Spaces not distinguishing pointwise and $\mathcal{I}$-quasinormal convergence

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Abstract. In this paper we extend the notion of quasinormal convergence via ideals and consider the notion of $\mathcal{I}$-quasinormal convergence. We then introduce the notion of $\mathcal{I}QN$ ($\mathcal{I}wQN$) space as a topological space in which every sequence of continuous real valued functions pointwise converging to 0, is also $\mathcal{I}$-quasinormally convergent to 0 (has a subsequence which is $\mathcal{I}$-quasinormally convergent to 0) and make certain observations on those spaces.

Keywords: ideal, filter, $\mathcal{I}$-quasinormal convergence, Chain Condition, AP-ideal, $\mathcal{I}QN$ space, $\mathcal{I}wQN$ space

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1. Introduction

We start by recalling the definition of asymptotic density as follows: If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$ then K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ stands for the cardinality of the set K_n. The asymptotic density of the subset K is defined by

$$d(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$$

provided the limit exists.

Using this idea of asymptotic density, the notion of convergence of a real sequence had been extended to statistical convergence by Fast [19] (see also [31]) as follows: A sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in a metric space (X, ρ) is said to be statistically convergent to ℓ if for arbitrary $\varepsilon > 0$, the set $K(\varepsilon) = \{k \in \mathbb{N} : d(x_k, \ell) \geq \varepsilon\}$ has asymptotic density zero. A lot of investigations have been done on this very interesting convergence and its topological consequences after the initial works by Fridy [20] and Šalat [30].

On the other hand, in [24] an interesting generalization of the notion of statistical convergence was proposed. Namely it is easy to check that the family $\mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$ forms a non-trivial admissible (or free) ideal of $\mathbb{N}$.

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A family $\mathcal{I} \subset 2^Y$ of subsets of a nonempty set Y is said to be an ideal in Y if (i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, (ii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$. Here we consider an ideal of $\mathbb{N}$ and without any loss of generality we also assume that $\bigcup_{A\in\mathcal{I}} A = \mathbb{N}$ which implies that $\{k\} \in \mathcal{I}$ for each $k \in \mathbb{N}$. Such ideals were sometimes called admissible ideals in the literature [24], [26], [14] (which are also called free ideals). If $\mathcal{I}$ is a proper ideal in Y (i.e. $Y \notin \mathcal{I}$, $\mathcal{I} \neq \{\emptyset\}$) then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset Y : \text{there exists } A \in \mathcal{I} : M = Y \setminus A\}$ is a filter in Y. It is called the filter associated with the ideal $\mathcal{I}$. Thus one may consider an arbitrary ideal $\mathcal{I}$ of $\mathbb{N}$ and define $\mathcal{I}$-convergence of a sequence by replacing the sets of density zero by the members of the ideal. Following the general line of [24] (see also [26]), ideals were used to study nets in topological and uniform spaces ([27], [12], [13]), to study certain variants of open covers and selection principles [16], [10], to study convergence of sequences of functions and its applications to measure theory ([2], [25], [28]).

The notion of quasinormal convergence was introduced by Bukovská in [3], [4] though it should be mentioned that Császár and Laczkovich [8] defined the same notion with the name 'equal convergence' in 1975 and again studied it in [9]. Bukovský, Reclaw and Repický introduced the notions of QN and wQN spaces in [5] as topological spaces not distinguishing pointwise and quasinormal convergence of real functions and established many fundamental and interesting properties of these spaces in [5], [6] and recently more work was done on these spaces relating them with certain covering properties by Bukovský and Hales [7]. A brief history of studies of spaces not distinguishing between two types of convergences and many important references can be found in the two beautifully written papers [5] and [6].

As a natural consequence we try to unify both these lines of investigations and first extend the notion of quasinormal convergence via ideals to $\mathcal{I}$-quasinormal convergence. Then we introduce the main notions of $\mathcal{I}QN$ ($\mathcal{I}wQN$) spaces as topological spaces in which every sequence of continuous real valued functions pointwise converging to 0, is $\mathcal{I}$-quasinormally convergent to 0 (has a subsequence which is $\mathcal{I}$-quasinormally convergent to 0). We make certain observations of these spaces basically following the line of investigation of [5].

2. Basic definitions and properties

Throughout the paper $\mathbb{N}$ will denote the set of all positive integers and $\mathcal{I}$ will stand for a non-trivial proper admissible ideal of $\mathbb{N}$.

Recall that the usual definition of convergence of a sequence was extended in two ways by using an ideal in [24] as follows: A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-convergent to $x \in \mathbb{R}$ if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}^*$-convergent to $x \in \mathbb{R}$ if there is a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m_1 < m_2 < \cdots < m_k < \dots\}$ such that $\lim_{k\to\infty} x_{m_k} = x$.

An ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is called an AP-ideal (or said to satisfy the property (AP) [24]) if for any sequence $\{A_1, A_2, \dots\}$ of mutually disjoint sets of $\mathcal{I}$ there is a sequence

$\{B_1, B_2, \dots\}$ of sets such that $A_i \Delta B_i$ $(i = 1, 2, \dots)$ is finite and $B = \bigcup_{j\in\mathbb{N}} B_j \in \mathcal{I}$. These types of ideals have also been called P-ideals (see [2], [25], [28], [14]). The ideal $\mathcal{I}_{fin}$ of all finite subsets of $\mathbb{N}$ as well as the ideal $\mathcal{I}_d$ are simple examples of AP-ideals. Other examples of AP-ideals can be seen from [25], [28]. Also a very useful fact is that the notions of $\mathcal{I}$ and $\mathcal{I}^*$-convergence of real sequences coincide if and only if the ideal $\mathcal{I}$ is an AP-ideal (see [24], [26] and for more applications [11]).

The following property of an ideal will play a very important role in many results of this paper.

We say that a subset $\mathcal{B}$ of an ideal $\mathcal{I}$ is a "basis" if every element of $\mathcal{I}$ is a subset of some element of $\mathcal{B}$. We say that $\mathcal{I}$ satisfies the "Chain Condition" if there exists a sequence $\{C_k\}_{k\in\mathbb{N}} \subset \mathcal{I}$ with $C_1 \subset C_2 \subset C_3 \subset \dots$ such that for any $A \in \mathcal{I}$ there exists $k \in \mathbb{N}$ such that $A \subset C_k$. Therefore an ideal satisfies the Chain Condition if and only if it possesses a countable basis. Note that the ideal $\mathcal{I}_{fin}$ clearly satisfies the Chain Condition. Another non-trivial example of an ideal with Chain Condition is the following. Let $\mathbb{N} = \bigcup_{j=1}^{\infty} A_j$ be a decomposition of $\mathbb{N}$ such that each A_j is infinite and $A_i \cap A_j = \emptyset$ for $i \neq j$. Let $\mathcal{I}_0$ denote the class of all $A \subset \mathbb{N}$ which intersect at most a finite number of A_j's. Then $\mathcal{I}_0$ is a non-trivial ideal satisfying the Chain Condition. But this ideal is not an AP-ideal as can be seen from [24], [26] where it was established that any metric space (or topological space) with at least one limit point has a sequence which is $\mathcal{I}_0$-convergent but not $\mathcal{I}_0^*$-convergent.

Following [24] the usual ideas of pointwise and uniform convergence of a sequence of functions were extended via ideals first in [2] and then studied in ([2], [25], [28]) which we now recall. Let X be a nonempty set and let f_n, f be real valued functions defined on X. A sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions is said to be $\mathcal{I}$-pointwise convergent to f if for each $x \in X$ and for each $\varepsilon > 0$ there exists an $A = A(x, \varepsilon) \in \mathcal{I}$ such that $n \in \mathbb{N} \setminus A$ implies $|f_n(x) - f(x)| < \varepsilon$ and in this case we write $f_n \xrightarrow{\mathcal{I}} f$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-uniformly convergent to f if for any $\varepsilon > 0$ there exists $A = A(\varepsilon) \in \mathcal{I}$ such that for all $n \in \mathbb{N} \setminus A$ and for all $x \in X$, $|f_n(x) - f(x)| < \varepsilon$. In this case we write $f_n \xrightarrow{\mathcal{I}-u} f$.

The important notion of quasinormal convergence (which was earlier introduced as equal convergence in [8]) was introduced in [3, 4] as follows. A function f is said to be the quasinormal limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$ if there is a sequence of positive reals $\varepsilon_n \to 0$ such that for every $x \in X$, there exists $n_0 = n_0(x)$ with $|f_n(x) - f(x)| < \varepsilon_n$ for $n \geq n_0$.

We are now in a position to introduce our main definitions.

Definition 2.1. Let X be a nonempty set and f_n, f be real valued functions defined on X. We say that $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-quasinormally convergent to f on X (written as $f_n \xrightarrow{\mathcal{I}QN} f$ on X) if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of nonnegative reals $\mathcal{I}$-converging to 0 such that for each $x \in X$, the set $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$.

This convergence can also be called $\mathcal{I}$-equal convergence following the terminology of [8] which has been very recently used to study certain properties concerning $\mathcal{I}$-equal limits of real functions in [15].

Definition 2.2. A topological space X is called an $\mathcal{I}QN$ space if for any sequence $\{f_n\}_{n\in\mathbb{N}}$ of continuous real valued functions pointwise converging to zero on X, we have $f_n \xrightarrow{\mathcal{I}QN} 0$.

Definition 2.3. A topological space X is called an $\mathcal{I}wQN$ space if for any sequence $\{f_n\}_{n\in\mathbb{N}}$ of continuous real valued functions pointwise converging to zero on X, there is an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers such that $f_{n_k} \xrightarrow{\mathcal{I}QN} 0$ on X.

Definition 2.4. A set $X \subset [0,1]$ is called an $\mathcal{I}QN$ set if X with the subspace topology induced from the usual topology is an $\mathcal{I}QN$ space.

Definition 2.5. A set $X \subset [0,1]$ is called an $\mathcal{I}wQN$ set if X with the subspace topology is an $\mathcal{I}wQN$ space.

We start with two results providing some necessary and sufficient (Theorem 2.1) and sufficient (Theorem 2.2) conditions for $\mathcal{I}$-quasinormal convergence which will play important roles throughout the paper.

Theorem 2.1. *Let $\mathcal{I}$ be an ideal satisfying the Chain Condition. Let f, f_n, $n = 1, 2, 3, \dots$ be real valued functions defined on a set X. The following conditions are equivalent.*

- (i) $f_n \xrightarrow{\mathcal{I}QN} f$ *on* X.
- (ii) *There are sets $X_k \subset X$ such that $X = \bigcup_{k\in\mathbb{N}} X_k$ and $f_n \xrightarrow{\mathcal{I}-u} f$ on X_k for every $k = 1, 2, 3, \dots$.*
- (iii) *There are sets $X_k \subset X$ such that $X = \bigcup_{k\in\mathbb{N}} X_k$, $X_1 \subset X_2 \subset X_3 \dots$ and $f_n \xrightarrow{\mathcal{I}-u} f$ on X_k for every $k = 1, 2, 3, \dots$.*

If X is a topological space and f_n, $n = 1, 2, 3, \dots$ are continuous, then (i), (ii), (iii) *are equivalent to:*

- (iv) *There are closed sets $X_k \subset X$, $k = 1, 2, 3, \dots$, $X = \bigcup_{k\in\mathbb{N}} X_k$, $X_1 \subset X_2 \subset X_3 \dots$ and $f_n \xrightarrow{\mathcal{I}-u} f$ on X_k for every $k = 1, 2, 3, \dots$.*

Proof: (i)$\Rightarrow$(iii) Assume (i), i.e. $f_n \xrightarrow{\mathcal{I}QN} f$. Then there is a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive real numbers with $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_n = 0$ and for every $x \in X$ there is a set $A_x \in \mathcal{I}$ such that $|f_n(x) - f(x)| < \varepsilon_n$ for all $n \in \mathbb{N} \setminus A_x$. Since $\mathcal{I}$ satisfies the Chain Condition, there exists a sequence $\{C_k\}_{k\in\mathbb{N}}$ in $\mathcal{I}$ with $C_1 \subset C_2 \subset C_3 \subset \dots$ such that for every $A \in \mathcal{I}$ there exists some $C_k \in \mathcal{I}$ with $A \subset C_k$. Now define $X_k = \{x \in X : |f_n(x) - f(x)| < \varepsilon_n \text{ for all } n \in \mathbb{N} \setminus C_k\}$, $k \in \mathbb{N}$. Then clearly $X_1 \subset X_2 \subset X_3 \subset \dots$. Further observe that for any $x \in X$, if $A_x \in \mathcal{I}$ is the set witnessing $\mathcal{I}$-quasinormal convergence as defined above, then $A_x \subset C_k$ for some $k \in \mathbb{N}$. Consequently $x \in X_k$. Hence $X = \bigcup_{k\in\mathbb{N}} X_k$. It is now easy to observe that $f_n \xrightarrow{\mathcal{I}-u} f$ on X_k. Indeed, take $\varepsilon > 0$. Let $B = \{n \in \mathbb{N} : \varepsilon_n \geq \varepsilon\}$. Then $B \in \mathcal{I}$, since $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_n = 0$. If $x \in X_k$, then $|f_n(x) - f(x)| < \varepsilon$ for $n \in (\mathbb{N} \setminus C_k) \cap (\mathbb{N} \setminus B) = \mathbb{N} \setminus (C_k \cup B)$ and $C_k \cup B \in \mathcal{I}$. This proves (iii).

(ii)⇒(i) Now assume (ii), i.e. suppose that $X = \bigcup_{k\in\mathbb{N}} X_k$ and $|f_n(x) - f(x)| \leq \varepsilon_{in}$ for all $x \in X_i$ when $n \notin M(i) \in \mathcal{I}$, where $\{\varepsilon_{in}\}_{n\in\mathbb{N}}$ is a sequence of positive reals depending on i such that $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_{in} = 0$ for a fixed i. We can select sets $M_k \in \mathcal{I}$ such that $M_1 \subset M_2 \subset \cdots \subset M_k \subset \ldots$ and $\varepsilon_{kn} < \frac{1}{k}$ whenever $n \notin M_k$, for $k = 1, 2, 3, \ldots$. Define

$$
\begin{aligned}
\varepsilon_n &= 1 \quad \text{if } n \in M_2 \\
&= \frac{1}{k} \quad \text{if } n \in M_{k+1} \setminus M_k \\
&= 0 \quad \text{if } n \notin \bigcup_{k\in\mathbb{N}} M_k.
\end{aligned}
$$

Then $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_n = 0$ and furthermore $|f_n(x) - f(x)| \leq \varepsilon_{in} < \varepsilon_n$ for $x \in X_i$ and if $n \notin M(i) \cup M_i \in \mathcal{I}$ which shows that $f_n \xrightarrow{\mathcal{I}QN} f$. So (i) follows. Since (iii)⇒(ii), so it now follows that (i), (ii) and (iii) are equivalent.

Now let X be a topological space and f_n, $n = 1, 2, 3, \ldots$ be continuous. Evidently (iv) implies (iii). Assume (i). Let us define $X_k = \{x \in X : |f_n(x) - f_m(x)| \leq \varepsilon_n + \varepsilon_m$ for all $m, n \in \mathbb{N} \setminus C_k\}$, $k \in \mathbb{N}$. Suppose as before $\mathcal{I}$ satisfies the Chain Condition with the sequence $\{C_k\}_{k\in\mathbb{N}}$ in $\mathcal{I}$. Clearly X_k is closed for $k = 1, 2, 3, \ldots$ as f_n's are continuous functions and $X_1 \subset X_2 \subset X_3 \subset \ldots$. If $x \in X$ then from the proof of (i)⇒(iii), it readily follows that $x \in X_k$ for some $k \in \mathbb{N}$ and $f_n \xrightarrow{\mathcal{I}-u} f$ on each X_k. So (iv) is proved. Hence (i), (ii) and (iii) are equivalent to (iv). □

Remark 2.1. The first part of the above theorem can be further generalized in the following manner: Let X be a topological space and f_n, $n = 1, 2, 3, \ldots$ be real valued continuous functions defined on X such that $f_n \xrightarrow{\mathcal{I}QN} f$ on X to some real valued function f defined on X. If the ideal $\mathcal{I}$ has a basis of cardinality κ, then there exists a family of sets $\mathcal{K}$ such that $|\mathcal{K}| = \kappa$, $X = \bigcup \mathcal{K}$ and $f_n \xrightarrow{\mathcal{I}-u} f$ on every $K \in \mathcal{K}$.

Note 2.1. Note that we require the additional hypothesis on the ideal to prove the necessity part but we do not require any additional assumption for the sufficiency part.

Corollary 2.1. *Let $X = \bigcup_{k\in\mathbb{N}} X_k$. If $f_n \xrightarrow{\mathcal{I}QN} f$ on each X_k, $k = 1, 2, 3, \ldots$, then $f_n \xrightarrow{\mathcal{I}QN} f$ on X.*

Example 2.1. This example shows that there exist functions f and f_n, $n = 1, 2, 3, \ldots$ such that $f_n \xrightarrow{\mathcal{I}} f$ but $f_n \overset{\mathcal{I}QN}{\nrightarrow} f$. Let $\mathcal{I}$ be an admissible ideal satisfying the Chain Condition and $\mathcal{I} \neq \mathcal{I}_{fin}$. Let C be an infinite member of $\mathcal{I}$. Let $\mathbb{Q} = \{r_k : k \in \mathbb{N} \cup \{0\}\}$ be a one to one enumeration of rational numbers. Let

$$\begin{aligned} f(x) = &\ 0 \qquad \text{if } \ x \in \mathbb{R} \setminus \mathbb{Q} \\ = &\ 2^{-k} \quad \text{if } \ x = r_k, \ k = 0, 1, 2, \dots . \end{aligned}$$

Clearly f is not continuous on any interval. For every $n \in \mathbb{N} \setminus C$ choose a positive real $\delta_n \leq 2^{-n}$ such that $\delta_n \leq \frac{1}{2}|r_i - r_j|$, $i = 0, 1, 2, \dots, n$, $j = 0, 1, 2, \dots, n$, $i \neq j$. Let

$$\begin{aligned} f_n(x) = &\ 0 && \text{if } \ x \in \mathbb{R} \setminus \bigcup_{i=0}^{n} (r_i - \delta_i, r_i + \delta_i) \\ = &\ 2^{-i} && \text{for } \ x = r_i, \ i = 0, 1, 2, \dots, n \\ = &\ 2^{-i}(1 - \frac{|x - r_i|}{\delta_i}) && \text{for } \ x \in (r_i - \delta_i, r_i + \delta_i), \ i = 1, 2, 3, \dots, n. \end{aligned}$$

for $n \in \mathbb{N} \setminus C$ and $f_n = n$ for each $n \in C$.

Clearly $f_n \xrightarrow{\mathcal{I}} f$ (though f_n does not converge to f pointwise) on $\mathbb{R}$. But $f_n \overset{\mathcal{I}QN}{\nrightarrow} f$ on $\mathbb{R}$. Otherwise if $f_n \xrightarrow{\mathcal{I}QN} f$ on $\mathbb{R}$ then by Theorem 2.1, $\mathbb{R} = \bigcup_{k=0}^{\infty} E_k$ where E_k's are closed and $f_n \xrightarrow{\mathcal{I}-u} f$ on every E_k for $k = 0, 1, 2, \dots$. By the Baire category theorem, there is k such that $Int\ E_k \neq \emptyset$, i.e. there are a, b, $a < b$ such that $[a, b] \subseteq E_k$. Since each f_n is continuous and $f_n \xrightarrow{\mathcal{I}-u} f$ on $[a, b]$, it follows that f being the $\mathcal{I}$-uniform limit of continuous functions on $[a, b]$ is continuous on $[a, b]$ (see [2]), which is a contradiction.

Example 2.2. This example shows that there exist f, f_n, $n = 0, 1, 2, \dots$ such that $f_n \xrightarrow{\mathcal{I}QN} f$ but $f_n \overset{\mathcal{I}-u}{\nrightarrow} f$. Let $\mathcal{I}$ be any admissible ideal and $\mathcal{I} \neq \mathcal{I}_{fin}$. Let C be any infinite member of $\mathcal{I}$. Take $f_n(x) = x^n$ if $n \notin C$ and $f_n(x) = n$ for all $x \in [0, 1]$ if $n \in C$. Let $f(x) = 0$ for $x \in [0, 1)$ and $f(1) = 1$. Clearly $f_n \xrightarrow{\mathcal{I}QN} f$ on $[0, 1]$. As f is not continuous, $f_n \overset{\mathcal{I}-u}{\nrightarrow} f$ on $[0, 1]$. Note that $\{f_n\}_{n \in \mathbb{N}}$ does not converge to f quasinormally.

A quasiordering $\leq^*$ is defined on $\mathbb{N}^{\mathbb{N}}$ by eventual dominance:

$$f \leq^* g \ \text{ if } \ f(n) \leq g(n) \ \text{ for all but finitely many } n.$$

We say that a subset Y of $\mathbb{N}^{\mathbb{N}}$ is bounded if there exists g in $\mathbb{N}^{\mathbb{N}}$ such that for each $f \in Y$, $f \leq^* g$. Otherwise we say that Y is unbounded. Moreover, $\mathfrak{b}$ is defined as

$$\mathfrak{b} = \min\{|B| : B \ \text{ is an unbounded subset of } \ \mathbb{N}^{\mathbb{N}}\}.$$

It is known that $\aleph_0 < \mathfrak{b} \leq \mathfrak{c}$ but not necessarily $\mathfrak{b} = \aleph_1$ ([31], see also [4], [5]).

Theorem 2.2. *Let $\mathcal{I}$ be an AP-ideal. Let $X = \bigcup_{s\in S} X_s$, $|S| < \mathfrak{b}$. If the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges $\mathcal{I}$-quasinormally to f on every X_s, $s \in S$, then it does so on X.*

PROOF: From hypothesis, for each $s \in S$, there is a sequence $\{{\varepsilon_n}^s\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converging to zero and witnessing $\mathcal{I}$-quasinormal convergence on X_s. Since $\mathcal{I}$ is an AP-ideal, $\{{\varepsilon_n}^s\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-convergent to zero. So we can actually take $\{{\varepsilon_n}^s\}_{n\in\mathbb{N}}$ to be a decreasing sequence of positive reals witnessing the $\mathcal{I}$-quasinormal convergence on X_s. Now let us define

$$h_s(k) = \min\{n \in \mathbb{N} : {\varepsilon_n}^s \le \frac{1}{k+1}\,,\ n > h_s(k-1)\}.$$

Since the family $\{h_s : s \in S\}$ is of power less than $\mathfrak{b}$, there exists a function $g : \mathbb{N} \to \mathbb{N}$ with the above described condition. Moreover, we can assume that g is strictly increasing. Define

$$\begin{aligned} \varepsilon_n =&\ 1 && \text{if } \ n < g(1), \\ =&\ \frac{1}{k+1} && \text{if } \ g(k) \le n < g(k+1). \end{aligned}$$

If $x \in X$, then $x \in X_s$ for some $s \in S$. Since $f_n \xrightarrow{\mathcal{I}QN} f$ on X_s we have $A = \{n \in \mathbb{N} : |f_n(x) - f(x)| \ge {\varepsilon_n}^s\} \in \mathcal{I}$. Consequently $\mathbb{N} \setminus A \in \mathcal{F}(\mathcal{I})$ and $n \in \mathbb{N} \setminus A$ implies $|f_n(x) - f(x)| < {\varepsilon_n}^s$. Also there is a natural number k such that $h_s(n) \le g(n)$ for $n \ge k$. Since we have already observed that $\{{\varepsilon_n}^s\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-convergent to zero, so there exists a set $B_s \in \mathcal{F}(\mathcal{I})$ such that $\{{\varepsilon_n}^s\}_{n\in B_s}$ converges to zero. Hence if $n \in (\mathbb{N}\setminus A)\cap B_s$ and $n \ge g(k)$ then $g(l) \le n < g(l+1)$ for some $l \ge k$. Since $g(l) \ge h_s(l)$, we have $|f_n(x) - f(x)| < {\varepsilon_n}^s \le \frac{1}{l+1} \le \varepsilon_n$ and this proves the theorem. □

Lemma 2.1. *Continuous image of an $\mathcal{I}QN$ space is an $\mathcal{I}QN$ space.*

The proof is omitted.

Lemma 2.2. *Continuous image of an $\mathcal{I}wQN$ space is an $\mathcal{I}wQN$ space.*

The proof is omitted.

Lemma 2.3. *Every countable space* (*more generally a space of cardinality less than* $\mathfrak{b}$) *is an $\mathcal{I}QN$ space* (*provided $\mathcal{I}$ is an AP-ideal*).

PROOF: Let X be countable and let $X = \{a_k : k \in \mathbb{N}\}$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous real valued functions on X pointwise converging to zero. Write $X = \bigcup_{k=1}^{\infty} X_k$, $X_k = \{a_k\}$ for $k = 1, 2, 3, \dots$. Each X_k is closed and $f_n \xrightarrow{\mathcal{I}-u} 0$ on each X_k as X_k is a singleton. Hence by Theorem 2.1, $f_n \xrightarrow{\mathcal{I}QN} 0$ on X and so X is an $\mathcal{I}QN$ space.

If X is of cardinality less than $\mathfrak{b}$, say $X = \{a_s : s \in S\}$, where $|S| < \mathfrak{b}$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous real valued functions on X pointwise converging to zero. Write $X = \bigcup_{s\in S} X_s$, where $X_s = \{a_s\}$ for $s \in S$. Now

$f_n \xrightarrow{\mathcal{I}-u} 0$ on each X_s and hence by Theorem 2.1 and Theorem 2.2, $f_n \xrightarrow{\mathcal{I}QN} 0$ on X. Hence X is an $\mathcal{I}QN$ space. $\square$

Let X be a perfectly normal topological space. We define

Definition 2.6. Let non($\mathcal{I}QN$ space) be the minimal cardinality of a perfectly normal space which is not an $\mathcal{I}QN$ space.

Definition 2.7. Let non($\mathcal{I}QN$ set) be the minimal cardinality of a subspace of $[0,1]$ which is not an $\mathcal{I}QN$ set.

Definition 2.8. Let add($\mathcal{I}QN$ space) be the minimal cardinal number α such that there is a perfectly normal non-$\mathcal{I}QN$ space (i.e. a perfectly normal space which is not an $\mathcal{I}QN$ space) which can be expressed as the union $X = \bigcup_{\xi<\alpha} X_\xi$, where X_ξ's are $\mathcal{I}QN$ spaces.

Definition 2.9. Let add($\mathcal{I}QN$ set) be the minimal cardinal number α such that there is a perfectly normal non-$\mathcal{I}QN$ set which can be expressed as the union $X = \bigcup_{\xi<\alpha} X_\xi$, where X_ξ's are $\mathcal{I}QN$ sets.

Theorem 2.3. *We have that*

(i) *add($\mathcal{I}QN$ set) $\geq$ add($\mathcal{I}QN$ space) $\geq$ $\mathfrak{b}$, where the second inequality holds provided $\mathcal{I}$ is an AP-ideal;*
(ii) *add($\mathcal{I}QN$ set) $\leq$ non($\mathcal{I}QN$ set).*

Proof: (i) If X is an $\mathcal{I}QN$ set then it is obviously an $\mathcal{I}QN$ space. Hence add($\mathcal{I}QN$ set)$\geq$ add($\mathcal{I}QN$ space). By Theorem 2.2, if $X = \bigcup_{s\in S} X_s$, $|S| < \mathfrak{b}$ and X_s is an $\mathcal{I}QN$ space for each $s \in S$, then X becomes an $\mathcal{I}QN$ space. So from definition of add($\mathcal{I}QN$ space) it follows that add($\mathcal{I}QN$ space) $\geq \mathfrak{b}$.

(ii) It follows directly from Definition 2.7 and Definition 2.9. $\square$

3. Some further observations on $\mathcal{I}QN$ and $\mathcal{I}wQN$ spaces

Theorem 3.1. *Let $\mathcal{I}$ be an AP-ideal.*

(a) *A closed subset of a perfectly normal $\mathcal{I}QN$ space is an $\mathcal{I}QN$ space.*
(b) *A closed subset of a perfectly normal $\mathcal{I}wQN$ space is an $\mathcal{I}wQN$ space.*
(c) *An F_σ subset of a perfectly normal $\mathcal{I}QN$ space is an $\mathcal{I}QN$ space.*

Proof: (a) Let X be a perfectly normal $\mathcal{I}QN$ space and $A \subseteq X$ is closed. Let $f_n : A \to \mathbb{R}$ be a sequence of continuous functions and $f_n \to 0$ on A. Since A is a closed subset of a perfectly normal space, there exist open sets $B_1 \supset B_2 \supset \dots$ such that $\bigcap_{n=1}^{\infty} B_n = A$. For each $n \in \mathbb{N}$, let $h_n : X \to \mathbb{R}$ be continuous such that $h_n|_A = f_n$ and $h_n(x) = 0$ for all $x \in X \setminus B_n$. Then $h_n \to 0$ on X and since X is an $\mathcal{I}QN$ space so $h_n \xrightarrow{\mathcal{I}QN} 0$ on X. Thus there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n \geq 0$ and $\varepsilon_n \xrightarrow{\mathcal{I}} 0$ such that for each $x \in X$, the set $\{n : |h_n(x)| \geq \varepsilon_n\} \in \mathcal{I}$. Thus for each $x \in A$, $\{n : |f_n(x)| \geq \varepsilon_n\} = \{n : |h_n(x)| \geq \varepsilon_n\} \in \mathcal{I}$. Hence $f_n \xrightarrow{\mathcal{I}QN} 0$ on A. Thus A is an $\mathcal{I}QN$ space.

(b) The proof is similar to that of (a) and so is omitted.

(c) By Theorem 2.3(i), add($\mathcal{I}QN$ space) $\geq \mathfrak{b}$ and $\mathfrak{b} > \aleph_0$, so it is sufficient to prove the assertions for closed subsets and by (a) the result holds. □

Remark 3.1. In [5] it was proved that $\mathfrak{b} \geq$ add(wQNset)$\geq$add(wQNspace)$\geq \mathfrak{h}$ (see [5, Theorem 3.3]) which was subsequently used to prove that an F_σ subset of a perfectly normal wQN space is a wQN space (see [5, Theorem 4.1]). We could neither prove nor disprove a result similar to [5, Theorem 3.3] for $\mathcal{I}wQN$ spaces and so we leave it as an open problem. It is easy to observe that if a similar result can be established then Theorem 3.1(c) is also true for $\mathcal{I}wQN$ spaces.

Theorem 3.2. *Let (X, ρ) be a separable metric space and let A be a subset of X without isolated points. If A is an $\mathcal{I}wQN$ space then A is meager in X, provided $\mathcal{I}$ satisfies the Chain Condition.*

Proof: Let $B = \{r_n : n \in \mathbb{N}\}$ be a countable dense subset of $\bar{A}$. For every $n \in \mathbb{N}$ choose a sequence $\{x_{n,m}\}_{m\in\mathbb{N}}$ from A such that $x_{n,m} \to r_n$, $x_{n,m} \neq r_n$ for each $m \in \mathbb{N}$. Let $f_{n,m} : X \to [0, \frac{1}{2^{n-1}}]$ be a continuous function such that $f_{n,m}(x_{n,m}) = \frac{1}{2^{n-1}}$ and $f_{n,m}(x) = 0$ for all those $x \in X$ for which $\rho(x, x_{n,m}) \geq \frac{1}{2}\rho(r_n, x_{n,m})$. Let us define $h_m(x) = \sum_{n=1}^{\infty} f_{n,m}(x)$, $x \in X$, $m = 1, 2, 3, \ldots$. Then every h_m is a continuous function from X into $[0, 2]$ and $h_m \to 0$ on X.

Suppose on the contrary that A is not meager in X though A is an $\mathcal{I}wQN$ space i.e. there exists a subsequence $\{h_{m_k}\}_{k\in\mathbb{N}}$ of the sequence $\{h_m\}_{m\in\mathbb{N}}$ converging $\mathcal{I}$-quasinormally to zero on A. By Theorem 2.1, there exist closed sets $A_l \subset X$, $l = 1, 2, 3, \ldots$, $A \subset \bigcup_{l=1}^{\infty} A_l$ such that

$$h_{m_k} \xrightarrow{\mathcal{I}-u} 0 \quad \text{on} \quad A \cap A_l,\ l = 1, 2, 3, \ldots \tag{1}$$

Moreover, we can assume that $A_l \subset \bar{A}$ (otherwise we can just replace A_l by $A_l \cap \bar{A}$). Since A is not meager, there exists a $p \in \mathbb{N}$ such that $Int(A_p) \neq \emptyset$. Since B is dense in $\bar{A}$, there is some $r_n \in Int(A_p)$. Consequently

$$x_{n,m} \in Int(A_p) \quad \text{for all} \quad m \geq m_1 \quad \text{for some} \quad m_1 \in \mathbb{N}. \tag{2}$$

Thus whenever $m \notin C$ where $C = \{1, 2, \ldots, m_1\} \in \mathcal{I}$, we have that $\sup\{h_m(x) : x \in A \cap A_p\} \geq h_m(x_{n,m}) \geq f_{n,m}(x_{n,m}) = \frac{1}{2^{n-1}}$. The first inequality follows from (2) and the second inequality follows from the definition of h_m. Now

$$\{m_k : \sup_{x\in A\cap A_p} h_{m_k}(x) \geq \frac{1}{2^{n-1}}\} = \mathbb{N} \setminus C \notin \mathcal{I}$$

as $C \in \mathcal{I}$. So $h_{m_k} \overset{\mathcal{I}-u}{\nrightarrow} 0$ on $A \cap A_p$ which is a contradiction to (1). This implies that A is not an $\mathcal{I}wQN$ space. This completes the proof of the theorem. □

Corollary 3.1. *If A is an $\mathcal{I}wQN$ subspace of a separable metric space, then A is perfectly meager. Especially, any $\mathcal{I}wQN$ set is perfectly meager, provided $\mathcal{I}$ is an ideal satisfying the Chain Condition.*

Proof: If P is a perfect set, then $P \cap A = A_0 \cup A_1$, where A_0 is countable and A_1 is dense in itself and closed in A. Since A_0 is countable, it is meager. Again since A_1 is a closed subset of the $\mathcal{I}wQN$ space A, hence by Theorem 3.1(b), A_1 is also an $\mathcal{I}wQN$ space. Observe that A_1 being dense in itself has no isolated points and hence by Theorem 3.2, A_1 is meager. Thus $P \cap A$ is the union of two meager sets and so it is meager. As $P \cap A$ is meager for any perfect set P, hence A is perfectly meager.

Similarly we can prove the assertion for any $\mathcal{I}wQN$ set because $[0,1]$ with the subspace topology is a separable metric space. □

Corollary 3.2. *If A is an $\mathcal{I}wQN$ set, then for the Lebesgue measure ν on $[0,1]$, the inner measure $\nu_*(A)$ of A is zero provided $\mathcal{I}$ is an ideal satisfying the Chain Condition.*

Proof: Suppose that $\nu_*(A) > 0$. Then from regularity we can find a compact set K such that $K \subset A$ and $0 < \nu_*(K) \leq \nu_*(A)$. The compact set K contains a perfect subset K_1. The perfect set K_1 is not perfectly meager and hence it is not an $\mathcal{I}wQN$ set. This is a contradiction to the fact that K_1 is an $\mathcal{I}wQN$ set by Theorem 3.1(b). Hence $\nu_*(A) = 0$. □

Remark 3.2. The above result is also true for every Radon measure on $[0,1]$ (by a Radon measure we mean a finite diffused regular Borel measure on $[0,1]$, see [22]).

Corollary 3.3. *If X is an $\mathcal{I}wQN$ set then X is zero dimensional provided $\mathcal{I}$ is an ideal satisfying the Chain Condition.*

The proof is similar to the proof of Corollary 3.2 and so we omit it.

Corollary 3.4. *If X is completely regular $\mathcal{I}wQN$ space, then X has a basis consisting of clopen sets. Moreover, if X is also perfectly normal then every open subset of X can be expressed as countable union of clopen sets provided $\mathcal{I}$ is an ideal satisfying the Chain Condition.*

Proof: Let A be an open subset of X and let $x \in A$. As X is completely regular, there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$, $f(y) = 1$ for $y \in X \setminus A$. Clearly $f(X) \subset [0,1]$ and as f is continuous so by Lemma 2.2, $f(X)$ is an $\mathcal{I}wQN$ set. Then by Corollary 3.3, $f(X)$ is zero dimensional. Since $f(X)$ is Hausdorff, there exists a basic open set U of $f(X)$ such that $0 \in U$ but $1 \notin U$. Also as $f(X)$ is zero dimensional, U can be chosen as clopen in $f(X)$. Now $f^{-1}(U)$ is a clopen subset of A (because $f(y) = 1$ for all $y \in X \setminus A$ and $1 \notin U$) and $x \in f^{-1}(U)$ (as $f(x) = 0$ and $0 \in U$). Thus for any $x \in X$ and for any open set $A \subset X$ containing x, there is a clopen subset of A containing x. Hence X has a basis consisting of clopen sets.

If X is perfectly normal then X is normal and every closed set in X is a G_δ set in X. We know that in a normal space Z, we can always find a continuous function $g : Z \to [0,1]$ such that $g(x) = 0$ for $x \in A$ and $g(x) > 0$ for $x \notin A$ if and only if A is a closed and G_δ set in Z. Let A be open in X. Then $X \setminus A$ is closed and so is a G_δ set in X. Now by the above stated property of the normal space, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ for $x \in X \setminus A$ and $f(x) < 1$ if $x \in A$. Let $c \in A$. Then $f(c) \in f(A)$ and so $f(c) < 1$ and consequently there exists a clopen set $U_c \subset f(X)$ such that $f(c) \in U_c$ and $1 \notin U_c$. Indeed, $f(X) \subset [0,1]$ is an $\mathcal{I}wQN$ space by Lemma 2.2, and so $f(X)$ is an $\mathcal{I}wQN$ set which implies that $f(X)$ is zero dimensional by Corollary 3.3. Now $f^{-1}(U_c)$ is a clopen subset of A containing c (because $1 \notin U_c$ and $f(x) = 1$ for $x \in X \setminus A$). Then we have $A = \bigcup_{c \in A} f^{-1}(U_c)$. Note that $X \setminus A = \bigcap_{n \in \mathbb{N}} G_n$ where G_n is open for $n = 1,2,3,\ldots$ (since $X \setminus A$ being closed is also G_δ). Clearly $A = \bigcup_{n \in \mathbb{N}} (X \setminus G_n)$ where $X \setminus G_n$ is open for $n = 1,2,3,\ldots$. Hence we can choose countably many $f^{-1}(U_{c_n})$, $n = 1,2,3,\ldots$ such that $A = \bigcup_{n \in \mathbb{N}} f^{-1}(U_{c_n})$ and so A is the union of a countably many clopen sets. □

Let $\alpha \leq \mathfrak{c}$ be a regular cardinal. We now consider the following definition.

Definition 3.1 ([5])**.** A set $X \subset [0,1]$ is called an α-Sierpiński set if $|X| \geq \alpha$ and for every zero Lebesgue measure set A, $|A \cap X| < \alpha$.

It is known that Martin axiom implies the existence of a $\mathfrak{c}$-Sierpiński set [5].

Though an Egoroff-like theorem was established in [27] for ideals, a notion of convergence weaker than $\mathcal{I}$-uniform convergence was used there. This result was called weak Egoroff's theorem and it was observed ([27, Theorem 3.1]) that for every analytic AP-ideal $\mathcal{I}$, weak Egoroff's theorem holds. Following the terminology of [27] we say that Egoroff's theorem holds for the ideal $\mathcal{I}$ if for any finite measure space $(X, \mathcal{S}, \nu)$ and for any real valued continuous functions f, f_n, $n = 1,2,3,\ldots$ defined almost everywhere on X such that $f_n \xrightarrow{\mathcal{I}} f$ almost everywhere on X, for every $\varepsilon > 0$ there is a measurable set H_ε such that $\nu(X \setminus H_\varepsilon) < \varepsilon$ and $f_n \xrightarrow{\mathcal{I}-u} f$ on H_ε.

Remark 3.3. In [27] it was further established that Egoroff's theorem holds true for a non-pathological ideal $\mathcal{I}$ if and only if it is isomorphic to $\mathcal{I}_{fin}$ or $\varphi \times \mathcal{I}_{fin}$ ([27, Theorem 3.4]). It is still an open problem whether there exists a pathological analytic AP-ideal for which Egoroff's theorem holds ([27, Problem 1]). We establish the following result for an ideal for which Egoroff's theorem holds. We do not know whether the result can be proved for ideals for which weak Egoroff's theorem hold and leave it as an open problem.

Theorem 3.3. *If X is $\mathfrak{b}$-Sierpiński set, then every subset is an $\mathcal{I}QN$ set, for an AP-ideal $\mathcal{I}$ for which Egoroff's theorem holds.*

Proof: As in [5, Theorem 4.7] let $A \subset X$ and $f_n : A \to \mathbb{R}$ be a continuous function for $n = 1,2,3,\ldots$ and $f_n \to 0$ on A. We can assume that all f_n are defined and continuous on a G_δ set $G \supset A$. Let $C \subset G$ be the Borel set of those $x \in G$ for which $f_n(x) \to 0$. Evidently $A \subset C$.

Now from our assumption of Egoroff's theorem for $\mathcal{I}$ on the finite measure space (C, ν), where ν stands here for the Lebesgue measure on C, for every $n \in \mathbb{N}$ we can choose a measurable set $H_n \subset C$ such that $f_n \xrightarrow{\mathcal{I}-u} 0$ on H_n and $\nu(C \setminus H_n) < \frac{1}{n}$. Define $H = \bigcup_{n\in\mathbb{N}} H_n$. Then $f_n \xrightarrow{\mathcal{I}QN} 0$ on H by Corollary 2.1 and $\nu(C \setminus H) = \nu(\bigcap_{n\in\mathbb{N}} C \setminus H_n) \leq \frac{1}{n}$ for each $n \in \mathbb{N}$ and so $\nu(C\setminus H) = 0$. Since $|A\cap(C\setminus H)| < \mathfrak{b}$, we have $f_n \xrightarrow{\mathcal{I}QN} 0$ on $A\cap(C\setminus H)$ by Theorem 2.2. Thus $f_n \xrightarrow{\mathcal{I}QN} 0$ on $A\cap(C\setminus H)$ and also $f_n \xrightarrow{\mathcal{I}QN} 0$ on $A \cap H$. Consequently $f_n \xrightarrow{\mathcal{I}QN} 0$ on A which implies that A is an $\mathcal{I}QN$ space. Clearly $A \subset X \subset [0, 1]$, i.e. A is an $\mathcal{I}QN$ set. □

We have already proved that continuous image of an $\mathcal{I}QN$ space ($\mathcal{I}wQN$ space) is also an $\mathcal{I}QN$ space ($\mathcal{I}wQN$ space) in Lemma 2.1 and Lemma 2.2. Below we prove a related result.

Theorem 3.4. *Let $f : X \to Y$ be a mapping from an $\mathcal{I}QN$ space X into a metric space Y. If f is an $\mathcal{I}$-quasinormal limit of a sequence of continuous mappings, then $f(X) \subseteq Y$ is an $\mathcal{I}QN$ space, provided $\mathcal{I}$ is an ideal satisfying the Chain Condition.*

Proof: Let $f_n : X \to Y$ be continuous functions for $n = 1, 2, 3, \ldots$ and $f_n \xrightarrow{\mathcal{I}QN} f$ on X. Then by Theorem 2.1, there exist closed sets X_k, $k = 1, 2, 3, \ldots$, $X = \bigcup_{k\in\mathbb{N}} X_k$ and $f_n \xrightarrow{\mathcal{I}-u} f$ on X_k, $k = 1, 2, 3, \ldots$. Now f being the $\mathcal{I}$-uniform limit of a sequence of continuous functions on X_k is continuous on each X_k, $k = 1, 2, 3, \ldots$ (see [2]). Since each X_k is closed in X which is an $\mathcal{I}QN$ space so X_k is also an $\mathcal{I}QN$ space by Theorem 3.1 and also $f(X_k) \subset Y$ is an $\mathcal{I}QN$ space by Lemma 2.1. As $f(X) = \bigcup_{k\in\mathbb{N}} f(X_k)$, $f(X)$ is an $\mathcal{I}QN$ space by Theorem 2.3(i). □

Concluding remarks. This is only an introduction into what seems to be an interesting line of investigation when one replaces the finiteness in a definition by members of an ideal as was previously done in ([2], [10]–[16], [24]–[28]) and a lot of investigation has to be done to understand the behaviors of the new notions. In particular we would like to raise the following questions which seem very natural.

Problem 3.1. We proved almost all the results under some assumption on the ideal (either taking it as an AP-ideal or requiring it to satisfy the Chain Condition). Are they essential? Can the results be proved for any admissible ideal (or at least under weaker assumption)?

Problem 3.2. At least under certain suitable assumption, many properties and behavior of QN spaces and $\mathcal{I}QN$ spaces (wQN spaces and $\mathcal{I}wQN$ spaces) appear to be the same. Then is every $\mathcal{I}QN$ space actually a QN space? And is a $\mathcal{I}wQN$ space a wQN space? We could neither prove nor disprove it.

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