-separable. We give an example of a countably compact space X such that X^ω is not d-separable.

On the other hand, we show that for any Lindelöf p-space X there exists a discrete subset $D \subset X \times X$ such that $\Delta = \{(x,x) : x \in X\} \subset \overline{D}$; in particular, the diagonal Δ is a retract of $\overline{D}$ and the projection of D on the first coordinate is dense in X. As a consequence, some properties that are not discretely reflexive in X become discretely reflexive in $X \times X$. In particular, if X is compact and $\overline{D}$ is Corson (Eberlein) compact for any discrete $D \subset X \times X$ then X itself is Corson (Eberlein). Besides, a Lindelöf p-space X is zero-dimensional if and only if $\overline{D}$ is zero-dimensional for any discrete $D \subset X \times X$.

Under CH, we give an example of a crowded countable space X such that every discrete subset of $X \times X$ is closed. In particular, the diagonal of X cannot be contained in the closure of a discrete subspace of $X \times X$.

Keywords: diagonal, discrete subspaces, d-separable space, discrete reflexivity, Lindelöf p-space, Lindelöf Σ-space, finite powers, Corson compact spaces, Eberlein compact spaces, countably compact spaces

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0. Introduction

A property $\mathcal{P}$ is called *discretely reflexive* if a space X has $\mathcal{P}$ whenever $\overline{D}$ has $\mathcal{P}$ for any discrete $D \subset X$. Tkachuk established in [Tk1] that a space X is compact if and only if the closure of every discrete subspace of X is compact, i.e., compactness is a discretely reflexive property. Alas, Tkachuk and Wilson proved in [ATW] that quite a few cardinal functions are reflected by closures of discrete subspaces. For example, a compact space X has countable character (tightness) if and only if the closure of every discrete subspace of X has countable character (or countable tightness respectively).

The paper [BT] provided some results on discrete reflexivity for countably compact spaces generalizing the corresponding theorems of [ATW] established for compact spaces. Such a generalization turned out to be possible for the case of weight less or equal to ω_1. It was proved, in particular, that countable tightness and countable character are discretely reflexive in countably compact spaces of weight less or equal to ω_1. It was also shown in [BT] that, at least, countable compactness cannot be omitted in these results.

It turns out that some non-discretely reflexive properties improve their behavior in $X \times X$ for an arbitrary X. It was proved in [BT] that for any countably compact space X, if $\overline{D}$ is metrizable for any discrete $D \subset X \times X$ then X is metrizable and hence compact. Another result from [BT] states that, for any topological property $\mathcal{P}$ preserved by continuous maps, if X is compact and $\overline{D}$ has $\mathcal{P}$ for any discrete $D \subset X^3$ then X has $\mathcal{P}$. It was asked in [BT] whether X^3 could be replaced with X^2. We give a positive answer to this question developing an idea of Juhász and Szentmiklóssy in [JSz] where they proved that for every compact space X, we can find a discrete $D \subset X \times X$ with $|D| = d(X)$.

We show that for every Lindelöf p-space X there exists a discrete $D \subset X \times X$ such that $\Delta \subset \overline{D}$ and therefore the projection of D onto the first coordinate is dense in X; in particular, the set Δ is a retract of $\overline{D}$. Here $\Delta = \{(x,x) : x \in X\}$ is the diagonal of the space X. As a consequence, if X is compact and $\overline{D}$ is Corson (Eberlein) compact for any discrete $D \subset X \times X$ then X itself is Corson (Eberlein), i.e., we give a positive answer to the corresponding questions from [BT]. The same conclusion cannot be derived if the closures of discrete subsets of X are Eberlein since there exist examples of compact spaces X which fail to be Corson while $\overline{D}$ is second countable for any discrete $D \subset X$. Another property, which is not discretely reflexive is zero-dimensionality. However, a Lindelöf p-space X is zero-dimensional if and only if $\overline{D}$ is zero-dimensional for any discrete $D \subset X \times X$.

As a generalization of the mentioned result of Juhász and Szentmiklóssy in another direction, we establish that for any Lindelöf Σ-space X we can find a discrete set $D \subset X \times X$ such that $|D| = d(X)$ and hence X^ω is d-separable. We also give an example of a countably compact space X such that X^ω is not d-separable. Another example constructed under CH shows that a countable space X can be maximal while all discrete subspaces of $X \times X$ are closed. Therefore, under CH, the diagonal of a countable space is not necessarily contained in the closure of a discrete set.

1. Notation and terminology

All spaces under consideration are assumed to be Tychonoff. Given a space X, the family $\tau(X)$ is its topology and $\tau^*(X) = \tau(X)\backslash\{\emptyset\}$. We denote by $\mathbb{R}$ the real line with the natural topology, the set $\mathbb{Q}$ consists of the rational points of $\mathbb{R}$ and $\mathbb{N} = \omega\backslash\{0\}$. The expression $X \simeq Y$ means that the spaces X and Y are homeomorphic. A space X is *Lindelöf p* if it is a perfect preimage of a second countable space. The space X is *crowded* if it has no isolated points. A crowded space X is called *maximal* if any strictly stronger (not necessarily Tychonoff) topology on X has isolated points. The space X is *ultradisconnected* if any two disjoint crowded subspaces of X have disjoint closures. We say that a space X is ω-bounded if $\overline{A}$ is compact for any countable $A \subset X$.

Given a space X, a family $\mathcal{N}$ of subsets of X is a *network modulo a cover* $\mathcal{C}$ of X if for every $C \in \mathcal{C}$ and $U \in \tau(X)$ with $C \subset U$ there exists $N \in \mathcal{N}$ such that $C \subset N \subset U$. A network of X modulo the cover $\{\{x\} : x \in X\}$ is called a network in X. The spaces with a countable network are called *cosmic*. A space X is Lindelöf Σ if it has a countable network modulo a compact cover of X. Furthermore, $nw(X) = \min\{|\mathcal{N}| : \mathcal{N}$ is a network in $X\}$. The cardinal $nw(X)$ is called *the network weight* of X. Let $s(X) = \sup\{|D| : D$ is a discrete subspace of $X\}$; the cardinal $s(X)$ is called *the spread of* X.

The set $\Delta_X = \{(x, x) : x \in X\}$ is called *the diagonal* of the space X; we denote the diagonal of X by Δ if X is clear. Given a topological property $\mathcal{P}$, we say that a space X is *discretely* $\mathcal{P}$ if $\overline{D}$ has $\mathcal{P}$ for any discrete subspace $D \subset X$. A property $\mathcal{P}$ is called *discretely reflexive* if a space X has $\mathcal{P}$ if and only if X is discretely $\mathcal{P}$.

The rest of our terminology is standard and can be found in [En] and [Tk2].

2. Large discrete subspaces of squares

This work was inspired by a theorem of Juhász and Szentmiklóssy about existence of large discrete subsets in $X \times X$ for compact spaces X. Our purpose is to show that the same result holds for Lindelöf Σ-spaces and strengthen it for the compact case.

2.1 Definition. If X is a space and $\mathcal{U} \subset \tau^*(X)$, we say that a set $A \subset X$ is dense in $\mathcal{U}$ if $A \cap U \neq \emptyset$ for any $U \in \mathcal{U}$.

2.2 Lemma. *Given a Lindelöf Σ-space X, suppose that κ is a cardinal and $\mathcal{U} \subset \tau^*(X)$ is a non-empty family of cozero subsets of X such that $nw(U) \geq \kappa$ for any $U \in \mathcal{U}$. Consider the following conditions:*

(a) *$|\mathcal{U}| \leq \kappa$;*
(b) *$|D| < \kappa$ for any discrete set $D \subset X \times X$;*
(c) *there exists a discrete set $D \subset X \times X$ dense in the family $\widehat{\mathcal{U}} = \{U \times U : U \in \mathcal{U}\}$.*

Then (a) *implies* (c) *and* (b) *implies* (c).

Proof: We will prove both implications simultaneously. If $\kappa < \omega$ then both (a) and (b) imply that $\mathcal{U}$ is finite so (c) trivially holds. Therefore we can assume, without loss of generality, that κ is an infinite cardinal and hence each $U \in \mathcal{U}$ is infinite. Observe first that every $U \in \mathcal{U}$ must be a Lindelöf Σ-space being an F_σ-subset of X. Choose an enumeration $\{G_\alpha : \alpha < \lambda\}$ for a family $\mathcal{U}$; if we prove that (a)$\Longrightarrow$(c) then $\lambda = \kappa$.

Take a point $x_0 \in G_0$; pick any $y_0 \in G_0 \backslash \{x_0\}$ and let $z_0 = (x_0, y_0)$. We can find disjoint open sets U_0 and V_0 such that $x_0 \in U_0$ and $y_0 \in V_0$. Proceeding by induction let $z_0 = (x_0, y_0)$ and assume that $\alpha < \kappa$ and we have a set $\{z_\beta : \beta < \alpha\}$ and a family $\{U_\beta, V_\beta : \beta < \alpha\} \subset \tau(X)$ with the following properties:

(1) $z_\beta = (x_\beta, y_\beta) \in (X \times X) \backslash \Delta$ for any $\beta < \alpha$;

(2) $x_\beta \in U_\beta$, $y_\beta \in V_\beta$ and $U_\beta \cap V_\beta = \emptyset$ for every $\beta < \alpha$;
(3) $z_\beta \notin \overline{\{z_\gamma : \gamma < \beta\}}$ for all $\beta < \alpha$;
(4) $z_\beta \notin H_\beta = \bigcup\{U_\gamma \times V_\gamma : \gamma < \beta\}$ for each $\beta < \alpha$;
(5) $\{z_\gamma : \gamma \leq \beta\} \cap (G_\beta \times G_\beta) \neq \emptyset$ for any $\beta < \alpha$.

Let $D_\alpha = \{z_\beta : \beta < \alpha\}$. If the set D_α is dense in $\widehat{\mathcal{U}}$ then we set $D = D_\alpha$ and stop the induction for both proofs. If D_α is not dense in $\widehat{\mathcal{U}}$, then the property (5) shows that $\gamma = \min\{\beta : D_\alpha \cap (G_\beta \times G_\beta) = \emptyset\} \geq \alpha$.

Suppose that the set $(G_\gamma \times G_\gamma)\backslash\Delta$ is contained in $H_\alpha = \bigcup\{U_\beta \times V_\beta : \beta < \alpha\}$. Then the family $\mathcal{H} = \{U_\beta \cap G_\gamma, V_\beta \cap G_\gamma : \beta < \alpha\}$ is T_2-separating in G_γ. If $\kappa = \omega$ then we obtain a finite T_2-separating family on an infinite set which is impossible. If $\kappa > \omega$ then it follows from [Gr, Corollary 7.10] that $nw(G_\gamma) \leq |\mathcal{H}| \cdot \omega < \kappa$ which is a contradiction.

Therefore $(G_\gamma \times G_\gamma)\backslash\Delta$ is not contained in H_α and hence we can find distinct points $x_\alpha, y_\alpha \in G_\gamma$ such that $z_\alpha = (x_\alpha, y_\alpha) \notin H_\alpha$. Choose disjoint open sets U_α, V_α such that $x_\alpha \in U_\alpha$ and $y_\alpha \in V_\alpha$. It is immediate that the conditions (1)–(5) are still satisfied for all $\beta \leq \alpha$.

If our inductive construction stops for some $\alpha < \kappa$ then D is dense in $\widehat{\mathcal{U}}$ for both proofs so assume that we have the set $D = \{z_\alpha : \alpha < \kappa\}$. If we prove (a)$\Longrightarrow$(c) then the set D is dense in $\widehat{\mathcal{U}}$ by the property (5). If we prove that (b)$\Longrightarrow$(c) then (b) holds and hence D cannot be discrete.

Therefore it suffices to show that D_α is discrete for any $\alpha \leq \kappa$. Fix any $\beta < \alpha$ and observe that it follows from (3) that $z_\beta \notin \overline{\{z_\gamma : \gamma < \beta\}}$. Besides, we have the inclusion $\{z_\gamma : \gamma > \beta\} \subset (X \times X)\backslash(U_\beta \times V_\beta)$ by the property (4) so $z_\beta \notin \overline{\{z_\gamma : \gamma > \beta\}}$ and hence the set D_α is discrete. $\square$

The following result generalizes Theorem 3 of [JSz].

2.3 Theorem. *If X is a Lindelöf Σ-space then there exists a discrete subspace $D \subset X \times X$ such that $|D| \geq d(X)$.*

Proof: Call a non-empty open set $U \subset X$ *adequate* if $nw(U) = nw(V)$ for any $V \in \tau^*(U)$. Every non-empty open subset of X contains a non-empty cozero set of minimal network weight; it is evident that such a set will be adequate. Therefore the family $\mathcal{B}$ of adequate cozero sets is a π-base of X. Let $\mathcal{G}$ be a maximal disjoint subfamily of $\mathcal{B}$; then $\bigcup\mathcal{G}$ is dense in X and hence $\sum\{d(G) : G \in \mathcal{G}\} = d(X)$.

If we find a discrete set $D_G \subset G \times G$ such that $|D_G| \geq d(G)$ for every $G \in \mathcal{G}$ then $D = \bigcup\{D_G : G \in \mathcal{G}\} \subset X \times X$ will be a discrete subspace of $X \times X$ such that $|D| = \sum\{|D_G| : G \in \mathcal{G}\} \geq d(X)$.

So take any set $G \in \mathcal{G}$ and observe that G is a Lindelöf Σ-space because it is an F_σ-subset of X. Let $\kappa = nw(G)$. If there is a discrete $D_G \subset G \times G$ with $|D_G| \geq \kappa$ then there is nothing to prove because $d(G) \leq nw(G) = \kappa$. If such a set does not exist then we can apply the implication (b)$\Longrightarrow$(c) of Lemma 2.2 for the family $\mathcal{U} = \tau^*(G)$ to see that there exists a discrete $D_G \subset G \times G$ such that D_G is dense in $\{U \times U : U \in \mathcal{U}\}$ and hence the projection of the set D_G on the first coordinate is dense in G so $|D_G| \geq d(G)$. □

2.4 Corollary. *If X is a Lindelöf Σ-space then X^ω is d-separable.*

PROOF: Apply Theorem 2.3 and Theorem 6 of [JSz]. □

Recall that X is *a Lindelöf p-space* if there exists a perfect map of X onto a second countable space. Evidently, all compact spaces and all second countable spaces are Lindelöf p-spaces. The following theorem also generalizes Theorem 3 of the paper [JSz].

2.5 Theorem. *For any Lindelöf p-space X there exists a discrete $D \subset X \times X$ such that $\Delta_X \subset \overline{D}$ and hence $p_1(D)$ is dense in X. Here $p_1(x, y) = x$ for any $(x, y) \in X \times X$, i.e., $p_1 : X \times X \to X$ is the projection onto the first coordinate.*

PROOF: Call a non-empty open set $U \subset X$ *adequate* if $nw(U) = nw(V)$ for any $V \in \tau^*(U)$. Every non-empty open subset of X contains a non-empty open F_σ-set of minimal network weight; it is evident that such a set will be adequate. Therefore the family $\mathcal{B}$ of adequate open F_σ-sets is a π-base of X. Let $\mathcal{G}$ be a maximal disjoint subfamily of $\mathcal{B}$; then $\bigcup \mathcal{G}$ is dense in X.

If we find a discrete set $D_G \subset G \times G$ such that $\Delta_{D_G} \subset \overline{D}_G$ for every $G \in \mathcal{G}$ then $D = \bigcup\{D_G : G \in \mathcal{G}\} \subset X \times X$ will be a discrete set such that $\Delta_X \subset \overline{D}$. So take any $G \in \mathcal{G}$ and observe that G is a Lindelöf Σ-space because it is Lindelöf p being a cozero subset of a Lindelöf p-space. Besides, $\pi w(G) \leq w(G) = nw(G)$ (see Corollary 4.3 of [Ar]) so we can choose a π-base $\mathcal{U}$ in G with $|\mathcal{U}| \leq \kappa = nw(G)$. The set G being adequate, we have $nw(U) = \kappa \geq |\mathcal{U}|$ for any $U \in \mathcal{U}$ so we can apply the implication (a)$\Longrightarrow$(c) of Lemma 2.2 to conclude that there exists a discrete $D_G \subset G \times G$ such that D_G is dense in $\{U \times U : U \in \mathcal{U}\}$ and hence $\Delta_{D_G} \subset \overline{D}_G$. □

2.6 Corollary. *Suppose that X is a Lindelöf p-space and $\mathcal{P}$ is a closed-hereditary topological property. If the closure of every discrete $D \subset X \times X$ has $\mathcal{P}$ then X also has $\mathcal{P}$.*

PROOF: It suffices to observe that the diagonal Δ_X of the space X is homeomorphic to X and apply Theorem 2.5. □

2.7 Corollary. *Suppose that X is a Lindelöf p-space and $\mathcal{P}$ is a topological property preserved by quotient maps. If the closure of every discrete $D \subset X \times X$ has $\mathcal{P}$ then X also has $\mathcal{P}$.*

PROOF: By Theorem 2.5 there exists a discrete $D \subset X \times X$ such that $\Delta \subset \overline{D}$. The set Δ is a retract of $X \times X$ so it is also a retract of $\overline{D}$. Finally observe that Δ is homeomorphic to X and every retraction is a quotient map. □

2.8 Corollary. *Given a cardinal κ suppose that X is a Lindelöf p-space and $\mathcal{P}$ is a property from the following list $\mathbb{M}_0 = \{$character$\le\kappa$, weight$\le\kappa$, i-weight$\le\kappa$, pseudocharacter$\le\kappa$, tightness$\le\kappa$, κ-monolithity$\}$. If the closure of every discrete subspace of $X \times X$ has $\mathcal{P}$ then X has $\mathcal{P}$.*

2.9 Corollary. *Suppose that X is a Lindelöf p-space and $\mathcal{P}$ is a property from the following list $\mathbb{M}_1 = \{$Fréchet-Urysohn property, sequentiality, k-property, zero-dimensionality, the covering dimension dim does not exceed n for some $n \in \mathbb{N}\}$. If the closure of every discrete subspace of $X \times X$ has $\mathcal{P}$ then X has $\mathcal{P}$.*

The following corollary answers Problems 4.8 and 4.9 from [BT].

2.10 Corollary. *Suppose that X is a compact space and $\overline{D}$ is Corson (Eberlein) compact for any discrete $D \subset X \times X$. Then X is Corson (Eberlein) compact.*

2.11 Example. Zero-dimensionality is not discretely reflexive even in compact metrizable spaces.

Proof: If $X = [0,1] \subset \mathbb{R}$ then the closure of every discrete subset of X is nowhere dense in X and hence zero-dimensional. However, X is not zero-dimensional. $\square$

We are going to show next that, at least, Theorem 2.5 cannot be proved for Lindelöf Σ-spaces. We precede the corresponding example with two technical lemmas.

2.12 Lemma. *If Z is any space and $A \subset Z$ is a countable metrizable crowded subspace of Z then, for any $a \in A$, we can find a crowded set $B \subset A$ such that $a \in B$ and B is nowhere dense in Z.*

Proof: It suffices to find a crowded $B \subset A$ such that $a \in B$ and B is nowhere dense in A. Using the fact that $A \simeq \mathbb{Q} \simeq \mathbb{Q} \times \mathbb{Q}$, it is easy to find a disjoint family $\{A_n : n \in \omega\}$ of nowhere dense crowded subspaces of A such that $A = \bigcup_{n\in\omega} A_n$. There exists $n \in \omega$ with $a \in A_n$ so we can take $B = A_n$. $\square$

2.13 Lemma. *Suppose that Z is a second countable crowded space and $D \subset Z \times Z$ is a discrete subspace of $Z \times Z$. Then*

(a) *if $x \neq y$ and $(x,y) \notin D$ then there exist disjoint nowhere dense crowded sets $P, Q \subset Z$ such that $(x,y) \in P \times Q \subset (Z \times Z)\backslash D$;*
(b) *if $x \in Z$ and $(x,x) \notin D$ then there exists a nowhere dense crowded set $P \subset Z$ such that $(x,x) \in P \times P \subset (Z \times Z)\backslash D$.*

Proof: Take a metric d on the space Z which generates the topology of Z and let $d(a, A) = \inf\{d(a,b) : b \in A\}$ for any $a \in Z$ and $A \subset Z$. To prove (a) let $x_0 = x$, $y_0 = y$ and consider the sets $P_0 = \{x_0\}$ and $Q_0 = \{y_0\}$. Proceeding by induction, assume that $n \in \omega$ and we have finite sets $P_0, \dots, P_n$ and $Q_0, \dots, Q_n$ with the following properties:

(6) $P_i \subset P_{i+1}$ and $Q_i \subset Q_{i+1}$ for any $i < n$;

(7) $P_i \cap Q_i = \emptyset$ for any $i \leq n$;
(8) $(P_i \times Q_i) \cap D = \emptyset$ for any $i \leq n$;
(9) if $i < n$ and $p \in P_i$ then there exists a point $p' \in P_{i+1}$ such that $p' \neq p$ and $d(p, p') < 2^{-i}$;
(10) if $i < n$ and $q \in Q_i$ then there exists a point $q' \in Q_{i+1}$ such that $q' \neq q$ and $d(q, q') < 2^{-i}$.

For any point $p \in P_n$ the set $\{z \in Z : (p, z) \in D\}$ is discrete and hence nowhere dense in Z. Therefore the set $H = \{z \in Z : (p, z) \in D$ for some $p \in P_n\}$ is the finite union of nowhere dense subsets of Z and therefore the set $E = Z \backslash (H \cup P_n \cup Q_n)$ is dense in Z. As a consequence, we can choose a finite set $Q' \subset E$ such that $d(q, Q') < 2^{-n}$ for any $q \in Q_n$; let $Q_{n+1} = Q_n \cup Q'$.

Analogously, the set $H' = \{z \in Z : (z, q) \in D$ for some $q \in Q_{n+1}\}$ is the finite union of nowhere dense subsets of Z and therefore the set $E' = Z \backslash (H' \cup P_n \cup Q_{n+1})$ is dense in Z. As a consequence, we can choose a finite set $P' \subset E'$ such that $d(p, P') < 2^{-n}$ for any $p \in P_n$; let $P_{n+1} = P_n \cup P'$.

It is straightforward that the conditions (6)–(10) are still satisfied if we replace n with $n+1$ so our inductive procedure can be continued to construct families $\{P_n : n \in \omega\}$ and $\{Q_n : n \in \omega\}$ for which the properties (6)–(10) hold for all $n \in \omega$. Letting $S = \bigcup_{n\in\omega} P_n$ and $T = \bigcup_{n\in\omega} Q_n$ we obtain crowded sets such that $(x, y) \in S \times T \subset (Z \times Z) \backslash D$. Use Lemma 2.12 to find nowhere dense crowded sets $P, Q \subset Z$ such that $x \in P \subset S$ and $y \in Q \subset T$. This gives the promised sets P and Q so (a) is proved.

To prove (b) make the evident changes in the above construction to obtain a family $\{P_n : n \in \omega\}$ satisfying the conditions (6), (8) and (9) for $P_i = Q_i$ for all $i \leq n$. It is immediate that $T = \bigcup_{n\in\omega} P_n$ is a crowded set such that $(x, x) \in T \times T \subset (Z \times Z) \backslash D$. Finally, use Lemma 2.12 to find a nowhere dense crowded set $P \subset Z$ such that $x \in P \subset T$; this gives the promised set P. $\square$

Recall that a space X is *maximal*, if it is crowded but no strictly stronger (not necessarily Tychonoff) topology on X is crowded.

2.14 Example. Under CH, there exists a maximal countable space X such that every discrete subspace of $X \times X$ is closed.

Proof: Let Y be the set $\mathbb{Q}$ and denote by τ_0 its usual topology of subspace of $\mathbb{R}$. We will also need the set L of all countably infinite limit ordinals. Let $\mathcal{E}$ be the family of all infinite subsets of Y and choose an enumeration $\{E_\alpha : \alpha \in L\}$ of the family $\mathcal{E}$.

It is easy to choose an enumeration $\{D_\alpha : \alpha \in \omega_1 \backslash L\}$ of the family $\mathcal{D}$ of all infinite subsets of $Y \times Y$ such that every $D \in \mathcal{D}$ occurs ω_1-many times in the family $\{D_{\alpha+n} : \alpha \in L\}$ for any $n \in \mathbb{N}$. Let $\{(x_n, y_n) : n \in \mathbb{N}\}$ be a faithful enumeration of the set $Y \times Y$.

We will inductively construct a collection $\{\tau_\alpha : \alpha < \omega_1\}$ of crowded regular second countable topologies on Y with the following properties:

(11) if $\alpha \leq \beta < \omega_1$ then $\tau_\alpha \subset \tau_\beta$;

(12) if $\alpha > 0$ is a limit ordinal then τ_α is the topology generated by $\bigcup\{\tau_\beta : \beta < \alpha\}$;

(13) if α is a limit ordinal and both E_α and $Y\backslash E_\alpha$ are crowded subsets of (Y, τ_α) then $\{E_\alpha, Y\backslash E_\alpha\} \subset \tau_{\alpha+1}$;

(14) if $n \in \mathbb{N}$, $\alpha \in L$, the set $D = D_{\alpha+n}$ is discrete in $(Y, \tau_{\alpha+n}) \times (Y, \tau_{\alpha+n})$ and $(x_n, y_n) \in \overline{D}\backslash D$ (the closure is taken in $(Y, \tau_{\alpha+n}) \times (Y, \tau_{\alpha+n})$) then there exist $U, V \in \tau_{\alpha+n+1}$ such that $(x_n, y_n) \in U \times V \subset (Y \times Y)\backslash D$.

We already have τ_0 so assume that $0 < \beta < \omega_1$ and we have constructed a topology τ_α for each $\alpha < \beta$. If β is a limit ordinal then let τ_β be the topology generated by the family $\bigcup\{\tau_\alpha : \alpha < \beta\}$ as a base. This guarantees that the property (12) will hold.

If $\beta = \alpha + 1$ for a limit ordinal α then look at the sets E_α and $Y\backslash E_\alpha$. If one of them is not crowded then we let $\tau_\beta = \tau_\alpha$. If they are both crowded then let τ_β be the topology generated by the family $\tau_\alpha \cup \{E_\alpha\} \cup \{Y\backslash E_\alpha\}$. It is evident that this guarantees that we will have the property (13).

Now, if $\beta = \alpha + n + 1$ for some limit ordinal α and $n \in \mathbb{N}$ then consider the set $D = D_{\alpha+n}$ and the point $z = (x_n, y_n)$. If the set D is not discrete or $z \notin \overline{D}\backslash D$ in $(Y, \tau_{\alpha+n}) \times (Y, \tau_{\alpha+n})$ then let $\tau_\beta = \tau_{\alpha+n}$. If the set D is discrete and $z \in \overline{D}\backslash D$ in $(Y, \tau_{\alpha+n}) \times (Y, \tau_{\alpha+n})$ then we can use Lemma 2.13(b) if $x_n = y_n$ (or Lemma 2.13(a) if $x_n \neq y_n$) to find a nowhere dense crowded set P (disjoint nowhere dense crowded sets P and Q) such that $z \in P \times P \subset (Y \times Y)\backslash D$ (or $z \in P \times Q \subset (Y \times Y)\backslash D$ respectively) and let τ_β be the topology generated by $\tau_{\alpha+n} \cup \{P\} \cup \{Y\backslash P\}$ as a subbase (or the topology generated by the family $\tau_{\alpha+n} \cup \{P\} \cup \{Q\} \cup \{Y\backslash(P \cup Q)\}$ respectively). Clearly, this shows the property (14) so we can construct the promised ω_1-sequence $\{\tau_\alpha : \alpha < \omega_1\}$.

Denote by τ the topology generated by the family $\bigcup\{\tau_\alpha : \alpha < \omega_1\}$ and consider the space $X = (Y, \tau)$.

Suppose that A is a crowded subspace of X such that $X\backslash A$ is also crowded. There is a limit ordinal $\alpha > 0$ such that $A = E_\alpha$; it follows from the property (13) that $A \in \tau_{\alpha+1} \subset \tau$ and $X\backslash A \in \tau_{\alpha+1} \subset \tau$ which shows that A is a clopen subset of X, i.e., X is ultradisconnected.

Suppose that some discrete subset D of $X \times X$ is not closed in $X \times X$ and fix a point $z \in \overline{D}\backslash D$. There exists a countable family $\mathcal{U} \subset \tau$ such that $\mathcal{U}$ witnesses discreteness of the set D, i.e., for every $d \in D$ there are $U, V \in \mathcal{U}$ such that $(U \times V) \cap D = \{d\}$. Since $\mathcal{B} = \bigcup\{\tau_\alpha : \alpha < \omega_1\}$ is a base of τ, we can assume, without loss of generality, that $\mathcal{U} \subset \mathcal{B}$ and hence there exists $\gamma < \omega_1$ such that $\mathcal{U} \subset \tau_\gamma$ and hence D is a discrete subset of $(Y, \tau_\alpha) \times (Y, \tau_\alpha)$ for any $\alpha \geq \gamma$. There exist $\alpha > \gamma$ and $n \in \mathbb{N}$ such that $z = (x_n, y_n)$ and $D = D_{\alpha+n}$. It follows from (14) that there exist $U, V \in \tau_{\alpha+n+1}$ such that $z \in U \times V \subset (Y \times Y)\backslash D$. The set

$U \times V$ is also open in $X \times X$ so $z \notin \overline{D}$ which is a contradiction.

Thus, every discrete subset of $X \times X$ is closed in $X \times X$ and therefore every discrete subspace of X is closed in X. This together with ultradisconnectedness of X implies that X is a maximal space (see Fact 1.15 and Theorem 2.2 of [vD]). Since all discrete subsets of $X \times X$ are closed, there is no discrete $D \subset X \times X$ such that $\Delta_X \subset \overline{D}$. □

2.15 Proposition. *If X is a pseudocompact space with $w(X) \leq \omega_1$ then there exists a discrete $D \subset X \times X$ such that $|D| \geq d(X)$.*

Proof: There is nothing to prove if $d(X) \leq \omega$. So assume that $d(X) = \omega_1$ and hence X has a left-separated subspace L with $|L| = \omega_1$. If $s(X \times X) \leq \omega$ and there is a right-separated $R \subset X$ with $|R| = \omega_1$ then it is easy to see that $L \times R$ has an uncountable discrete subspace which is a contradiction. Therefore X is hereditarily Lindelöf and hence compact. So we can apply Theorem 3 of [JSz] to conclude that $s(X \times X) \geq \omega_1$ which is a contradiction. □

2.16 Corollary. *If X is a pseudocompact space and $w(X) \leq \omega_1$ then the space X^ω is d-separable.*

Proof: Apply Theorem 2.11 and Theorem 6 of [JSz]. □

We will show next that in some models of ZFC there exist countably compact spaces X such that X^ω is not d-separable. Since the proofs depend heavily on the construction of certain special trees, let us recall some basic notions and facts about trees.

A tree is a partially ordered set $(S, <)$ such that for any $s \in S$ the set $\{t \in S : t < s\}$ is well ordered by $<$. We write S instead of $(S, <)$. A subset $S' \subset S$ is called *a subtree* of the tree S if $\{t \in S : t < s\} \subset S'$ for any $s \in S'$. A subset $C \subset S$ of a tree S is called *a chain* if $s < s'$ or $s' < s$ for any distinct $s, s' \in C$. A set $A \subset S$ is *an antichain* if any distinct $a, b \in A$ are incomparable, i.e., neither $a < b$ nor $b < a$ is true. If S is a tree and $s \in S$ then $\mathrm{ht}(s)$ is the order type of the well ordered set $\{t \in S : t < s\}$ and $S_\alpha = \{s \in S : \mathrm{ht}(s) = \alpha\}$ for any ordinal α. Given a tree S and an $s \in S$ let $T_S(s) = \{t \in S : s \leq t\}$.

If S is a tree and $\{s_n : n \in \omega\} \subset S_\alpha$ for some α, let $T = \prod_* \{T_S(s_n) : n \in \omega\} = \{f : \omega \to S : f(\omega) \subset S_\beta \text{ for some } \beta \text{ and } f(n) \geq s_n \text{ for all } n \in \omega\}$. If $f, g \in T$ then $f < g$ if $f(n) < g(n)$ for all $n \in \omega$. The pair $(T, <)$ is called *the tree product of the trees* $\{T_S(s_n) : n \in \omega\}$.

In this paper we will work with the tree $\omega_1^{<\omega_2} = \{f : f \text{ is a function from } \alpha \text{ to } \omega_1 \text{ for some ordinal } \alpha < \omega_2\}$ and its subtrees with the order defined by $f < g$ if g extends f.

Recall that a subset $C \subset \omega_2$ is *closed unbounded* if it is closed in the order topology on ω_2 and cofinal in ω_2. A subset $B \subset \omega_2$ is called *stationary* if it intersects any closed unbounded subset of ω_2. Let $\omega_2^1 = \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega_1\}$. *The set-theoretic principle* $\diamondsuit(\omega_2^1)$ says that for each $\alpha < \omega_2$ there exists a set

$A_\alpha \subset \alpha$ such that for any $A \subset \omega_2$ the set $\{\alpha \in \omega_2^1 : A \cap \alpha = A_\alpha\}$ is stationary. It is well known that $\diamondsuit(\omega_2^1)$ is consistent with CH and the usual axioms of ZFC. It is not difficult to prove that $\diamondsuit(\omega_2^1)$ is equivalent to the following statement:

For an arbitrary set A of cardinality ω_2 and any $\alpha < \omega_2$ there exists a function $f_\alpha : \alpha \to A$ such that for any map $f : \omega_2 \to A$ the set $\{\alpha \in \omega_2^1 : f|\alpha = f_\alpha\}$ is stationary. The family $\{f_\alpha : \alpha < \omega_2\}$ is called the *$\diamondsuit(\omega_2^1)$-sequence for A.*

2.17 Example. If we assume CH and $\diamondsuit(\omega_2^1)$ then there exists an ω-bounded (and hence countably compact) space X such that $d(X) = \omega_2$ and $s(X^\omega) \leq \omega_1$. Therefore there is no discrete $D \subset X \times X$ with $|D| = d(X)$ and X^ω is not d-separable.

Proof: It was proved in Theorem 4.19 of [DTTW] that under CH and $\diamondsuit(\omega_2^1)$ there exists a tree $S = \bigcup\{S_\alpha : 0 < \alpha < \omega_2\} \subset \omega_1^{<\omega_2}$ with the following properties:

(15) $S_\alpha \subset \omega_1^\alpha$ and $|S_\alpha| = \omega_1$ for any $\alpha \in [1, \omega_2)$;
(16) the tree S has neither chains or antichains of cardinality ω_2;
(17) if $0 < \alpha < \beta < \omega_2$ and $x \in S_\alpha$ then x has ω_1-many successors in S_β;
(18) if $0 < \alpha < \omega_2$ and $\{x_n : n \in \omega\} \subset S_\alpha$ then the tree $\prod_*\{T_S(x_n) : n \in \omega\}$ does not have antichains of cardinality ω_2.

Let τ be the topology generated on S by the family

$$\mathcal{S} = \{T_S(s) : s \in S_{\alpha+1},\ \alpha \in \omega_2\} \cup \{S \backslash T_S(s) : s \in S_{\alpha+1},\ \alpha \in \omega_2\}$$

as a subbase. Clearly, for each $\alpha < \omega_2$ and $s \in S_{\alpha+1}$ the set $T_S(s)$ is clopen in (S, τ). It is an easy exercise to see that (S, τ) is a Tychonoff space. In what follows we identify S with the topological space (S, τ) and the subsets of S with the respective subspaces of (S, τ). Let $X = \{z \in \beta S :$ there is a countable $A_z \subset S$ such that $z \in \overline{A}_z\}$. It is clear that X is ω-bounded and hence countably compact. Observe that, given $\alpha \in [1, \omega_2)$ and $s \in S_\alpha$, if we take any infinite subset $Q \subset T_S(s) \cap S_{\alpha+1}$ then the family $\{T_S(t) : t \in Q\}$ is a local π-base of s in S.

Claim. For any $n \in \mathbb{N}$, each discrete subset of X^n has cardinality at most ω_1.

Proof of the claim: We will proceed by induction on n. Let X^0 be a singleton (the empty function) so it is clear that it has no big discrete subsets. Assume that $n \in \mathbb{N}$ and we proved our claim for all $k < n$. For each $i < n$ let $p_i : X^n \to X$ be the projection onto the i-th factor. The set $\Delta_n = \{x \in X^n : p_i(x) = p_j(x)$ for some distinct $i, j < n\}$ is easily seen to be the finite union of subsets homeomorphic to X^k for some $k < n$ so it suffices to show that $s(X^n \backslash \Delta_n) \leq \omega_1$.

Suppose that a set $D \subset X^n \backslash \Delta_n$ is discrete and $|D| = \omega_2$. For every point $d = (d_0, \ldots, d_{n-1}) \in D$ we can find a disjoint family $\{U_0^d, \ldots, U_{n-1}^d\}$ of open neighborhoods of the points $d_0, \ldots, d_{n-1}$ respectively in such a way that $U_d \cap D = \overline{U}_d \cap D = \{d\}$ where $U_d = U_0^d \times \ldots \times U_{n-1}^d$. Using the remark about the π-bases, for each point $d = (d_0, \ldots, d_{n-1}) \in D$ and $s \in A_{d_i} \cap U_i^d$ choose a countably infinite set $P_s \subset S$ such that $\{T_S(p) : p \in P_s\}$ is a π-base at s in S and $\bigcup\{T_S(p) : p \in P_s\} \subset U_{d_i}$ for all $i < n$. Let $B_{d_i} = \bigcup\{P_s : s \in A_{d_i} \cap U_{d_i}\}$ for every $i < n$. We have the following properties for each $d = (d_0, \ldots, d_{n-1}) \in D$:

(19) $\bigcup\{T_S(s) : s \in B_{d_i}\} \subset U_{d_i}$ for all $i < n$;

(20) the family $\{T_S(s) : s \in B_{d_i}\}$ is a π-network at d_i for each $i < n$;

(21) if $\mathcal{E}_d = \{T_S(t_0) \times \ldots \times T_S(t_{n-1}) : t_i \in B_{d_i}$ for all $i < n\}$ then $d \in \overline{\bigcup \mathcal{E}_d}$ but $e \notin \overline{\bigcup \mathcal{E}_d}$ for any $e \in D \backslash \{d\}$.

Observe that if $s \le t$ then $T_S(t) \subset T_S(s)$. As a consequence, if $d \in D$ and for every $i < n$ and $s \in B_{d_i}$ we choose in a non-limit level of S an element $f(s) \ge s$ then the family $\{T_S(f(s)) : s \in B_{d_i}\}$ is still a π-network at the point d_i and we have the inclusion $\bigcup\{T_S(f(s)) : s \in B_{d_i}\} \subset \bigcup\{T_S(s) : s \in B_{d_i}\}$. The property (17) for the tree S implies that for any $d = (d_0, \ldots, d_{n-1}) \in D$ and any $s \in \bigcup_{i<n} B_{d_i}$ there exists $f(s) \in S_{\alpha+1} \cap T_S(s)$, where $\alpha = \sup\{ht(s) : s \in B_{d_i}$ and $i < n\}$. Therefore $B'_{d_i} = \{f(s) : s \in B_{d_i}\} \subset S_{\alpha+1}$, the family $\{T_S(s) : s \in B'_{d_i}\}$ is a π-network at the point d_i and $\bigcup\{T_S(s) : s \in B'_{d_i}\} \subset \bigcup\{T_S(s) : s \in B_{d_i}\} \subset U_{d_i}$ for every $i < n$. This shows that, without loss of generality, we can assume that the set $B_{d_0} \cup \ldots \cup B_{d_{n-1}}$ is contained in some $S_{\alpha+1}$. For any $d = (d_0, \ldots, d_{n-1}) \in D$, fix an ordinal $\mu(d) < \omega_2$ such that $\bigcup_{i<n} B_{d_i} \subset S_{\mu(d)+1}$.

Our plan is to find distinct points $d, e \in D$ such that for every $i < n$ and $s \in B_{e_i}$ there is $t \in B_{d_i}$ such that $t < s$. This will imply $\bigcup \mathcal{E}_e \subset \bigcup \mathcal{E}_d$ and hence $e \in \overline{\bigcup \mathcal{E}_d}$ which is a contradiction with the property (21).

By CH, there are only ω_1-many countable subsets contained in each level of S and therefore the set $\{\mu(d) : d \in D\}$ is cofinal in ω_2. This makes it possible to choose an ω_2-sequence of ordinals $\{\gamma_\alpha : \alpha < \omega_2\}$ so that the following properties hold:

(22) $\gamma_\alpha = \mu(d^\alpha)$ for some $d^\alpha = (d_0^\alpha, \ldots, d_{n-1}^\alpha) \in D$;

(23) $\gamma_\beta > \sup\{\gamma_\alpha : \alpha < \beta\}$ for each $\beta < \omega_2$.

Note that it follows from the properties (22) and (23) that $d^\alpha \ne d^\beta$ if $\alpha \ne \beta$. Let C be the closure of $\{\gamma_\alpha : \alpha < \omega_2\}$ in ω_2 (considered with the interval topology). Then C is a closed unbounded subset of ω_2 and therefore $E = C \cap \omega_2^1$ is stationary. For each $\lambda \in E$ consider the ordinal $\nu(\lambda) = \min\{\gamma_\alpha : \lambda < \gamma_\alpha\}$; for every $i < n$ let $F_\lambda^i = B_{d_i^\alpha}$ and $e_\lambda^i = d_i^\alpha$, where α is determined by the condition $\gamma_\alpha = \nu(\lambda)$. Note that $\lambda, \delta \in E$, $\lambda < \delta$ implies that $\nu(\lambda) < \nu(\delta)$ and therefore the points $e_\lambda = (e_\lambda^0, \ldots, e_\lambda^{n-1})$ and $e_\delta = (e_\delta^0, \ldots, e_\delta^{n-1})$ are distinct elements of D.

For any $\lambda \in \omega_2$ denote by $\pi_\lambda : S \backslash (\bigcup\{S_\alpha : \alpha < \lambda\}) \to S_\lambda$ the restriction map defined by the formula $\pi_\lambda(s) = s|\lambda$ for any $s \in S$ with $ht(s) \ge \lambda$. For each $\lambda \in E$ choose a set $G_\lambda^i \subset F_\lambda^i$ such that $\pi_\lambda | G_\lambda^i : G_\lambda^i \to \pi_\lambda(F_\lambda^i)$ is a bijection. Since λ has cofinality ω_1, there exists $\beta(\lambda) < \lambda$ such that $\pi_{\beta(\lambda)} | (\bigcup_{i<n} G_\lambda^i) : \bigcup_{i<n} G_\lambda^i \to S_{\beta(\lambda)}$ is an injective map. By the Pressing-Down Lemma, there is $\delta < \omega_1$ such that the set $\{\lambda \in E : \beta(\lambda) = \delta\}$ is stationary. Represent ω as $\Omega_0 \cup \ldots \cup \Omega_{n-1}$ where the sets $\{\Omega_i : i < n\}$ are disjoint and infinite.

Using CH, we can find a set $P_i = \{s^i_k : k \in \Omega_i\} \subset S_\delta$ such that the set $R = \{\lambda \in E : \pi_\delta(G^i_\lambda) = P_i$ for each $i < n\}$ has cardinality ω_2. The sets $\{G^i_\lambda : i < n\}$ are disjoint by the property (19) so the family $\{P_i : i < n\}$ is also disjoint, the set $\bigcup_{i<n} P_i$ being an injective image of $\bigcup\{G^i_\lambda : i < n\}$ for any $\lambda \in R$.

Thus, we can choose a bijection $f_\lambda : \omega \to \bigcup_{i<n} G^i_\lambda$ such that for each $i < n$ we have the equalities $\{f_\lambda(k) : k \in \Omega_i\} = G^i_\lambda$ and $\pi_\delta(f_\lambda(k)) = s^i_k$ for any $k \in \Omega_i$ and $\lambda \in R$. Then $F = \{f_\lambda : \lambda \in R\} \subset H = \prod_*\{T_S(s^i_k) : i < n$ and $k \in \Omega_i\}$ cannot be an antichain in the tree H by property (18) of the tree S.

Therefore there are distinct $\lambda, \beta \in R$, say $\lambda < \beta$, such that $f_\lambda(k) < f_\beta(k)$ for all $k \in \omega$. If $s \in F^i_\beta$, then $s|\beta = f_\beta(k)|\beta$ for some $k \in \omega$ and therefore $t = f_\lambda(k) \in F^i_\lambda$ and $t = f_\lambda(k) < f_\beta(k)|\beta \leq s$. Recall that, for every $i < n$, we have $F^i_\beta = B_{e_i}$ and $F^i_\lambda = B_{d_i}$ for some distinct elements $d = (d_0, \ldots , d_{n-1})$ and $e = (e_0, \ldots , e_{n-1})$ of the set D. Consequently, for each $s \in B_{e_i}$ there is $t \in B_{d_i}$ such that $t < s$. We saw already that this is a contradiction which completes the proof of our Claim.

As an immediate consequence, we have $s(X^n) \leq \omega_1$ for any $n \in \omega$ and hence $s(X^\omega) \leq \omega_1$. It follows from the definition of X that, for any point $x \in X$ there exists a countable $A_x \subset S$ with $x \in \overline{A}_x$. Therefore, for any set $B \subset X$ with $|B| \leq \omega_1$ we can find $\alpha < \omega_2$ such that $B \subset \overline{\bigcup_{0<\beta<\alpha} S_\beta}$. Take any $y \in S_{\alpha+1}$ and observe that $T_S(y)$ is a clopen subset of S which contains y and does not meet $\bigcup_{0<\beta<\alpha} S_\beta$. As an easy consequence, y does not belong to the closure in X of the set B.

Therefore $d(X) = \omega_2$ so our Claim guarantees that $X \times X$ has no discrete subsets of cardinality $d(X)$. Suppose that $Z = \bigcup\{Z_n : n \in \omega\}$ is a dense subset of X^ω and every Z_n is discrete. Since $d(X^\omega) \geq d(X) = \omega_2$, we must have $|Z| = \omega_2$ and hence there is $n \in \omega$ such that $|Z_n| = \omega_2$ which is a contradiction with $s(X^\omega) \leq \omega_1$. Therefore X^ω is not d-separable. $\square$

3. Open problems

The authors feel that the topic of this paper is far from being exhausted. An evidence of this is the fact that there are more unsolved problems here than solved ones. We list below the most interesting questions we could not answer.

3.1 Problem. *Given a Tychonoff space X assume that $\overline{D}$ is zero-dimensional for any discrete $D \subset X \times X$. Must X be zero-dimensional?*

3.2 Problem. *Given a Lindelöf Σ-space X assume that $\overline{D}$ is zero-dimensional for any discrete $D \subset X \times X$. Must X be zero-dimensional?*

3.3 Problem. *Given a space X with a countable network, assume that $\overline{D}$ is zero-dimensional for any discrete $D \subset X \times X$. Must X be zero-dimensional?*

3.4 Problem. *Given a space X assume that $\overline{D}$ is countable for any discrete set $D \subset X \times X$. Must X be countable?*

3.5 Problem. *Suppose that X is a space such that $\overline{D}$ has countable i-weight for any discrete $D \subset X \times X$. Must X have countable i-weight?*

3.6 Problem. *Suppose that X is a space such that $\overline{D}$ is σ-compact for any discrete $D \subset X \times X$. Must X be σ-compact?*

3.7 Problem. *Suppose that X is a Lindelöf Σ-space such that $\overline{D}$ is σ-compact for any discrete $D \subset X \times X$. Must X be σ-compact?*

3.8 Problem. *Suppose that X is a cosmic space such that $\overline{D}$ is σ-compact for any discrete $D \subset X \times X$. Must X be σ-compact?*

3.9 Problem. *Suppose that X is a space such that $\overline{D}$ is σ-compact for any discrete $D \subset X \times X$. Must X be Lindelöf?*

3.10 Problem. *Suppose that X is a cosmic space such that $\overline{D}$ is analytic for any discrete $D \subset X \times X$. Must X be analytic?*

3.11 Problem. *Suppose that X is a space such that $\overline{D}$ is Čech-complete for any discrete $D \subset X \times X$. Must X be Čech-complete?*

3.12 Problem. *Suppose that X is a Lindelöf Σ-space. Does there exist a discrete $D \subset X \times X$ such that the projection of D onto the first coordinate is dense in X?*

3.13 Problem. *Does there exist in ZFC a countable maximal space X such that all discrete subspaces of $X \times X$ are closed?*

3.14 Problem. *Does there exist in ZFC a countably compact space X such that X^ω is not d-separable?*

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