Involutive birational transformations of arbitrary complexity in Euclidean spaces

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Abstract. A broad family of involutive birational transformations of an open dense subset of $\mathbb{R}^n$ onto itself is constructed explicitly. Examples with arbitrarily high complexity are presented. Construction of birational transformations such that $\phi^k = \mathrm{Id}$ for a fixed integer $k > 2$ is also presented.

Keywords: rational mapping, birational transformation, involutive transformation

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1. Introduction

Already in the secondary school, students meet an example of an involutive birational transformation (i.e., $\phi^2 = \mathrm{Id}$) of the plane without origin onto itself, namely that given by the formulas

$$\bar{x} = \frac{x}{x^2 + y^2}, \qquad \bar{y} = \frac{y}{x^2 + y^2}.$$

Geometrically, this transformation arises if we project a sphere into the plane passing through its center from the north (or south) pole. The aim of this paper is to show that there exist involutive birational transformations (in more than two real variables) of arbitrarily high "complexity", i.e., containing terms of arbitrarily high degrees in numerators or denominators of the coordinate components of such transformations. Birational transformations (i.e., rational transformations whose inverse is also rational) over general fields have been studied extensively, see e.g. [5]. Involutive birational transformations have been studied in different contexts in [4] or [6].

In the paper [2], the first author studied scalar invariants of representations of the group $\mathrm{SL}(2,\mathbb{R})$ on the spaces of equiaffine connections in the plane. During the procedure of looking for the invariants, remarkable involutive birational transformations appeared. They were constructed using particular invariants of the operators generating the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$.

In the paper [3], the authors studied standard representations of the group $\mathrm{SL}(2,\mathbb{R})$ on spaces of homogeneous polynomials in two variables (binary forms)

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and constructed similar involutive birational transformations in arbitrary dimension. These transformations were still constructed from particular invariants and they have a geometrical meaning. In the same paper, we constructed examples of involutive birational transformations of arbitrary complexity in a purely algebraic way.

In the present paper, we come out of the "purely algebraic" examples from [3], and we try to develop a more general scheme of construction of involutive birational transformations in Euclidean spaces on a purely algebraic basis. Simple series of examples are given whose complexity grows to infinity with raising dimension n. We also construct series of examples for any fixed n, whose complexity grows to infinity with raising auxiliary parameter s. We have realized that our involutions cannot be derived from the well known De Jonquières involutions (see for example [1]) because polynomials involved in our paper are non-homogeneous. At the end, an example of birational transformation of order 3 is given. This method can be easily adapted for the construction of birational transformations of arbitrary order k. All the constructions can be easily modified from real numbers to complex numbers or even to more general fields. Because the idea of our paper is elementary one, we are using also elementary language, not the exact language of algebraic varieties as in [5].

2. Explicit construction of birational transformations

For each integer $i \geq 1$, let $P_i(b_0, \dots, b_{i-1})$ and $Q_i(b_0, \dots, b_{i-1})$ be nonzero polynomials in variables $b_0, \dots, b_{i-1}$ and let

$$y_i = P_i + Q_i b_i. \tag{1}$$

In particular, we put $y_0 = b_0$. Further, let us fix $n > 1$ and consider the new variables $y_0, \dots, y_n$ as polynomials of variables $b_0, \dots, b_n$ given above. For any permutation $p \in S_{n+1}$ of the variables $b_0, \dots, b_n$, we define $x_i = y_i \circ p$, which means the application of the permutation p to the variables of polynomials y_i.

Proposition 1. *For each $i = 1, \dots, n$, the variable b_i can be written as a rational function of $y_0, \dots, y_i$. The definition domain of the transformation $(y_0, \dots, y_n) \mapsto (b_0, \dots, b_n)$ is $\mathbb{R}^{n+1}[y_0, \dots, y_n] \setminus D$, where D is a subset of measure zero.*

Proof: We have $b_0 = y_0$ and

$$b_1 = \frac{y_1 - P_1(b_0)}{Q_1(b_0)}\,, \tag{2}$$

which is a rational function $b_1(y_0, y_1)$. Now let us suppose that the statement holds up to $i - 1$, i.e., we have fixed the expressions of $b_0, \dots, b_{i-1}$ as rational functions of $y_0, \dots, y_n$. Then we get from (1)

$$b_i \quad = \quad \frac{y_i - P_i(b_0, \dots, b_{i-1})}{Q_i(b_0, \dots, b_{i-1})} \tag{3}$$

and after the substitution of $b_0(y_0), \dots, b_{i-1}(y_0, \dots, y_{i-1})$ in place of $b_0, \dots, b_{i-1}$ and a simplification, we obtain the rational function $b_i(y_0, \dots, y_i)$ in the form of one single fraction whose numerator and denominator have no nontrivial common factor. Let us denote by $D_i[y_0, \dots, y_n]$ the subset of $\mathbb{R}^{n+1}[y_0, \dots, y_n]$ where the denominator of $b_i(y_0, \dots, y_n)$ is zero. Each of the sets $D_i[y_0, \dots, y_n]$ is of measure zero and the rational mapping is defined on the set $\mathbb{R}^{n+1}[y_0, \dots, y_n] \setminus \bigcup_{i=0}^{n} D_i$. $\square$

Proposition 2. *For each permutation $p \in S_{n+1}$, there is a unique birational transformation of the variables $x_0, \dots, x_n$ through the variables $y_0, \dots, y_n$ depending only on the choice of the polynomials P_i and Q_i, $i = 1, \dots, n$ in* (1). *The definition domain of the transformation ϕ: $(y_0, \dots, y_n) \mapsto (x_0, \dots, x_n)$ is again $\mathbb{R}^{n+1}[y_0, \dots, y_n] \setminus D$, where D is a subset of measure zero.*

Proof: First we have $x_i(b_0, \dots, b_n) = y_i(b_{p(0)}, \dots, b_{p(n)})$ for $i = 0, \dots, n$, by definition. Due to the formula (1) we have $y_i(b_{p(0)}, \dots, b_{p(n)}) = P_i(b_{p(0)}, \dots, b_{p(n)}) + Q_i(b_{p(0)}, \dots, b_{p(n)})b_{p(i)}$ for each i. If we substitute the rational functions $b_i(y_0, \dots, y_i)$ from (3) in place of $b_0, \dots, b_n$ in the last polynomial expressions, we obtain, after a simplification, the unique rational expressions $x_i(y_0, \dots, y_n)$, $i = 0, \dots, n$, for the transformation ϕ: $(y_0, \dots, y_n) \mapsto (x_0, \dots, x_n)$. The definition domain of the transformation ϕ is obvious. $\square$

Proposition 3. *Let the order of the permutation p be equal to* 2. *Then the birational transformation from variables x_i to y_i (and vice versa) is involutive. The intersection of the definition domain with the range of each transformation is of the form $\mathbb{R}^{n+1}[y_0, \dots, y_n] \setminus (D \cup D')$, or $\mathbb{R}^{n+1}[x_0, \dots, x_n] \setminus (C \cup C')$, respectively. Here C, D, C', D' are subsets of measure zero and $C \cup C'$ can be naturally identified with $D \cup D'$.*

Proof: Because $x_i = y_i \circ p$ and also $y_i = x_i \circ p$, the expressions for the transformation φ: $(x_0, \dots, x_n) \mapsto (y_0, \dots, y_n)$ and the transformation ϕ: $(y_0, \dots, y_n) \mapsto (x_0, \dots, x_n)$ are the same. We denote by C the singular subset for the transformation φ, analogous to the subset D for the transformation ϕ, according to Proposition 2. Now, we put $D' = \phi^{-1}(C)$, $C' = \varphi^{-1}(D)$ and shrink definition domains appropriately. Then we have $\varphi^2 = \mathrm{Id}$, $\phi^2 = \mathrm{Id}$ and both these transformations are involutive. $\square$

3. Complexity of birational transformations

We now estimate the complexity of the above maps in a special situation.

Definition 4. Let ϕ be a rational mapping of $\mathbb{R}^{n+1} \setminus D$, where D is a subset of measure zero, into $\mathbb{R}^{n+1}$. Let ϕ_i, $i = 0, 1, \dots, n$ be its components expressed in a reduced form (i.e., there are no nontrivial common factors in numerators and corresponding denominators). By the complexity of the mapping ϕ we mean the highest degree occuring among monomials involved in the numerators of the components ϕ_i.

Let us suppose that the polynomials Q_i and P_i are special monomials, namely $Q_1 = b_0$, $Q_i = b_0 b_1 \dots b_{i-1}$ and $P_1 = {b_0}^2$, $P_i = {b_0}^2 b_1 \dots b_{i-1}$. Then we have $\deg(y_i) = i+1$. For the further convenience, let us define the polynomials S_i as $S_0 = y_0$ and $S_i = S_{i-1} y_0 + (-1)^i y_i$. We easily derive the general formula

$$S_i \quad = \quad {y_0}^{i+1} + \sum_{k=1}^{i} (-1)^k {y_0}^{i-k} y_k. \tag{4}$$

Proposition 5. *For the polynomials P_i and Q_i as above, the variables b_i expressed using Proposition 1 are*

$$b_0 = S_0, \qquad b_i = -\frac{S_i}{S_{i-1}}, \qquad i = 1, \dots, n. \tag{5}$$

Explicitly, using formula (4), we have

$$\begin{aligned} b_0 &= y_0, \\ b_i &= -\frac{{y_0}^{i+1} + \sum_{k=1}^{i} (-1)^k {y_0}^{i-k} y_k}{{y_0}^{i} + \sum_{k=1}^{i-1} (-1)^k {y_0}^{i-1-k} y_k}, \qquad i = 1, \dots, n. \end{aligned} \tag{6}$$

Proof: The statement holds clearly for b_0 and b_1. Assume that it holds up to b_{i-1}. From the formula $y_i = P_i + Q_i b_i$, we obtain

$$b_i = \frac{y_i - P_i}{Q_i} = \frac{y_i - {b_0}^2 b_1 \dots b_{i-1}}{b_0 b_1 \dots b_{i-1}}\,. \tag{7}$$

Now we use the formula (5) and the induction assumption up to b_{i-1}. The formula simplifies as

$$b_i = \frac{y_i - (-1)^{i-1} S_{i-1} y_0}{(-1)^{i-1} S_{i-1}} = -\frac{S_i}{S_{i-1}}\,. \tag{8}$$

□

Corollary 6. *Let the permutation $p \in S_{n+1}$ act on $b_0, \dots b_n$ in a way that $p(b_i) = b_{n-i}$. Then the rational mapping ϕ is an involutive birational transformation, and the particular component $x_0(y_i) = b_n(y_i)$ is of the form*

$$x_0 \quad = \quad -\frac{{y_0}^{n+1} + \sum_{k=1}^{n} (-1)^k {y_0}^{n-k} y_k}{{y_0}^{n} + \sum_{k=1}^{n-1} (-1)^k {y_0}^{n-1-k} y_k}\,. \tag{9}$$

Proof: Follows immediately from the formula (6). □

Corollary 7. *The complexity of the birational transformation considered in the special situation above is at least $n+1$.*

With a small modification, we can construct involutive birational transformations such that the complexity increases with a new integral parameter s, for *fixed* n.

Proposition 8. *Let us consider, for fixed n, the polynomials $y_i = P_i + Q_i b_i$ for $i = 0, \dots, n-1$ as before and $y_n = {P_n}^s + Q_n b_n$, for some positive integer s. Then the component $x_0(y_0, \dots, y_n)$ is*

$$(10) \qquad x_0 \;=\; (-1)^n \frac{(-1)^{s(n-1)} {y_0}^s \left({y_0}^n + \sum_{k=1}^{n-1} (-1)^k {y_0}^{n-1-k} y_k \right)^s - y_n}{{y_0}^n + \sum_{k=1}^{n-1} (-1)^k {y_0}^{n-1-k} y_k} .$$

Hence, the complexity is at least $(n+1)s$.

Proof: In the same way as in the proof of Proposition 5, we obtain

$$(11) \qquad b_n = \frac{y_n - {P_n}^s}{Q_n} = \frac{y_n - ({b_0}^2 b_1 \dots b_{n-1})^s}{b_0 b_1 \dots b_{n-1}} = \frac{y_n - (-1)^{s(n-1)} {S_{n-1}}^s y_0^s}{(-1)^{n-1} S_{n-1}}.$$

The final formula is obtained by the substitution of the formula (4) for S_{n-1} and the simplification of signs. □

Let us remark that the complexity of our mappings depends on the choice of the permutation p (of order 2 in our case). We will show just a simple example of this phenomenon.

Example 9. We start with P_1, P_2, Q_1, Q_2 as before and y_0, y_1, y_2 according to formula (1). We have

$$(12) \qquad \begin{aligned} y_0 &= b_0, \\ y_1 &= {b_0}^2 + b_0 b_1, \\ y_2 &= {b_0}^2 b_1 + b_0 b_1 b_2. \end{aligned}$$

For the expressions $b_i(y_0, y_1, y_2)$, we have according to (5) and (6) the formulas

$$(13) \qquad \begin{aligned} b_0 &= y_0, \\ b_1 &= -\frac{{y_0}^2 - y_1}{y_0}, \\ b_2 &= -\frac{{y_0}^3 - y_0 y_1 + y_2}{{y_0}^2 - y_1}. \end{aligned}$$

We choose the permutation p_1 as in Corollary 6, hence $p_1(0, 1, 2) = (2, 1, 0)$. We obtain formulas for $x_i = y_i \circ p_1$ in the form

$$(14) \qquad \begin{aligned} x_0 &= b_2, \\ x_1 &= {b_2}^2 + b_1 b_2, \\ x_2 &= {b_2}^2 b_1 + b_0 b_1 b_2 \end{aligned}$$

and, after the substitution of (13) into (14), the formulas for the involutive transformation $(y_0, y_1, y_2) \mapsto (x_0, x_1, x_2)$ are

$$
\begin{aligned}
x_0 &= -\frac{{y_0}^3 - y_0 y_1 + y_2}{{y_0}^2 - y_1},\\
x_1 &= \frac{({y_0}^3 - y_0 y_1 + y_2)(2{y_0}^4 - 3{y_0}^2 y_1 + y_0 y_2 + {y_1}^2)}{y_0({y_0}^2 - y_1)^2},\\
x_2 &= -\frac{y_2({y_0}^3 - y_0 y_1 + y_2)}{y_0({y_0}^2 - y_1)}.
\end{aligned}
\tag{15}
$$

The complexity according to Definition 4 is equal to 7.

Now, we choose another permutation p_2 as $p_2(0,1,2) = (0,2,1)$. For $x_i = y_i \circ p_2$, we have the formulas

$$
\begin{aligned}
x_0 &= b_0,\\
x_1 &= {b_0}^2 + b_0 b_2,\\
x_2 &= {b_0}^2 b_2 + b_0 b_1 b_2.
\end{aligned}
\tag{16}
$$

By the substitution of formulas (13) into these formulas, we obtain the involutive transformation $(y_0, y_1, y_2) \mapsto (x_0, x_1, x_2)$ in the form

$$
\begin{aligned}
x_0 &= y_0,\\
x_1 &= -\frac{y_0 y_2}{{y_0}^2 - y_1},\\
x_2 &= -\frac{y_1({y_0}^3 - y_0 y_1 + y_2)}{{y_0}^2 - y_1}
\end{aligned}
\tag{17}
$$

and its complexity according to Definition 4 is equal to 4.

4. Birational transformations of higher order

We show on the example that permutations of order k give rise to birational transformations of order k. We construct just a simple example of birational transformation of order 3 on an open dense subset of $\mathbb{R}^3[x_0, x_1, x_2]$. Examples of higher order can be obtained by an obvious modification.

Let us use the same polynomials P_i and Q_i, $i = 0, 1, 2$ as in the previous section. They give the same polynomials y_0, y_1, y_2. Now, we use the permutation $p \in S_3$, such that $p(0,1,2) = (1,2,0)$. Hence, we have for $x_i = y_i \circ p$

$$
\begin{aligned}
x_0 &= b_1,\\
x_1 &= {b_1}^2 + b_1 b_2,\\
x_2 &= {b_1}^2 b_2 + b_0 b_1 b_2.
\end{aligned}
\tag{18}
$$

We have the formulas for $b_i(y_0, y_1, y_2)$ again in the form (13). After the substitution of formulas (13) into formulas (18), we obtain formulas for $x_i(y_0, y_1, y_2)$ in the form

$$
\begin{aligned}
x_0 &= -\frac{{y_0}^2 - y_1}{y_0}\,,\\
x_1 &= \frac{2{y_0}^4 - 3{y_0}^2 y_1 + y_0 y_2 + {y_1}^2}{{y_0}^2}\,,\\
x_2 &= \frac{y_1({y_0}^3 - y_0 y_1 + y_2)}{{y_0}^2}\,.
\end{aligned}
\tag{19}
$$

By the iteration of this mapping, we easily verify that the formulas for the second iteration are

$$
\begin{aligned}
z_0 &= -\frac{{y_0}^3 - y_0 y_1 + y_2}{{y_0}^2 - y_1}\,,\\
z_1 &= \frac{y_2({y_0}^3 - y_0 y_1 + y_2)}{({y_0}^2 - y_1)^2}\,,\\
z_2 &= \frac{(2{y_0}^4 - 3{y_0}^2 y_1 + y_0 y_2 + {y_1}^2)({y_0}^3 - y_0 y_1 + y_2)}{({y_0}^2 - y_1)^2}
\end{aligned}
\tag{20}
$$

and the third iteration is the identity. In this way, any nontrivial permutation of order k on a set $\{0, 1, \ldots, n-1\}$ such that $k \leq n-1$, gives rise to a birational transformation of order k on an open dense subset of $\mathbb{R}^n[x_0, \ldots, x_{n-1}]$.

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