Lonely points revisited

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Abstract. In our previous paper, we introduced the notion of a lonely point, due to P. Simon. A point $p \in X$ is lonely if it is a limit point of a countable dense-in-itself set, it is not a limit point of a countable discrete set and all countable sets whose limit point it is form a filter. We use the space $\mathcal{G}_\omega$ from a paper of A. Dow, A.V. Gubbi and A. Szymański [*Rigid Stone spaces within ZFC*, Proc. Amer. Math. Soc. **102** (1988), no. 3, 745–748] to construct lonely points in ω^*. This answers the question of P. Simon posed in our paper *Lonely points in* ω^*, Topology Appl. **155** (2008), no. 16, 1766–1771.

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1. Introduction

1.1 Definition. A *topological type* in a space X is a subset $T \subseteq X$ which is invariant under homeomorphisms.

An example of a topological type are discrete points in a space X. Another more interesting type is given in the following definition. The first part is due to W. Rudin ([10]), the second to K. Kunen ([8]).

1.2 Definition (Rudin, Kunen)**.** A point $x \in X$ is a *P-point* if a countable intersection of neighbourhoods of x is again a neighbourhood of x. It is a *weak P-point* if it is not a limit point of a countable subset of X.

Clearly any isolated point is a P-point, and a P-point is a weak P-point. However none of the implications can be reversed.

If a space contains two distinct topological types, then it is not homogeneous. The motivation for finding topological types in ω^* was given by the following surprising result of Z. Frolík ([5], [4]):

1.3 Theorem (Frolík)**.** *ω^* is not homogeneous.*

His proof used a clever combinatorial argument but it gave no intrinsically topological reason for the non-homogeneity of ω^*. This motivated the question whether one can find a "topologically defined" topological type — an "honest"

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proof of nonhomogeneity. Under CH, this was answered already by W. Rudin in [10] where he proved that P-points exist in ω^*. However in ZFC the question remained open for some twenty years.

In his seminal paper [8], K. Kunen proved in ZFC that ω^* contains a weak P-point:

1.4 Theorem (Kunen). *ω^* contains a weak P-point.*

Since it obviously contains non weak P-points, this is an "honest" proof of nonhomogeneity. In [9], J. van Mill exploited the techniques of K. Kunen to prove, in ZFC, the existence of sixteen distinct topological types in ω^*! One of the types he introduced is given in the following theorem:

1.5 Theorem (van Mill). *There is a point $p \in \omega^*$ which is a limit point of a countable discrete set and the countable sets whose limit point it is form a filter.*

Proof: (Idea) Use Kunen's result to construct a weak P-point $p \in \omega^* \subseteq \beta\omega$. Now use a theorem of P. Simon (see Theorem 2.5) to embed $\beta\omega$ into ω^* as a weak P-set. Then the image of p via the embedding will be as required, since p clearly has the property in $\beta\omega$ and the embedding does not destroy it since the image of $\beta\omega$ is a weak P-set. $\square$

This motivated P. Simon to define the following notion, which we have called a lonely point in [12]. We want essentially the same type of point as in the above theorem only replacing the countable discrete set whose limit point it is by a crowded set:

1.6 Definition. A point $p \in X$ is a *lonely point* provided:

(i) p is ω-discretely untouchable, i.e. not a limit point of a countable discrete set,

(ii) p is a limit point of a countable crowded (i.e. without isolated points) set and

(iii) the countable sets whose limit point p is form a filter.

In the paper we were able to show that lonely points exist in some open dense subspace of ω^*. Here we prove that they actually exist in ω^*:

1.7 Theorem. *ω^* contains a lonely point.*

The idea is to construct a countable, perfectly disconnected space X with an $\aleph_0$-bounded remainder and then embed it as a weak P-set into ω^*. Any point of X will then be a lonely point of βX and, since βX is a weak P-set in ω^*, also a lonely point of ω^*.

2. Basic definitions and theorems

2.1 Definition (Kunen). $F \subseteq X$ is a weak P-set of X if any countable $D \subseteq X$ disjoint from F has closure disjoint from F.

2.2 Observation. If $F \subseteq X$ is a weak P-set of X and $x \in F$ is a lonely point of F then it is also a lonely point of X.

2.3 Definition. A space X is *extremally disconnected* (or ED for short) if the closure of any open set is open.

The following is standard, see e.g. [3]:

2.4 Theorem. *If X is ED then so is βX.*

We shall also need the following theorem of P. Simon (see [11]):

2.5 Theorem (Simon)**.** *The Čech-Stone compactification of any T_3 ED space of weight $\leq 2^{\aleph_0}$ can be embedded into ω^* as a closed weak P-set.*

3. Irresolvable spaces

In this section, unless otherwise stated, we assume all spaces to be crowded (i.e. without isolated points). The following definitions were introduced in [1]:

3.1 Definition (van Douwen)**.** A crowded space X is *perfectly disconnected* if no point of X is a limit point of two disjoint subsets of X. It is irresolvable, if it contains no disjoint dense sets. It is open-hereditarily-irresolvable (OHI for short), provided each open subspace is irresolvable. A crowded space is *maximal regular* if each finer topology either contains an isolated point or is not regular.

Irresolvable spaces were constructed by E. Hewitt ([6]) and independently by M. Katětov ([7]). They were extensively studied in [1] where the following theorems may be found:

3.2 Theorem ([1, 1.7, 1.11])**.** *Maximal regular spaces are zero dimensional, ED and OHI.*

3.3 Theorem ([1, 1.4, 1.6])**.** *If A, B are disjoint crowded subspaces of a maximal regular space, then $\overline{A}$ and $\overline{B}$ are disjoint.*

3.4 Theorem ([1, 2.2])**.** *If X is ED and OHI and each nowhere dense subset of X is closed then X is perfectly disconnected.*

The following theorem is not explicitly stated in van Douwen's paper, but its proof is essentially given in his Lemma 3.2 and Example 3.3.

3.5 Theorem (van Douwen)**.** *Any countable maximal regular space X contains an open perfectly disconnected subspace.*

Proof: For each $Z \subseteq X$ let

$$A_Z = \{x \in Z : x \text{ is a limit point of a relatively discrete subset of } Z\}.$$

Claim $A_Z \neq Z$ for each open subset Z of X.

Assume otherwise. Enumerate Z as $\langle x_n : n < \omega\rangle$. By induction construct pairwise disjoint, relatively discrete sets $\langle D_n : n < \omega\rangle$ such that:

(i) $\bigcup_{i<n} D_i \subseteq \overline{D_n}$ for all $n < \omega$ and
(ii) $x_n \in \overline{D_n}$ for $n < \omega$.

This will lead to a contradiction with the irresolvability of Z (by Theorem 3.2, X is OHI, so Z is irresolvable). Indeed, $\bigcup_{n<\omega} D_{2n}$ and $\bigcup_{n<\omega} D_{2n+1}$ are disjoint dense subsets of Z. To see that the construction can be carried out let $D_0 = \{x_0\}$ and assume we have constructed D_i for $i \leq n$. Let $Y = D_n \cup Z \setminus \overline{D_n}$. Since D_n is relatively discrete, Y is open. Since Z is regular and D_n is countable and relatively discrete, there is a pairwise disjoint collection of open sets $\{U_x : x \in D_n\}$ such that $x \in U_x \subseteq Y$. Since we assume $A_Z = Z$ we can choose for each $x \in D_n$ a relatively discrete set D_x such that $D_x \subseteq U_x$ and $x \in \overline{D_x} \setminus D_x$. Let $D'_{n+1} = \bigcup_{x\in D_n} D_x$. If x_{n+1} is a limit point of D'_{n+1} let $D_{n+1} = D'_{n+1}$, otherwise let $D_{n+1} = D'_{n+1} \cup \{x_{n+1}\}$. Then D_{n+1} is as required.

Claim int $A_X = \emptyset$.

For any clopen U, $A_X \cap U = A_U$. Since X is regular and countable, it is zero dimensional. Suppose U is clopen and $U \subseteq A_X$. By the previous claim $U \setminus A_U \neq \emptyset$ but then $U \setminus A_X \neq \emptyset$, which is a contradiction.

Claim A_X is nowhere dense.

Take any open $U \subseteq X$. Then $U \setminus A_X$ is dense in U, since int $A_X = \emptyset$. Since X is OHI (by Theorem 3.2) and U is irresolvable, A_X cannot be dense in U and so $U \not\subseteq \overline{A_X}$. Thus we have that int $\overline{A_X} = \emptyset$.

Claim If $A \subseteq X$ is nowhere dense then there is a discrete $D \subseteq A$ dense in A.

Let $D = \{x \in A : x \text{ is isolated in } A\}$. Since X is regular and countable D is relatively discrete. Since A is nowhere dense, D is discrete. Let $E = A \setminus \overline{D}$. Then E has no isolated points. Also $X \setminus E$ has no isolated points. By Theorem 3.3 E must be open which contradicts that A is nowhere dense.

Let $\vartheta = \{x \in X : x \text{ is not a limit point of a nowhere dense subset of } X\}$.

By the previous claim (and by the fact that each discrete subset of X is nowhere dense) we have that

$$\vartheta = \{x \in X : x \text{ is not a limit point of a discrete set}\}$$

Then $X \setminus \vartheta \subseteq A_X$ so $X \setminus \vartheta$ is nowhere dense, so int ϑ is nonempty. We finally show that int ϑ is perfectly disconnected. By the definition of ϑ any nowhere dense subset of int ϑ is closed. Now it remains to apply Theorem 3.4 remembering that by Theorem 3.2 int ϑ is ED and OHI (any open subspace of a maximal regular space is maximal regular). □

4. Proof of the main theorem

The following definition and theorem is taken from [2]:

4.1 Definition. Let $p \in \omega^*$ be a weak P-point. The space $\mathcal{G}_\omega$ is the space $\omega^{<\omega}$ of all finite sequences of natural numbers with $G \subseteq \omega^{<\omega}$ being open precisely when for each $\sigma \in G$ the set $\{n : \sigma^\frown n \in G\}$ is in p.

4.2 Theorem (Dow, Gubbi, Szymanski). *The remainder of $\mathcal{G}_\omega$ is $\aleph_0$-bounded. Moreover $\mathcal{G}_\omega$ is a T_2, zero dimensional, ED space.*

Notice that if a space X has an $\aleph_0$-bounded remainder, any finer topology also has an $\aleph_0$-bounded remainder:

4.3 Proposition. *If $(X,\tau)^*$ is a zero dimensional $\aleph_0$-bounded space and $\sigma \supseteq \tau$ is also zero dimensional, then $(X,\sigma)^*$ is $\aleph_0$-bounded.*

Proof: Note that any $p \in (X,\tau)^*$ corresponds to a closed subset of $(X,\sigma)^*$ (denoted by $[p]$). Now given $\{q_n : n<\omega\} \subseteq (X,\sigma)^*$ we can find $\{p_n : n<\omega\} \subseteq (X,\tau)^*$ such that $\{q_n : n<\omega\} \subseteq \bigcup\{[p_n] : n<\omega\}$. Since $(X,\tau)^*$ is $\aleph_0$-bounded, $\overline{\{p_n : n<\omega\}}^{\beta(X,\tau)} \cap X = \emptyset$ so also $\overline{\{q_n : n<\omega\}}^{\beta(X,\sigma)} \cap X = \emptyset$ which implies that $(X,\sigma)^*$ is $\aleph_0$-bounded. □

4.4 Theorem. *There is a countable, ED, perfectly disconnected space X with an $\aleph_0$-bounded remainder.*

Proof: Take the space $\mathcal{G}_\omega$ from Theorem 4.2, and refine the topology to a maximal regular topology. Then, by the previous proposition, this space still has an $\aleph_0$-bounded remainder and so does its open perfectly disconnected subspace given by Theorem 3.5. Let X be this subspace. □

4.5 Theorem. *ω^* contains a lonely point.*

Proof: Let X be the space from the previous theorem. Since it is crowded perfectly disconnected, each of its points is a lonely point of X. Since its remainder is $\aleph_0$-bounded, each of its points is also a lonely point of βX. Since it is ED, βX is also ED and since it is countable, βX has weight at most $2^{\aleph_0}$. Hence, by Theorem 2.5, βX can be embedded as a weak P-set into ω^* and each point of X will be a lonely point of ω^* (by Observation 2.2). □

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References

[1] van Douwen E.K., *Applications of maximal topologies*, Topology Appl. **51** (1993), 125–139.
[2] Dow A., Gubbi A.V., Szymański A., *Rigid Stone spaces within ZFC*, Proc. Amer. Math. Soc. **102** (1988), no. 3, 745–748.
[3] Engelking R., *General Topology*, Sigma Series in Pure Mathematics, Heldermann Verlag, Berlin, 1989.
[4] Frolík Z., *Nonhomogeneity of $\beta P - P$*, Comment. Math. Univ. Carolin. **8** (1967), 705–709.
[5] Frolík Z., *Sums of ultrafilters*, Bull. Amer. Math. Soc. **73** (1967), 87–91.
[6] Hewitt E., *A problem of set-theoretic topology*, Duke Math. J. **10** (19430), 309–333.

[7] Katětov M., *On topological spaces containing no disjoint dense subsets*, Mat. Sbornik N.S. **21(63)** (1947), 3–12.
[8] Kunen K., *Weak P-points in N^**, Colloq. Math. Soc. János Bolyai, 23, North-Holland, Amsterdam-New York, 1980, pp. 741–749.
[9] van Mill J., *Sixteen types in $\beta\omega - \omega$*, Topology Appl. **13** (1982), 43–57.
[10] Rudin W., *Homogeneity problems in the theory of Čech compactifications*, Duke Math. J. **23** (1956), 409–419.
[11] Simon P., *Applications of independent linked families*, Colloq. Math. Soc. János Bolyai, 41, North-Holland, Amsterdam, 1985, pp. 561–580.
[12] Verner J., *Lonely points in ω^**, Topology Appl. **155** (2008), no. 16, 1766–1771.

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