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Frame monomorphisms and a feature of the l-group of Baire functions on a topological space

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Dedicated to the 120th birthday anniversary of Eduard Čech.

Abstract. The "kernel functor" $W \xrightarrow{k}$ LFrm from the category W of archimedean lattice-ordered groups with distinguished weak unit onto LFrm, of Lindelöf completely regular frames, preserves and reflects monics. In W, monics are one-to-one, but not necessarily so in LFrm. An embedding $\varphi \in W$ for which $k\varphi$ is one-to-one is termed kernel-injective, or KI; these are the topic of this paper. The situation is contrasted with kernel-surjective and -preserving (KS and KP). The W-objects every embedding of which is KI are characterized; this identifies the LFrm-objects out of which every monic is one-to-one. The issue of when a W-map $G \xrightarrow{\varphi} \cdot$ is KI is reduced to when a related epicompletion of G is KI. The poset $EC(G)$ of epicompletions of G is reasonably well-understood; in particular, it has the functorial maximum denoted βG, and for $G = C(X)$, the Baire functions $B(X) \in EC(C(X))$. The main theorem is: $E \in EC(C(X))$ is KI iff $B(X) \overset{*}{\leq} E \overset{*}{\leq} \beta C(X)$ in the order of $EC(C(X))$. This further identifies in a concrete way many LFrm-monics which are/are not one-to-one.

Keywords: Baire functions, archimedean lattice-ordered group, Lindelöf frame, monomorphism

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1. Introduction

General references are, for l-groups [16] and [20], for frames [29], for category theory [28]. For present purposes, the most useful single reference is [9], which see for much of what now follows. (A considerable amount of [9] is reprise of work of Madden and Vermeer [30] and Madden [32]. See [9] for a careful discussion.)

W is the category whose objects are archimedean l-groups A with distinguished positive weak order unit e_A ($e_A \wedge |a| = 0$ implies $a = 0$), and a morphism $A \xrightarrow{\varphi} B$ is an l-group homomorphism with $\varphi(e_A) = e_B$. Prototypical examples are: for X a Tychonoff space, the l-group part of $C(X)$ the "ring of continuous functions" [23], with unit the constant function 1. ([24] discuss aspects of W and the more general connection with rings with identity.)

LFrm is the category with objects Lindelöf completely regular frames, with frame homomorphisms.

These two categories are connected by the adjunction $W \underset{c}{\overset{k}{\leftrightarrows}} \mathrm{LFrm}$ with k on the left, as we now describe following [9].

Let $A \in W$. If $A \xrightarrow{\varphi} B \in W$, then $\{a \mid \varphi(a) = 0\}$ is called a W-kernel of A. These are the convex sub-l-groups I of A for which A/I is archimedean and $e_A + I$ is a weak unit in A/I. (See [9] for further details.) The collection of these is denoted kA. For $S \subseteq A$, there is a least W-kernel containing S, denoted $\langle S \rangle_A$. After some work, one finds $kA \in \mathrm{LFrm}$, the operations being $\bigwedge I_i = \bigcap I_i$ and $\bigvee I_i = \langle \bigcup I_i \rangle_A$. Then, for $A \xrightarrow{\varphi} B \in W$, $kA \xrightarrow{k\varphi} kB$ is the function $k\varphi(I) = \langle \varphi(I) \rangle_B$.

The functor $W \xrightarrow{k} \mathrm{LFrm}$ results, and a right adjoint $W \underset{c}{\leftarrow} \mathrm{LFrm}$ is given (on objects) by $C(F) = \{\mathbb{R} \xrightarrow{f} F \mid f \in \mathrm{Frm}\} \in W$. A consequence of this (e.g. since k is full, faithful and dense) is that k preserves and reflects monics [28]. I.e.

1.1 Proposition. *φ is monic in W (which means 1-1) iff $k\varphi$ is monic in* LFrm *(which means "dense", i.e., $k\varphi(I) = 0$ implies $I = 0$).*

In LFrm, 1-1 implies dense, but not conversely.

We call $\varphi \in W$ kernel-injective (resp., -surjective, -preserving) if $k\varphi$ is 1-1 (resp., onto, 1-1 and onto), and denote this KI (resp., KS, KP).

Now, φ KI implies φ is one-to-one (since $k\varphi$ one-to-one implies φ monic, i.e., one-to-one). And, one may study KS by just studying one-to-one KS maps (since any $\varphi \in W$ has the factorization $\varphi = is$, s onto and i one-to-one, and φ is KS iff i is).

Consequently, all remarks about $A \xrightarrow{\varphi} B$, being or not, KI, KS, or KP can be addressed to one-to-one φ, which we may indicate just as $A \overset{\varphi}{\leq} B$, or $A \leq B$.

We shall frequently avail ourselves of the frame-theoretic adjoint situations [29] where, for $A \xrightarrow{\varphi} B$, we have $kA \underset{t}{\overset{k\varphi}{\leftrightarrows}} kB$, where $t = t(\varphi)$ stands for "trace": $t(J) \equiv \varphi^{-1}J$ (and if $A \overset{\varphi}{\leq} B$, $t(J) = J \cap A$). (t would frequently be denoted $(k\varphi)^*$.) We have then these descriptions.

1.2 Proposition. *Let $A \overset{\varphi}{\leq} B \in W$ with adjoint t. These are equivalent.*

(a) *φ is KI (resp., KS, KP).*
(b) *$k\varphi$ is 1-1 (resp., onto, 1-1 and onto).*
(c) *t is onto (resp., 1-1, onto and 1-1).*

We note further connections with categorical monic (m with $mf = mg \Rightarrow f = g$), epic ($e$ with $fe = ge \Rightarrow f = g$), and essential monic (monic m with fm monic $\Rightarrow f$ monic). (See [28].) In W: as noted earlier, monic means 1-1. An essential monic is, of course, a $A \leq B$ with $0 \neq J \in kB$ implying $A \cap J \neq 0$. It is not hard to see that this is equivalent to essentiality in all abelian l-groups, as described in [16]. Epic is a much more complicated story. This theory is described in [3] and [8] (also [10]), underlies later developments here but the technicalities can be avoided.

The following is true in much more general contexts (as the proofs will show), but we ignore this.

1.3 Proposition. *In* W

(a) *if* φ *is KI then* φ *is monic,*
(b) *if* φ *is KS then* φ *is epic,*
(c) *if* φ *is KS and monic then* φ *is essential monic,*
(d) *if* φ *is KP then* φ *is essential monic, and epic.*

PROOF: In the following, write $A \xrightarrow{\varphi} B$, and t is the adjoint of φ (the "trace").

(a) We noted this earlier, from features of the functor k. However, again: If φ is not 1-1, there is $a > 0$ with $\varphi(a) = 0$. Then $\langle a\rangle_A \neq 0 = \langle 0\rangle_A$ but $\langle\langle\varphi(a)\rangle_A\rangle_B = \langle\langle 0\rangle_A\rangle_B = 0$, so $k\varphi$ is not 1-1.

(b) For any $B \xrightarrow{fj} C$, $j = 1, 2$, $kf_1 = kf_2$ implies $f_1 = f_2$ by uniqueness of (1-1)$\circ$ onto factorization. If $f_1\varphi = f_2\varphi$, then $\varphi^{-1}kf_1 = \varphi^{-1}kf_2$, and if φ is KS, then t is 1-1 and then $kf_1 = kf_2$.

(c) $A \overset{\varphi}{\leq} B$ essential means $0 \neq J \in kB$ implies $0 \neq J \cap A$. This is so when t is one-to-one.

(d) Follows from (a), (b) and (c). □

We note a gross distinction between, on the one hand KP and KS, and on the other KI. Let $A \in W$. Then, $A \leq C(kA)$ is the maximum KP extension [9]. And, there is the maximum KS extension $A \leq aA$ [6]. While, for any cardinal m there are KI extensions $A \leq B$ with $m \leq |B|$ (4.4 below). Further details appear in Section 3 below.

Most what we have hinted at in this introduction seems to be required to get to the Main Result about $B(X)$ stated in the abstract.

2. Yosida representation and spaces with filter

Our proofs will be couched in terms of the classical Yosida representation $A \leq D(YA)$ [36], augmented to functoriality [24], further augmented to the representation of kernels and the frames kA (thus of any $F \in$ LFrm) [7]. We now explain this.

The extended real line is $\mathbb{R} \cup \{\pm\infty\} = [-\infty, +\infty]$ with the obvious order and topology. For X a Tychonoff space, $D(X) \equiv \{f \in C(X, [-\infty, +\infty]) \mid f^{-1}\mathbb{R}$ dense$\}$. In the pointwise order, this is a lattice, is closed under $f \mapsto nf$ $\forall n \in \mathbb{Z}$, and the constant function 1 has the property $|f| \wedge 1 = 0$ implies $f = 0$. But, pointwise addition is only partly defined (likewise, multiplication). We write $A \leq D(X)$ if $A \subseteq D(X)$, $1 \in A$, and A is closed under the full and partial operations of $D(X)$ requisite to making $A \in W$ (with $e_A = 1$). Comp denotes the category of compact Hausdorff spaces with continuous maps.

2.1 Theorem. (a) *For $A \in W$, there is $YA \in \mathrm{Comp}$ and a 1-1 map $A \xrightarrow{\eta_A} D(YA)$ such that $\eta_A(A) \leq D(YA)$, $A \xrightarrow{\eta_A} \eta_A(A)$ is a W-isomorphism, and $\eta_A(A)$ separates points of YA. For those attributes, YA is unique (up to homeomorphism).*

(b) *For $A \xrightarrow{\varphi} B \in W$, there is unique $YA \xleftarrow{Y\varphi} YB \in \mathrm{Comp}$ for which, for each $a \in A$ $\eta_B(\varphi(a)) = \eta_A(a) \circ Y\varphi$. φ is 1-1 (monic in W) iff $Y\varphi$ is onto (epic in Comp).*

(c) *$W \xrightarrow{Y} \mathrm{Comp}$ is a functor called the Yosida functor.*

We identify each A with its $\eta_A(A)$ and just write $A \leq D(YA)$. Let $A^{-1}\mathbb{R} \equiv \{a^{-1}\mathbb{R} \mid a \in A\}$; this a filter base of dense cozero-sets in YA. Note the following consequences of 2.1(b) regarding a $A \xrightarrow{\varphi} B$ with its $YA \xleftarrow{Y\varphi} YB$: For $S \in A^{-1}\mathbb{R}$, we have $(Y\varphi)^{-1}S \in B^{-1}\mathbb{R}$. This has spawned the following, first mentioned in [3], and analyzed further in [7], [8], [12], [14].

The specific statements 2.3 and 2.4 are re-phrasing of results in [13].

2.2 Definition. The category SpFi (of Spaces with Filter) has: objects $(X, \mathscr{H})$, $X \in \mathrm{Comp}$ and $\mathscr{H}$ a filter base of dense open sets; and morphisms $(X, \mathscr{H}) \xleftarrow{f} (Y, \mathscr{L})$ with $f \in \mathrm{Comp}$ with the feature that $f^{-1}F \in \mathscr{L}$ for each $F \in \mathscr{H}$.

We usually write just X for $(X, \mathscr{H})$, the filter base being understood.

For $X = (X, \mathscr{H}) \in \mathrm{SpFi}$, and closed $S \subseteq X$, let $S \cap \mathscr{H} = \{S \cap F \mid F \in \mathscr{H}\}$. If $S \cap \mathscr{H}$ consists of dense subsets of S, then $(S, S \cap \mathscr{H}) \in \mathrm{SpFi}$, the inclusion $S \subseteq X$ yields a SpFi-morphism, and then (abusing language) we say "S is a SpFi-subspace of X".

$\mathrm{sub}\, X$ denotes the collection of such subspaces.

$L\,\mathrm{SpFi}$ is the full subcategory of SpFi of objects $(X, \mathscr{H})$ for which $\mathscr{H}$ consists of cozero-sets.

2.3 Proposition. *Let $X \in \mathrm{SpFi}$.*

(a) *If $\{S_i\}_I \subseteq \mathrm{sub}\, X$ (I arbitrary), then $\overline{\bigcup S_i} \in \mathrm{sub}\, X$.*

(b) *For K any closed subset of X there is a largest $K' \in \mathrm{sub}\, X$ with $K' \subseteq K$.*

In the following "Frm" is the category of completely regular frames and co-Frm is Frm, with objects oppositely ordered.

2.4 Theorem. (a) *For $X \in \mathrm{SpFi}$, $\mathrm{sub}\, X$ ordered by inclusion k a completely regular co-frame, with $\bigvee S_i = \overline{\bigcup S_i}$ and $\bigwedge S_i = (\bigcap S_i)'$. (I.e. reversing the order gives a frame (or equivalently) $\mathrm{cosub}\, X = \{X - S \mid S \in \mathrm{sub}\, X\}$ ordered by inclusion is a frame.)*

(b) *If $X \xleftarrow{\tau} Y \in \mathrm{SpFi}$, then a co-frame morphism $\mathrm{sub}\, X \xrightarrow{\mathrm{sub}\,\tau} \mathrm{sub}\, Y$ is given by $(\mathrm{sub}\,\tau)(S) = (\tau^{-1}S)'$. Its "co-frame adjoint" τ_0 is just forward image: $\tau_0(S) = \tau(S)$.*

(c) *"sub" is a functor and there is an adjunction F as $\mathrm{SpFi} \underset{F}{\overset{\mathrm{sub}}{\leftrightarrows}} \text{co-Frm}$ with sub on the right. In consequence of which*

(d) *Every completely regular frame F, when oppositely ordered, is isomorphic to a* sub X*, and F is Lindelöf iff $X \in L\,\mathrm{SpFi}$.*

One may interpret 2.4(d) as putting points back into point-free topology.

We return to the association described before 2.2: For $A \in W$, we have $(YA, A^{-1}\mathbb{R}) \in L\,\mathrm{SpFi}$, which we now denote SYA and for $A \xrightarrow{\varphi} B \in W$, we have $(YA, A^{-1}\mathbb{R}) \xleftarrow{Y\varphi} (YB, B^{-1}\mathbb{R}) \in L\,\mathrm{SpFi}$, which we now denote $SYA \xleftarrow{SY\varphi} SYB$. Of course

2.5 Theorem ([7], [8])**.** *This defines a faithful contravariant functor $W \xrightarrow{SY} L\,\mathrm{SpFi}$.*

It is noted in [8, 4.7] that this functor is not dense ("onto objects"). We do not know a characterization of objects in the range. (One is purported in [8, 4.8], but it alludes to an erroneous statement elsewhere.)

Now consider the composition $W \xrightarrow{SY} L\,\mathrm{SpFi} \xrightarrow{\mathrm{cosub}} \mathrm{LFrm}$. From the following, this is the kernel functor.

For $A \in W$, as $A \leq D(YA)$, let $Za = \{y \in YA \mid a(y) = 0\}$.

The following is from [7]. In this set-up, we have: For $A \in W$, the associated $kA \in \mathrm{LFrm}$, $SYA \in L\,\mathrm{SpFi}$, and $\mathrm{sub}\, SYA \in$ co-LFrm. For $A \xrightarrow{\varphi} B \in W$, the associated $kA \xrightarrow{k\varphi} kB \in \mathrm{LFrm}$, and $SYA \xrightarrow{SY\varphi} SYB \in L\,\mathrm{SpFi}$.

2.6 Theorem. (a) *Suppose $A \in W$. A frame/co-frame isomorphism is given by mutually inverse functions*

$$\begin{tikzcd} kA \arrow[r, shift left, "Z_A"] & \mathrm{sub}\, SYA \arrow[l, shift left, "Z_A^{-1}"] \end{tikzcd}$$

which are $Z_A I \equiv \bigcap\{Za \mid a \in I\}$, and $Z_A^{-1} S \equiv \{a \mid Za \supseteq S\}$.

(b) *Suppose $A \xrightarrow{\varphi} B \in W$. The following commutes.*

$$\begin{tikzcd} kA \arrow[r, "k\varphi"] \arrow[d, "Z_A"'] & kB \arrow[d, "Z_B"] \\ \mathrm{sub}\, SYA \arrow[r, "\mathrm{sub}\, SY\varphi"'] & \mathrm{sub}\, SYB \end{tikzcd}$$

The interpretation of (b) is: Since Z_A and Z_B are "anti-isomorphisms", $\mathrm{sub}\, SY\varphi$ "represents" $k\varphi$, and in the same way, the adjoint $t(\varphi)$ (trace) of φ corresponds to and is represented by the adjoint of $\mathrm{sub}\, SY\varphi$, denoted τ_0 (forward image).

This association between W-kernels of A and the subsets of YA generalizes all of the successively more general descriptions for kernels of homomorphisms $A_1 \to A_2$ (1) $A_i = C(X_i)$, $X_i \in$ Comp (folk? Gelfand-Kolmogorov?), (2) $A_i = C(X_i)$, X_i Tychonoff [34], (3) A_i reduced archimedean f-algebras with identity

[25]. In each of these situations "homomorphism" usually meant "ring or ring and lattice homomorphism preserving identity". These are W-homomorphisms, and [24] shows that any W-homomorphism between these objects is one of them.

The following sums up the situation, for easy reference to details to be used later.

2.7 Corollary. *For $A \xrightarrow{\varphi} B \in W$, the following are equivalent.*

(a) *φ is KI (resp., KS, KP).*
(b) *$k\varphi$ is 1-1 (resp., onto, 1-1 and onto).*
(c) *$\operatorname{sub}\tau$ is 1-1 (resp., onto, 1-1 and onto).*
(d) *$t(\varphi)$ is onto (resp., 1-1, onto and 1-1).*
(e) *τ_0 is onto (resp., 1-1, onto and 1-1).*

We now provide even more detailed interpretation of 2.7 for KI, to make clearer how the proofs in Section 5 work. In fact, the following lemma is considerably more general than for KI embeddings in W, being phrased in forms of $L\,\mathrm{SpFi}$ surjections τ in the opposite direction which have $\operatorname{sub}\tau$ 1-1.

For $(X, \mathscr{H}) \in L\,\mathrm{SpFi}$, $\mathscr{H}_\delta = \{\bigcap F_n \mid F_1, F_2, \dots \in \mathscr{H}\}$. The members of $\mathscr{H}_\delta$ are dense by the Baire Category Theorem, and Lindelöf (see [22]). Recall the subspace operator $(\cdot)'$ from 2.3.

2.8 Lemma. *The following are equivalent for a surjection $X = (X, \mathscr{H}) \xleftarrow{\tau} (Y, \mathscr{L}) = Y$ in $L\,\mathrm{SpFi}$, with associated co-frame map $\operatorname{sub} X \xrightarrow{\operatorname{sub}\tau} \operatorname{sub} Y$ given as $(\operatorname{sub}\tau)(S) = (\tau^{-1}S)'$.*

(1) *$\operatorname{sub}\tau$ is 1-1 (or, its adjoint is onto).*
(1′) *$(\tau^{-1}S)' = \emptyset$ implies $S = \emptyset$ ($S \in \operatorname{sub} X$).*
(2) *For all $S \in \operatorname{sub} X$ and $G \in \mathscr{H}_\delta$, $(\tau G) \cap S$ is dense in S.*
(3) *For all $\emptyset \neq S \in \operatorname{sub} X$ and $G \in \mathscr{H}_\delta$, $(\tau G) \cap S \neq \emptyset$.*
(3′) *If $S \in \operatorname{sub} X$ and $G \in \mathscr{H}_\delta$ have $(\tau G) \cap S = \emptyset$, then $S = \emptyset$.*

PROOF: (1) $\Leftrightarrow$ (1′) because the frames are regular; see [15].

(3) and (3′) are contrapositive.

(1′) $\Rightarrow$ (3). Suppose $\emptyset \neq S$ and $G \in \mathscr{H}_\delta$. By (1′), $(\tau^{-1}S)' \neq \emptyset$ so $(\tau^{-1}S) \cap G \neq \emptyset$. Thus, $\emptyset \neq \tau((\tau^{-1}S) \cap G) \subseteq \tau\tau^{-1}S \cap \tau G = S \cap \tau(G)$.

(3) $\Rightarrow$ (2). Suppose (2) false: we have a situation $(\tau G) \cap S$ not dense in S. Then there is a regular closed U with $U \cap S \neq \emptyset$ and $((\tau G) \cap S) \cap U = \emptyset$. Now $U \in \operatorname{sub} X$, and because U is regular closed, we have $S_0 = U \cap S \in \operatorname{sub} X$, with $S_0 \cap \tau G = \emptyset$ and (3) fails.

(2) $\Rightarrow$ (1). If (2) holds, we shall show $(*)$ $\emptyset \neq S \in \operatorname{sub} X$ implies $\tau((\tau^{-1}S)') = S$ (i.e., for the adjoint τ_0 to $\operatorname{sub}\tau$, $\tau_0 \circ \operatorname{sub}\tau$ is the identity, which means (1)).

For such S: Fix $x \in S$. For any $G \in \mathscr{H}_\delta$, we have $T(G) \equiv \overline{G \cap \tau^{-1}S} \cap \tau^{-1}x \neq \emptyset$. (If this set is $\emptyset$, there is open $U \supseteq \tau^{-1}x$ with $U \cap (G \cap \tau^{-1}S) = \emptyset$. Then, $V = X - \tau(Y - U)$ is open, $x \in V$, and $V \cap S \cap \tau G = \emptyset$; this contradicts (2).)

The family of compact sets $T(G)$ has the finite intersection property (indeed, the countable i.p.). Thus, $\emptyset \neq \bigcap_G T(G) = (\tau^{-1}S)' \cap \tau^{-1}x$.

This shows $(*)$. $\square$

3. KP, KS, and absolute KI

The point of this section is to compare the properties that a W-monic might have: KP, KS, and KI. We call $A \in W$ KS-complete (resp., absolutely KI) if any $A \overset{\varphi}{\leq} B$ which is KS is an isomorphism (resp., any $A \overset{\varphi}{\leq} B$ is KI); these A shall be characterized, and the results used later.

Some preliminaries are needed.

Suppose $X \in$ Comp and $\mathscr{H}$ is a filter base of dense sets in X. We denote by $C[\mathscr{H}]$ the direct limit in W of the system $\{C(F) \mid F \in \mathscr{H}\}$ (the bonding maps being, for $F_1 \supseteq F_2$, the restriction $C(F_1) \ni f \mapsto f \mid F_2 \in C(F_2)$). Then, $C[\mathscr{H}] \in W$ and $YC[\mathscr{H}]$ is the inverse limit in Comp of the system $\{\beta F \mid F \in \mathscr{H}\}$ — this β being Čech-Stone compactification (the bonding maps being, for $F_1 \supseteq F_2$, the natural surjection $\beta F_1 \twoheadleftarrow \beta F_2$); we denote $YC[\mathscr{H}]$ as $\beta[\mathscr{H}]$.

If $\mathscr{H}_1 \subseteq \mathscr{H}_2$, we have $C[\mathscr{H}_1] \leq C[\mathscr{H}_2]$ in W, and $\beta[\mathscr{H}_1] \twoheadleftarrow \beta[\mathscr{H}_2]$ in Comp. For any such $\mathscr{H}$, if $\mathscr{H}_\delta \equiv \{\bigcap F_n \mid F_1, F_2, \dots \in \mathscr{H}\}$ consists of dense sets, then $C[\mathscr{H}] \leq C[\mathscr{H}_\delta]$.

An important special case is: Let $\mathscr{C}(X)$ be the filter base of all dense cozero-sets of X. The spaces $\beta[\mathscr{C}(X)]$ and $\beta[\mathscr{C}(X)_\delta]$ are the same, called the quasi-F cover of X, denoted QFX. A space Y is called quasi-F (QF) if each $F \in \mathscr{C}(Y)$ is C^*-embedded; this is equivalent to $D(Y) \in W$ [26].

The natural surjection $X \overset{\alpha}{\twoheadleftarrow}$ QFX is irreducible, the space QFX is QF and whenever $X \overset{\beta}{\twoheadleftarrow} Y$ is irreducible with Y QF, there is δ with $\alpha\delta = \beta$. We have $C[\mathscr{C}(X)_\delta] = D(\text{QFX})$. All this originates in [21] and [37], is explained further in [27], is exploited and generalized in [14], and explained "framically" in [31] and [33] (see 3.6(e) below).

For $A \in W$, using $YA = X$ above, we have $A \leq C[A^{-1}\mathbb{R}]$, and if $A^{-1}\mathbb{R} \subseteq \mathscr{H}$, then $A \leq C[\mathscr{H}]$ is essential monic in W. The cases of $\mathscr{H} = A^{-1}\mathbb{R}_\delta$ and $\mathscr{C}(YA)_\delta$ are to the present point. The extension $A \leq C[A^{-1}\mathbb{R}_\delta]$ originates in [1], has come to be called $A \leq c^3A$ and is the "same" as $A \overset{\mu_A}{\leq} C(kA)$ (noted in [9], inter alia). We label the $A \leq D(\text{QF}YA)$ as $A \overset{\alpha_A}{\leq} aA$. Evidently, $A \leq c^3A \leq aA$.

As advertised at the end of Section 1 we have the following quite parallel facts. Further remarks on the "parallelism", etc., are made in 3.6 below.

3.1 Theorem. *Let $A \overset{\varphi}{\leq} B \in W$.*

(a) ([7, 4.13] and [9, 2.3.3]) *φ is KP iff there is monic $B \xrightarrow{\psi} c^3A$ with $\psi\varphi = \mu_A$.*

(b) ([14, 2.19]) *$A \leq aA$ is KS, and if φ is KS, then there is monic $B \xrightarrow{\psi} aA$ with $\psi\varphi = \alpha_A$.*

(If φ is KS, it is epic (1.4), and the converse in 3.1(b) fails just because $\psi\varphi = \alpha_A$ does not imply φ epic. Perhaps this converse holds if φ is assumed epic.)

3.2 Corollary. *A is KS complete iff $A = aA$, i.e., YA is QF and $A = D(YA)$.*

Below, we keep in mind features of the adjunction $W \underset{c}{\overset{k}{\leftrightarrows}} \text{LFrm} : A \overset{\mu_A}{\leq} C(kA) = c^3A$ is essential monic. Any $L_1 \xrightarrow{f} L_2$ in LFrm has $C(L_1) \xrightarrow{Cf} C(L_2)$ with $kCf = f$. For $A \xrightarrow{\varphi} B \in W$, and any factorization $k\varphi : kA \xrightarrow{f} F \xrightarrow{g} kB$, there is the unique factorization $Ck\varphi = C(kA) \xrightarrow{Cf} C(F) \xrightarrow{Cg} C(kB)$, and thus a unique factorization $\varphi = A \xrightarrow{\alpha} D \xrightarrow{\beta} B$ with $k\alpha = f$ and $k\beta = g$ obtained with $D \equiv Cf(\mu_A(A)) \leq C(F)$ and α, β the obvious modifications of Cf and Cg.

3.3 Lemma. *Any $A \overset{\varphi}{\leq} B$ in W has the unique factorization $\varphi = \beta\alpha$ with α epic and KS, β monic and KI.*

Proof: Consider the (1-1)$\circ$onto factorization $k\varphi = is$, then take $\varphi = \beta\alpha$ with $k\alpha = s$, $k\beta = i$. □

3.4 Theorem. (a) *If A is KS complete, then A is absolutely KI.*
(b) *A is absolutely KI iff $c^3A = aA$.*
(c) *$F \in$ LFrm has every monic $F \to \cdot$ 1-1 iff $F = kD(X)$ for QF $X \in$ Comp.*

Proof: (a) Use 3.3 supposing A is KS-complete: The α in 3.3 is an isomorphism.

(b) From 3.1(a), $A \overset{\varphi}{\leq} B$ is KI iff the $c^3A \overset{\overline{\varphi}}{\leq} c^3B$ is KI. If $c^3A = aA$, then c^3A is KS-complete (by 3.2), so any $c^3A \leq H$ is KI (by (a)), so any $A \leq B$ is KI.

(c) Evidently, F has all monics $F \to \cdot$ 1-1 iff $C(F)$ is absolutely KI (since $kC(F) = F$), which means $c^3C(F) = aC(F)$ by (b). But $c^3C(F) = C(F)$, so that means $C(F) = aC(F)$, which means $C(F) = D(YC(F))$, with $YC(F)$ QF. □

3.5 Corollary. *For Tychonoff X, the following are equivalent.*

(1) *$C(X)$ is KS-complete.*
(2) *$C(X)$ is absolutely KI.*
(3) *X is an almost P-space (no proper dense cozeros).*

Proof: By 3.2., (1) means $C(X) = aC(X)$. Clearly, (3) implies that, and conversely, if $\operatorname{coz} f$ is dense and proper, then $1/|f| \in aC(X) - C(X)$.
Now, $C(X) = c^3C(X)$ always, so, 3.4(b) and the previous gives (2) $\Leftrightarrow$ (3). □

We shall mention 3.5 again in Section 5.

3.6 Remarks. We conclude this section with a number of inter-connected remarks about 3.1.

(a) 3.1(a) was announced in [7, 4.13(c)], without details of proof; the proof envisioned there was via the apparatus surrounding $W \xrightarrow{SY} L\,\text{SpFi}$ (especially 2.6), which apparatus is sketched out there. A full proof of 3.1(a), purely in Frames, appears in [9].

(b) 3.1(b) can be found in [6], but a reading of the proofs is required: what is asserted is a property called "akd", and what 3.1(b) asserts would be called "Wkd". The later paper [14, p. 13] recognizes 3.1(b). The proofs in [6] and [14] are a mixture of topology and l-group theory.

A frame-theoretic proof of 3.1(b) would be desirable.

(c) The paper [14] concerns exactly extensions in W (and in fact in archimedean l-groups (without specified unit)) which are "m-KP", i.e., preserve the frames of "m-kernels", for regular m with $\omega_0 \leq m \leq \infty$. (An m-kernel in A is an ideal closed under existing suprema of size $< m$, these are all W-kernels of m-complete W-maps.) The case $m = \omega_1$ concerns us here; let us call the ω_1-kernels "σW-kernels", and the ω_1-KP property of extensions "σ-KP". [14, 3.10] shows σ-KP $=$ σ-KS (contrasting with ω-KP $=$ KP $\neq$ KS), and the present $A \overset{\alpha_A}{\leq} aA$ has the feature (5.1): $A \overset{\varphi}{\leq} B$ is σ-KP iff there is monic $B \xrightarrow{\psi} aA$ with $\psi g = \alpha_A$. In fact, this is shown for m-KP with aA replaced by $D(X_m)$, X_m denoting the QF_m-cover of YA. The proofs are in the terms of the adaptation to m-kernels and "m-SpFi sets" of the apparatus $W \xrightarrow{SY} L\,\mathrm{SpFi}$; it is not clear if this adaptation actually generalizes SY. Again, a purely frame-theoretic proof of all this would be desirable.

(d) It is shown in [10] that in W, the class of $A = c^3A$ comprises the least essentially monoreflective subcategory. This can be shown to follow from 3.1(a).

Analogously, consider $W_\sigma \equiv$ the category with all W-objects, but only σ-homomorphisms (countably complete homomorphisms). It can be shown that the class of all $A = aA$ is the smallest essentially reflective subcategory of W_σ, that for any A, $\sigma kA = \sigma kaA$ ($\sigma(\cdot)$ denoting the frame of σ-kernels); further $\sigma kaA = kaA$ appears in [5], and $aA = C(\sigma kA)$ follows. All of this is true replacing the ω_1 implicit in σ by ∞, where $\infty k(\cdot)$ is the Boolean algebra of polars, the correspondent of aA is Conrad's essential completion, and X_∞ is the ED cover of X. Whether ω_1 can be replaced by general m in all this is not clear. Writing all this down carefully would be desirable in whatever terms; also, we would like to see purely frame-theoretic proofs copying [9] as far as possible.

Also, the condition $c^3A = aA$ is equivalent to: Any W-homomorphism $A \to \cdot$ is a σ-homomorphism [18]. Of course, this is not a surprise granted "$aA = C(\sigma kA)$", but the proof of that has not been written down.

(e) Finally, a propos of 3.4(c), we ask: What is an intrinsic description of the frames $F = kD(X)$, X QF, equivalently (via (d) above), the frames σkA? According to Madden [31] and Molitor [33], these F have the localic description: F^{op} has exactly the form $lX \equiv$ the minimum dense Lindelöf sublocale of some $X \in$ Comp and the cover $\mathrm{QFX} = \beta(lX)$, β being localic Čech-Stone compactification; and this is shown for "$\mathrm{QF}_m X = \beta(l_m X)$", $l_m(\cdot)$ being minimum dense m-Lindelöf sublocale.

4. KI epicompletions

$E \in W$ is called epicomplete if the only $E \to \cdot$ which are monic and epic are isomorphisms. From [4], [5], and [30]: Each $A \in W$ has an epicomplete monoreflection $A \overset{\beta_A}{\leq} \beta A$. If E is epicomplete and $E \twoheadrightarrow A$ is a surjection, then A is epicomplete ("epicompleteness is H-closed"). (The notation "β" was chosen because of the analogy with Čech-Stone compactification.)

This is all we need for the moment. More detail is required below, and yet more in Section 5.

4.1 Theorem. *Each $A \overset{\beta_A}{\leq} \beta A$ is KI.*

Proof: Recall from Section 1 that $k\beta_A(P) = \langle P\rangle_{\beta A}$ $(P \in kA)$. The main step in the proof is

$$(*) \qquad \langle P\rangle_{\beta A} = k\overline{\psi},$$

for the $\overline{\psi}$ in the diagram

$$(1) \qquad \begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\psi"'] & \beta A \arrow[d, dashed, "\overline{\psi}"] & (\beta \text{ is } \beta_A) \\ A/P \arrow[r, "\beta'"'] & \beta(A/P) & (\beta' \text{ is } \beta_{A/P}) \end{tikzcd}$$

where ψ is the quotient, and $\overline{\psi}$ exists by reflectivity.

In $(*)$, $\subseteq$ is clear. For the reverse, diagram (1) is expanded to the following "completely commutative" diagram.

$$(2) \qquad \begin{tikzcd} A \arrow[rr, "\beta"] \arrow[dd, "\psi"'] & & \beta A \arrow[d, "f"] \\ & & \beta A/\langle P\rangle_{\beta A} \arrow[d, dashed, "g"] \\ A/P \arrow[r, "\beta'"'] \arrow[urr, dashed, "h"] & \beta(A/P) \arrow[r, equal] \arrow[ur, dashed, "k"'] & \beta A/k\beta\psi \end{tikzcd}$$

in which: the "=" is because $\overline{\psi}$ is epic, and thus the (1-1)$\circ$ onto factorization $\overline{\psi} : \beta A \overset{s}{\twoheadrightarrow} S \overset{i}{\to} \beta(A/P)$ has i epic and S epicomplete, so i is an isomorphism.

Then $\overline{\psi} = gf$ as shown, because $\langle P\rangle_{\beta A} \subseteq k\overline{\psi}$ and a homomorphism theorem.

Then there is the h by a homomorphism theorem, which is epic because $h\psi = f\beta$ is epic. Then, there is k by reflectivity, since $\beta A/\langle P\rangle_{\beta A}$ is epicomplete.

We have the equations $h = kgh$ and $\beta' = gk\beta'$ which imply kg is an identity (since h is epic), and this implies g is 1-1, which means $\langle P\rangle_{\beta A} = k\overline{\psi}$. So $(*)$ is proved.

Now, it is easy to see that (any) $G \leq H$ is KI iff $P \nsubseteq Q$ in kG implies $\langle P\rangle_H \nsubseteq \langle Q\rangle_H$.

To show $A \leq \beta A$ is KI, suppose $P \nsubseteq Q$ in kA. In the manner of diagram (1), we have

(3)
$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\sigma"'] & \beta A \arrow[d, "\overline{\sigma}"] \\
A/P \arrow[r] \arrow[d, "\delta"'] & \beta(A/P) \arrow[d, "\overline{\delta}"] \\
A/Q \arrow[r] & \beta(A/Q)
\end{tikzcd}$$

The desired $\langle P\rangle_{\beta A} \nsubseteq \langle Q\rangle_{\beta A}$ is exactly the statement "$\overline{\delta}$ is not 1-1" using $(*)$ above. Since δ is not 1-1 (i.e., $P \nsubseteq Q$), neither is $\overline{\delta}$. □

4.2 Remarks. (a) The proof above seems valid in considerable generality, proving roughly this: In a concrete category "with kernels" if $\mathscr{R}$ is a monoreflective subcategory for which $\mathscr{R} \ni R \twoheadrightarrow A$ implies $A \in \mathscr{R}$, then all reflections $A \xrightarrow{r_G} rA$ are KI.

(b) In W, since epicompleteness is monoreflective, for any monoreflective subcategory $\mathscr{R}$, each reflection map $A \to rA$ is an initial factor of $A \to \beta A$, thus is KI. Cf. (a) above.

(c) It appears that [30, 3.2] asserts 4.1 here, after sorting through several categorical equivalences.

In W, an epicompletion of A is a monic epic $A \leq E$, with E epicomplete. Most A have many such; see the next section.

We now reduce a general $A \leq B$ KI to a unique epicompletion $A \leq E$ which is KI. We need to know (see [5]) that: W-maps φ have unique factorization $\varphi = ie$, with e epic and i extremal monic ($A \leq B$ is extremal monic means A has no proper epic extension within B). And, if $E_1 \leq E_2$ is extremal monic with E_2 epicomplete, then so is E_1 (this follows from the monoreflectivity). Also, recall from [4] and [5] that in W, E is epicomplete iff YE is basically disconnected (BD) and $E = D(YA)$. Since BD implies QF, this means $E = aE$ and thus E is absolutely KI (3.2 and 3.4).

The following is entré to the main theorem of the paper, in Section 5, about the KI epicompletions of $C(X)$. Note the "simple proof" using 4.1, knowledge of epicompleteness in W, 3.1(b), and many features of W.

4.3 Proposition. *Let $A \xrightarrow{\varphi} B \in W$, consider $A \xrightarrow{\varphi} B \xrightarrow{\beta_B} \beta B$ and let $A \xrightarrow{e} E \xrightarrow{i} \beta B$ be the (extr. mono) $\circ$ epi factorization of $\beta_B\varphi$ mentioned above.*

Then φ is KI iff e is KI (thus a KI epicompletion of A).

Proof: Suppose φ is KI. By 4.1, β_B is KI, the composition of KIs is KI, so $\beta_B\varphi$ is KI. An initial factor of a KI is KI, thus E is KI.

Suppose e is KI. Since E is epicomplete, i is KI, thus so is $ie = \beta_B\varphi$, and so is φ. □

If $A \overset{\varphi}{\leq} E$ is an epicompletion, then there is $\beta A \overset{\delta}{\twoheadrightarrow} E$ with $\delta\beta_A = \varphi$ by reflectivity and H-closed. This "upper bounds" the epicompletions of A, thus the KI epicompletions. In fact, [4, 10.1] says $|E| = |A|^\omega$.

On the other hand, and finally for this section, we show that there is no upper bound on the KI extensions of any object. This contrasts with KP and KS (3.1).

4.4 Proposition. *For any $A \in W$ and any cardinal m, there is $A \leq B$ which is KI and $m \leq |B|$.*

Proof: A map $X_1 \overset{f}{\twoheadleftarrow} X_2$ in Comp is called skeletal if F dense open in X_1 implies $f^{-1}F$ dense in F_2.

For any $Y, X \in$ Comp, projection $Y \overset{\pi}{\twoheadleftarrow} Y \times X$ is skeletal.

Any irreducible surjection is skeletal, and thus for any $U \in$ Comp, the absolute (projective cover) $U \overset{p}{\twoheadleftarrow} pU$ has p skeletal. Here pU is extremally disconnected (ED), ED $\Rightarrow$ BD $\Rightarrow$ QF, so $D(pU) \in W$. The topological weights satisfy $wU \leq wpU$.

If $V \in$ Comp is zero-dimensional, then $wV = |\operatorname{clop} V| \leq |D(V)|$. (Here, $\operatorname{clop}(\cdot)$ is the collection of clopen sets.) This applies to ED spaces, thus to the $D(pU)$ above.

Now let $A \in W$. Then, $A \overset{\beta}{\leq} \beta A$ is KI (4.1), any $\beta A \overset{\varphi}{\leq} B$ is KI (by 3.2 etc., as noted before 4.3) and thus $A \overset{\varphi\beta}{\leq} B$ is KI.

Consider $YA \overset{Y\beta}{\longleftarrow} Y\beta A$. Take any X with $m \leq wX$, and consider $Y\beta A \overset{\pi}{\twoheadleftarrow} Y\beta A \times X \overset{p}{\twoheadleftarrow} p(Y\beta A \times X) \equiv K$. For $b \in \beta A = D(Y\beta A)$, $b \circ (\pi \circ p) \in D(K)$ because p, π, thus $\pi \circ p$ are skeletal. This defines $\beta A \overset{\varphi}{\leq} D(K)$ which is KI. So $A \overset{\varphi\beta}{\leq} D(K)$ is KI, while $m \leq |D(K)|$ follows from the above. □

Doubtless there are other ways to prove 4.4.

5. KI epicompletions of $C(X)$ and the Baire functions

We come to the main theorem of the paper.

For $A \in W$, the set $EC(A)$ of (equivalence classes of) epicompletions of A is partially ordered by: For $A \overset{\varphi_i}{\leq} E_i$ $(i = 1, 2)$, $E_1 \overset{*}{\geq} E_2$ means there is $E_1 \overset{\psi}{\twoheadrightarrow} E_2$ with $\psi\varphi_1 = \varphi_2$. The maximum in $EC(A)$ is $A \leq \beta A$, of course. More detail appears below.

For X a Tychonoff space, we have the much-studied W-object $C(X)$ (e.g., [23]), and its W-extension $C(X) \leq B(X)$ to the Baire functions, and this is an epicompletion (more detail below). (The issue $B(X) \overset{?}{=} \beta C(X)$ is addressed in [11] and commented on below in 5.5(2).)

5.1 Theorem. *Let X be any Tychonoff space and $E \in EC(C(X))$. Then, $C(X) \leq E$ is KI iff $B(X) \overset{*}{\leq} E$.*

This result answers the question 10.6(f) in [4]. We do not know if there is a similar theorem for general $A \in W$. A speculation is that an answer might be found in the first author's frame-theoretic development of pointwise convergence [2].

In proving 5.1, we shall have to say quite a bit about the Yosida representations of the $E \in EC(C(X))$ as developed in [4]. This will be explained as needed.

First, for any Tychonoff space X, the Baire field $\mathscr{B}(X)$ is the σ-field of subsets of X generated by $\{\operatorname{coz} f \mid f \in C(X)\}$ ($\operatorname{coz} f = \{x \mid f(x) \neq 0\}$), and the W-object of Baire functions is $B(X) = \{f \in \mathbb{R}^X \mid f^{-1}(U) \in \mathscr{B}(X) \text{ for } U \text{ open}\}$, with unit 1. Evidently $C(X) \overset{b}{\leq} B(X)$ (b is a label).

We explain the Yosida representations.

Of course $YC(X) = \beta X$, Čech-Stone compactification, with $C(X) \leq D(\beta X)$ achieved by extension and $SYC(X) \in L\,\mathrm{SpFi}$ has the filter $C(X)^{-1}\mathbb{R} = \{F \mid F$ is dense cozero in βX and $F \supseteq X\}$.

Now, $YB(X) = S\mathscr{B}(X)$, the Stone space of the Boolean algebra $\mathscr{B}(X)$. This space is BD, and is a compactification of the space $X_P \equiv$ the set X with the weak topology generated by $B(X)$. (X_P is the P-space coreflection of the space X.) We have $B(X) \leq C(X_P)$ (equality is rare), and the Yosida representation $B(X) = D(S\mathscr{B}(X))$ is continuous extension, with equality because $B(X)$ is epicomplete. For $SYB(X) \in L\,\mathrm{SpFi}$, the filter is $B(X)^{-1}\mathbb{R} = \{F \mid F$ is dense cozero in $S\mathscr{B}(X)\}$. (See [35] about Boolean algebras, and [8] for a careful discussion of $S\mathscr{B}(X)$.)

The image under SY of $C(X) \overset{b}{\leq} B(X)$ is the surjection SYb relabeled as $\beta X \overset{\rho}{\twoheadleftarrow} S\mathscr{B}(X)$, which is extension of the "identity" $X \leftarrow X_P$. The Stone representation of the Boolean isomorphism $\mathscr{B}(X) \approx \operatorname{clop} S\mathscr{B}(X)$ is $\mathscr{B}(X) \ni A \mapsto \overline{A}_P \in \operatorname{clop} S\mathscr{B}(X)$, where A_P is A with the topology from X_P and $\overline{(\cdot)}$ is closure in $S\mathscr{B}(X)$; we have $\overline{\varrho^{-1}A} = \overline{A}_P$.

Proof of 5.1$\Leftarrow$: Note that, in general, if $E_1 \overset{*}{\leq} E_2$ in an $EC(A)$, then, if $A \leq E_1$ is KI, so is E_2: Label the situation as $A \overset{\varphi}{\leq} E_2 \overset{\psi}{\twoheadrightarrow} E_1$ with $A \overset{\psi\varphi}{\leq} E_1$. Here $k(\psi\varphi) = (k\psi)(k\varphi)$ is supposed 1-1, thus so is the first factor $k\varphi$.

So it suffices to show that $C(X) \overset{b}{\leq} B(X)$ is KI. We show that $SYb = \rho$ satisfies condition $(3')$ in 2.8.

Suppose $S \in \operatorname{sub} \beta X$, $G \in (B(X)^{-1}\mathbb{R})_\delta$ in $S\mathscr{B}(X)$ and $\rho(G) \cap S = \emptyset$. This $G \supseteq X_P$, so $\rho(G) \supseteq \rho(X_P) = X$. This G is Lindelöf [22], and so is ρG. By Smirnov's Theorem on normal placement [22], there is a cozero-set F of βX with $\rho G \subseteq F$ and $F \cap S = \emptyset$. Thus, immediately, $(S =)\ S' \neq \emptyset$. $\square$

The proof of the converse requires more machinery again synopsized from [4].

For $A \in W$, we have YA and $A \leq D(YA)$ and thus $\operatorname{coz} A \equiv \{\operatorname{coz} g \mid g \in A\}$, and the Baire field $\mathscr{B}(YA)$, which is actually the σ-field on YA generated by $\operatorname{coz} A$ (because for any $f \in C(YA)$ there is $\{g_n\} \subseteq A$ with $\bigcup \operatorname{coz} g_n = \operatorname{coz} f$, in consequence of 2.1(a)). For $g \in A$, let $\infty(g) = g^{-1}(\pm\infty)$, and let $\infty(A)$ be the σ-ideal in $\mathscr{B}(YA)$ generated by $\{\infty(g) \mid g \in A\}$.

The poset $EC(A)$ is in bijective order-reversing correspondence with the family of σ-ideals $\mathscr{D}$ in $\mathscr{B}(YA)$ for which

$$\text{(i) } \infty(A) \subseteq \mathscr{D}, \text{ and (ii) } \operatorname{coz} A \cap \mathscr{D} = \{\emptyset\},$$

as follows.

Given $\mathscr{D}$ as above, let $I = \{f \in B(YA) \mid \operatorname{coz} f \in \mathscr{D}\}$. This is a σ-ideal in $B(YA)$ which is (thus) a W-kernel, and $A \leq B(YA)/I \equiv E(\mathscr{D})$ is a typical epicompletion of A. (Here, (i) creates an epic $A \to E(\mathscr{D})$, and (ii) makes it 1-1.) The orderings are: $\mathscr{D}_1 \subseteq \mathscr{D}_2$ iff $E(\mathscr{D}_1) \overset{*}{\geq} E(\mathscr{D}_2)$.

Given $E \in EC(A)$, the associated $\mathscr{D}$ is denoted $\mathscr{D}(E)$. For βA (the top of $EC(A)$) we have $\mathscr{D}(\beta A) = \infty(A)$ (the bottom of the poset of $\mathscr{D}$s).

The Yosida representation of an $A \leq E(\mathscr{D})$ in $EC(A)$ has $YE(\mathscr{D}) = S(\mathscr{B}(YA)/\mathscr{D})$ (the Stone space of the quotient Boolean algebra), which is BD, and $E(\mathscr{D}) = D(S(\mathscr{B}(YA)/\mathscr{D}))$ and the filter is $E(\mathscr{D})^{-1}\mathbb{R}$ = all dense cozero-sets. The Boolean theory of quotients [35] applied here shows

$$S(\mathscr{B}(YA)/\mathscr{D}) = S\mathscr{B}(YA) - \bigcup\{\overline{D}_P \mid D \in \mathscr{D}\}, \tag{5.2}$$

where $D \in \mathscr{D} \subseteq \mathscr{B}(YA)$, D_P is D with the topology from $(YA)_P$, and $\overline{(\cdot)}$ is closure in $S\mathscr{B}(YA)$.

The discussion preceding the proof of 5.1$\Leftarrow$ above applies using for the X there a present YA: We have $C(YA) \leq B(YA)$ whose image under SY we denote $YA \overset{\rho}{\twoheadleftarrow} S\mathscr{B}(YA)$. Then, for $A \leq E(\mathscr{D})$ in $EC(A)$ we have the image under SY, $YA \overset{\tau}{\twoheadleftarrow} YE(\mathscr{D}) = S(\mathscr{B}(YA)/\mathscr{D})$ and the commutative diagram

(5.3)
$$\begin{tikzcd} YA & S\mathscr{B}(YA) \arrow[l, two heads, "\rho"'] \\ YE(\mathscr{D}) \arrow[u, two heads, "\tau"] \arrow[ur, hook] & \end{tikzcd}$$

For the situation under discussion $A = C(X)$, where $YC(X) = \beta X$, we have (as noted) $B(X) \in EC(C(X))$ and $\mathscr{D}(B(X)) = \{M \in \mathscr{B}(\beta X) \mid M \subseteq \beta X - X\}$.

Proof of 5.1$\Rightarrow$: For $A = C(X)$, and $E \in EC(C(X))$, the diagram (5.3) becomes

$$\begin{tikzcd} \beta X & & S\mathscr{B}(\beta X) \arrow[ll, two heads, "\rho"'] \\ YE = S\mathscr{B}(\beta X) - \bigcup\{\overline{D}_P \mid D \in \mathscr{D}(E)\} \arrow[u, two heads, "\tau"] \arrow[urr, hook] & & \end{tikzcd} \tag{5.4}$$

Suppose $E \overset{*}{\geq} B(X)$ fails. This means $\mathscr{D}(E) \nsubseteq \mathscr{D}(\beta A)$. So there is $M \in \mathscr{D}(E)$ with $M \nsubseteq \beta X - X$. In any space, any Baire set is a union of zero-sets [19], so there is a zero-set Z of βX with $Z \subseteq M$ and $Z \nsubseteq \beta X - X$. Thus, $Z \cap X \neq \emptyset$ and $\overline{Z \cap X} \subseteq Z'$ (closure in βX, Z' is the largest SpFi-subspace contained in Z (Section 2), and $\subseteq$ here because $\overline{Z \cap X} \in \operatorname{sub} \beta X$). Using $Z \cap X$ as a D_P in (5.4), we have $\rho^{-1}Z = \overline{Z \cap X}$ (since Z is closed) and $\tau^{-1}Z = \rho^{-1}Z \cap YE \subseteq \overline{Z \cap X} \cap (S\mathscr{B}(\beta X) - \overline{Z \cap X}) = \emptyset$. Since $Z' \subseteq Z$, $(\tau^{-1}Z)' = \emptyset$ as well.

Thus, Z' witnesses violation of 2.8(1′), so $\operatorname{sub}\tau$ is not 1-1, and $C(X) \leq E$ is not KI. $\square$

5.5 Remarks. We briefly consider two extreme cases.

(1) The first case is (E) Every epicompletion of A is KI. From 4.3, (E) holds iff A is absolutely KI. Now, any A has a unique minimal (not minimum) and unique essential epicompletion $A \leq \lambda A$ [4, Section 9]. Obviously, (E) implies $A \leq \lambda A$ is KI. We do not know about the converse in general, but for $A = C(X)$, it does hold because $C(X) \leq \lambda C(X)$ KI implies $B(X) \overset{*}{\leq} \lambda C(X)$ (by 5.1), and this condition is equivalent to "$C(X) \leq B(X)$ is essential", which can be shown to hold iff X is almost P, thus $C(X)$ is absolutely KI (3.5).

(2) The second case is (U) A has Unique KI epicompletion (which must be βA, by 4.1). What this means in general, we do not know, but for $A = C(X)$, it means that $B(X) = \beta C(X)$ (by 5.1). This condition is studied in [11], there called "X is an ε-space": it holds if X is pseudocompact or a P-space, or the irrationals, or the Sorgenfrey line; it fails if X is the rationals, or a certain locally compact space. Overlooked in [11] is that this holds for X almost-P, because then $C(X)$ has unique epicompletion [4, Section 9].

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