# Universal meager $F_\sigma$-sets in locally compact manifolds

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*Dedicated to the 120th birthday anniversary of Eduard Čech.*

*Abstract.* In each manifold $M$ modeled on a finite or infinite dimensional cube $[0,1]^n$, $n \le \omega$, we construct a meager $F_\sigma$-subset $X \subset M$ which is universal meager in the sense that for each meager subset $A \subset M$ there is a homeomorphism $h : M \to M$ such that $h(A) \subset X$. We also prove that any two universal meager $F_\sigma$-sets in $M$ are ambiently homeomorphic.

*Keywords:* universal nowhere dense subset, Sierpiński carpet, Menger cube, Hilbert cube manifold, $n$-manifold, tame ball, tame decomposition

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In this paper we shall construct and characterize universal meager $F_\sigma$-sets in $\mathbb{I}^n$-manifolds.

A meager subset $A$ of a topological space $X$ is called *universal meager* if for each meager subset $B \subset X$ there is a homeomorphism $h : X \to X$ such that $h(B) \subset A$. So, each universal meager subset of $X$ contains homeomorphic copies of all other meager subsets of $X$.

In fact, the notion of a universal meager set is a special case of a more general notion of a $\mathcal{K}$-universal set for some family $\mathcal{K}$ of subsets of a topological space $X$. Namely, we define a set $U \in \mathcal{K}$ to be $\mathcal{K}$-*universal* if for each set $K \in \mathcal{K}$ there is a homeomorphism $h : X \to X$ such that $h(K) \subset U$.

$\mathcal{K}$-Universal sets for various classes $\mathcal{K}$ often appear in topology. A classical example of such set is the Sierpiński Carpet $M^2_1$, known to be a $\mathcal{K}$-universal set for the family $\mathcal{K}$ of all (closed) nowhere dense subsets of the square $\mathbb{I}^2 = [0,1]^2$ (see [14]). The Sierpiński Carpet $M^2_1$ is one of the Menger cubes $M^n_k$, which are $\mathcal{K}$-universal for the family $\mathcal{K}$ of all $k$-dimensional compact subsets of the $n$-dimensional cube $\mathbb{I}^n$ (see [15], [8, §4.1]). An analogue of the Sierpiński Carpet exists also in the Hilbert cube $\mathbb{I}^\omega$, which contains a $\mathcal{Z}_0$-universal set for the family $\mathcal{Z}_0$ of closed nowhere dense subsets of $\mathbb{I}^\omega$ (see [3]).

Many $\mathcal{K}$-universal spaces arise in infinite-dimensional topology. For example, the pseudo-boundary $B(\mathbb{I}^\omega) = [0,1]^\omega \setminus (0,1)^\omega$ of the Hilbert cube $\mathbb{I}^\omega$ is known to be $\sigma\mathcal{Z}_\omega$-universal for the family $\sigma\mathcal{Z}_\omega$ of $\sigma Z_\omega$-subsets of $\mathbb{I}^\omega$. What is surprising, up

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to an ambient homeomorphism, $B(\mathbb{I}^\omega)$ is a unique $\sigma\mathcal{Z}_\omega$-universal set in $\mathbb{I}^\omega$. In this paper we shall show that such a uniqueness theorem also holds for $\sigma\mathcal{Z}_0$-universal subsets in the Hilbert cube $\mathbb{I}^\omega$.

Let us recall the definition of the families $\sigma\mathcal{Z}_\omega$ and $\sigma\mathcal{Z}_0$. They consist of $\sigma Z_\omega$-sets and $\sigma Z_0$-sets, respectively.

A closed subset $A$ of a topological space $X$ is called a *$Z_n$-set* in $X$ for a (finite or infinite) number $n \le \omega$ if the set $\{f \in C(\mathbb{I}^n, X) : f(\mathbb{I}^n) \cap A = \emptyset\}$ is dense in the space $C(\mathbb{I}^n, X)$ of all continuous functions $f : \mathbb{I}^n \to X$, endowed with the compact-open topology. Here by $\mathbb{I} = [0,1]$ we denote the unit interval and by $\mathbb{I}^n$ the $n$-dimensional cube. For $n = \omega$ the space $\mathbb{I}^n = \mathbb{I}^\omega$ is the Hilbert cube.

A subset $A \subset X$ is called a *$\sigma Z_n$-set* in $X$ if $A$ can be written as the union $A = \bigcup_{k\in\omega} A_k$ of countably many $Z_n$-sets $A_k \subset X$. Let us observe that a subset $A \subset X$ is a $Z_0$-set in $X$ if and only if it is closed and nowhere dense in $X$, and $A$ is a $\sigma Z_0$-set if and only if $A$ is a meager $F_\sigma$-set in $X$.

For a topological space $X$ by $\mathcal{Z}_n$ and $\sigma\mathcal{Z}_n$ we denote the families of $Z_n$-sets and $\sigma Z_n$-sets in $X$, respectively.

A characterization of $\mathcal{Z}_\omega$-universal sets in the Hilbert cube is quite simple and can be easily derived from the $Z$-Set Unknotting Theorem 11.1 from [7]:

**Proposition 1.** *A subset $A \subset \mathbb{I}^\omega$ is $\mathcal{Z}_\omega$-universal in $\mathbb{I}^\omega$ if and only if $A$ is a $Z_\omega$-set in $\mathbb{I}^\omega$, containing a topological copy of the Hilbert cube $\mathbb{I}^\omega$.*

A characterization of $\sigma\mathcal{Z}_\omega$-universal sets in the Hilbert cube is also well-known and can be given in many different terms (skeletoid of Bessaga-Pelczynski [4], capsets of Anderson [1], [6], absorptive sets of West [16], pseudoboundaries of Geoghegan and Summerhill [11], [12]). For our purposes the most appropriate approach is that of West [16] and Geoghegan and Summerhill [12]. To formulate this approach, we need to recall some notation.

Let $\mathcal{U}$, $\mathcal{V}$ be two families of sets of a topological space $X$. Put

$$\mathcal{U} \wedge \mathcal{V} = \{U \cap V : U \in \mathcal{U},\ V \in \mathcal{V},\ U \cap V \ne \emptyset\} \text{ and}$$
$$\mathcal{U} \vee \mathcal{V} = \{U \cup V : U \in \mathcal{U},\ V \in \mathcal{V},\ U \cap V \ne \emptyset\}.$$

We shall write $\mathcal{U} \prec \mathcal{V}$ and say that $\mathcal{U}$ *refines* $\mathcal{V}$ if each set $U \in \mathcal{U}$ is contained in some set $V \in \mathcal{V}$. Let $\mathcal{S}t(\mathcal{U},\mathcal{V}) = \{\mathcal{S}t(U,\mathcal{V}) : U \in \mathcal{U}\}$ where $\mathcal{S}t(U,\mathcal{V}) = \bigcup\{V \in \mathcal{V} : U \cap V \ne \emptyset\}$. Put $\mathcal{S}t(\mathcal{U}) = \mathcal{S}t(\mathcal{U},\mathcal{U})$ and $\mathcal{S}t^{n+1}(\mathcal{U}) = \mathcal{S}t(\mathcal{S}t^n(\mathcal{U}))$ for each $n > 0$. We shall say that two maps $f, g : Z \to X$ are *$\mathcal{U}$-near* and denote it by $(f,g) \prec \mathcal{U}$ if the family $(f,g) = \{\{f(z), g(z)\} : z \in Z\}$ refines the family $\mathcal{U} \cup \{\{x\} : x \in X\}$. For a family $\mathcal{F}$ of subsets of a metric space $(X,d)$ we put $\mathrm{mesh}(\mathcal{F}) = \sup_{F\in\mathcal{F}} \mathrm{diam}(F)$.

Let $\mathcal{K}$ be a family of closed subsets of a Polish space $X$ and $\sigma\mathcal{K} = \{\bigcup_{n\in\omega} A_n : A_n \in \mathcal{K},\ \ n \in \omega\}$. We shall say that $\mathcal{K}$ is *topologically invariant* if $\mathcal{K} = \{h(K) : K \in \mathcal{K}\}$ for each homeomorphism $h : X \to X$.

A subset $B \subset X$ is called *$\mathcal{K}$-absorptive* in $X$ if $B \in \sigma\mathcal{K}$ and for each set $K \in \mathcal{K}$, open set $V \subset X$, and open cover $\mathcal{U}$ of $V$ there is a homeomorphism $h : V \to V$

such that $h(K \cap V) \subset B \cap V$ and $(h, \mathrm{id}) \prec \mathcal{U}$. An important observation is that each set $A \in \sigma\mathcal{K}$ containing a $\mathcal{K}$-absorptive subset of $X$ is also $\mathcal{K}$-absorptive.

The following powerful uniqueness theorem was proved by West [16] and Geoghegan and Summerhill [12, 2.5].

**Theorem 1** (Uniqueness Theorem for $\mathcal{K}$-absorptive sets)**.** *Let $\mathcal{K}$ be a topologically invariant family of closed subsets of a Polish space $X$. Then any two $\mathcal{K}$-absorptive sets $B, B' \subset X$ are ambiently homeomorphic. More precisely, for any open set $V \subset X$ and any open cover $\mathcal{U}$ of $V$ there is a homeomorphism $h : V \to V$ such that $h(V \cap B) = V \cap B'$ and $h$ is $\mathcal{U}$-near to the identity map of $V$.*

Two subsets $A, B$ of a topological space $X$ are called *ambiently homeomorphic* if there is a homeomorphism $h : X \to X$ such that $h(A) = B$. This happens if and only if the pairs $(X, A)$ and $(X, B)$ are homeomorphic. We shall say that two pairs $(X, A)$ and $(Y, B)$ of topological spaces $A \subset X$ and $B \subset Y$ are *homeomorphic* if there is a homeomorphism $h : X \to Y$ such that $h(A) = B$. In this case we say that $h : (X, A) \to (Y, B)$ is a homeomorphism of pairs.

According to the following corollary of Theorem 1, each $\mathcal{K}$-absorptive set is $\sigma\mathcal{K}$-universal.

**Corollary 1.** *Let $\mathcal{K}$ be a topologically invariant family of closed subsets of a Polish space. If a $\mathcal{K}$-absorptive set $B$ in $X$ exists, then a subset $A \subset X$ is $\sigma\mathcal{K}$-universal in $X$ if and only if $A$ is $\mathcal{K}$-absorptive.*

Proof: Assume that a subset $A$ of $X$ is $\mathcal{K}$-absorptive. The definition implies that $A \in \sigma\mathcal{K}$. To show that $A$ is $\sigma\mathcal{K}$-universal, fix any subset $K \in \sigma\mathcal{K}$. The definition of a $\mathcal{K}$-absorptive set implies that the union $A \cup K$ is $\mathcal{K}$-absorptive. By the Uniqueness Theorem 1, there is a homeomorphism of pairs $h : (X, A \cup K) \to (X, A)$. This homeomorphism embeds the set $K$ into $A$, witnessing that the $\mathcal{K}$-absorptive set $A$ is $\sigma\mathcal{K}$-universal.

Now assume that a set $A \subset X$ is $\sigma\mathcal{K}$-universal. Since the $\mathcal{K}$-absorptive set $B$ belongs to the family $\sigma\mathcal{K}$, there is a homeomorphism $h$ of $X$ such that $h(B) \subset A$. The topological invariance of the class $\mathcal{K}$ implies that the set $h(B)$ is $\mathcal{K}$-absorptive, and so is the set $A \supset h(B)$. □

Corollary 1 reduces the problem of studying $\sigma\mathcal{K}$-universal sets in a Polish space $X$ to studying $\mathcal{K}$-absorptive sets in $X$ (under the assumption that a $\mathcal{K}$-absorptive set in $X$ exists). The problem of the existence of $\mathcal{K}$-absorptive sets was considered in several papers. In particular, Geoghegan and Summerhill [12] proved that each Euclidean space $\mathbb{R}^n$ contains a $\mathcal{Z}_0$-absorptive set and such a set is unique up to ambient homeomorphism.

Unfortunately, the methods of constructing $\mathcal{Z}_0$-absorptive sets in Euclidean spaces used in [12] do not work in case of the Hilbert cube or Hilbert cube manifolds (in spite of the fact that the paper [12] was written to demonstrate applications of methods of infinite-dimensional topology in the theory of finite-dimensional manifolds). Known results on $\mathcal{Z}_\omega$-absorptive sets in the Hilbert cube $\mathbb{I}^\omega$ and $\mathcal{Z}_0$-absorptive sets in Euclidean spaces allow us to make the following:

**Conjecture 1.** *The Hilbert cube contains a $\mathcal{Z}_n$-absorptive set for every $n \le \omega$.*

This conjecture is true for $n = \omega$ as witnessed by the pseudoboundary $B(\mathbb{I}^\omega) = \mathbb{I}^\omega \setminus (0,1)^\omega$ of $\mathbb{I}^\omega$ which is a $\mathcal{Z}_\omega$-absorptive set in $\mathbb{I}^\omega$. In this paper we shall confirm Conjecture 1 for $n = 0$. In fact, our proof works not only for the Hilbert cube but also for any $\mathbb{I}^k$-manifold of finite or infinite dimension. By a *manifold modeled on a space $E$* (briefly, an *$E$-manifold*) we understand any paracompact space $M$ admitting a cover by open subsets homeomorphic to open subspaces of the model space $E$. In this paper we consider only manifolds modeled on (finite or infinite dimensional) cubes $\mathbb{I}^n$, $n \le \omega$. So, from now on, by a *manifold* we shall understand an $\mathbb{I}^n$-manifold for some $0 < n \le \omega$. If a manifold $X$ is finite-dimensional, then its *boundary* $\partial X$ consists of all points $x \in X$ which do not have neighborhoods homeomorphic to Euclidean spaces. If $X$ is a Hilbert cube manifold, then we put $\partial X = \emptyset$.

Our approach to constructing $\mathcal{Z}_0$-absorptive sets in manifolds is based on the notion of a tame $G_\delta$-set, which is interesting by itself, see [2]. First we recall some definitions.

A family $\mathcal{F}$ of subsets of a topological space $X$ is called *vanishing* if for each open cover $\mathcal{U}$ of $X$ the family $\mathcal{F}' = \{F \in \mathcal{F} : \forall U \in \mathcal{U},\ \ F \not\subset U\}$ is locally finite in $X$. It is easy to see that a countable family $\mathcal{F} = \{F_n\}_{n\in\omega}$ of subsets of a compact metric space $(X,d)$ is vanishing if and only if $\lim_{n\to\infty} \operatorname{diam}(F_n) = 0$.

An open subset $B$ of an $\mathbb{I}^n$-manifold $X$ is called a *tame open ball* in $X$ if its closure $\bar{B}$ has an open neighborhood $O(\bar{B})$ in $X$ such that the pair $(O(\bar{B}), \bar{B})$ is homeomorphic to the pair $(\mathbb{R}^n, \mathbb{I}^n)$ if $n < \omega$ and to the pair $(\mathbb{I}^\omega \times [0,\infty), \mathbb{I}^\omega \times [0,1])$ if $n = \omega$. Tame balls form a neighborhood base at each point $x \in X$, which does not belong to the boundary $\partial X$ of $X$ (this is trivial for $n < \omega$ and follows from Theorem 12.2 of [7] for $n = \omega$).

A subset $U$ of a manifold $X$ is called a *tame open set in $X$* if $U = \bigcup \mathcal{U}$ for some vanishing family $\mathcal{U}$ of tame open balls having pairwise disjoint closures in $X$. Observe that the family $\mathcal{U}$ is unique and coincides with the family $\mathcal{C}(U)$ of connected components of the set $U$. By $\bar{\mathcal{C}}(U) = \{\bar{C} : C \in \mathcal{C}(U)\}$ we shall denote the family of the closures of the connected components of $U$ in $X$.

A subset $G \subset X$ is called a *tame $G_\delta$-set in $X$* if $G = \bigcap_{n\in\omega} U_n$ for some decreasing sequence $(U_n)_{n\in\omega}$ of tame open sets such that the family $\mathcal{C} = \bigcup_{n\in\omega} \mathcal{C}(U_n)$ is vanishing and for every $n \in \omega$ the family $\bar{\mathcal{C}}(U_{n+1})$ refines the family $\mathcal{C}(U_n)$ of connected components of $U_n$.

Tame open and tame $G_\delta$-sets can be equivalently defined via tame families of tame open balls. A family $\mathcal{U}$ of non-empty open subsets of a topological space $X$ is called *tame* if $\mathcal{U}$ is vanishing and for any distinct sets $U, V \in \mathcal{U}$ one of three possibilities hold: either $\bar{U} \cap \bar{V} = \emptyset$ or $\bar{U} \subset V$ or $\bar{V} \subset U$. For a family $\mathcal{U}$ of subsets of a set $X$ by

$$\bigcup^\infty \mathcal{U} = \bigcap \big\{ \bigcup(\mathcal{U} \setminus \mathcal{F}) : \mathcal{F} \text{ is a finite subfamily of } \mathcal{U} \big\}$$

we denote the set of all points $x \in X$ which belong to infinite number of sets $U \in \mathcal{U}$.

**Proposition 2.** *A subset $T$ of a manifold $X$ is tame open (resp. tame $G_\delta$) if and only if $T = \bigcup \mathcal{T}$ (resp. $T = \bigcup^\infty \mathcal{T}$ ) for a suitable tame family $\mathcal{T}$ of tame open balls in $X$.*

Proof: The "only if" part follows directly from the definition of a tame open (resp. tame $G_\delta$) set. To prove the "if" part, assume that $\mathcal{T}$ is a tame family of tame open balls in $X$. Endow the family $\mathcal{T}$ with a partial order $\le$ defined by the reverse inclusion relation, that is $U \le V$ if and only if $U \supset V$. The vanishing property of $\mathcal{T}$ guarantees that for each set $U \in \mathcal{T}$ the set $\downarrow U = \{V \in \mathcal{T} : V \le U\}$ is finite. This allows us to define the ordinal $\mathrm{rank}(U)$ letting $\mathrm{rank}(U) = |\downarrow U|$. For each number $n \in \omega$ let $\mathcal{T}_n = \{U \in \mathcal{T} : \mathrm{rank}(U) = n+1\}$. It follows from the definition of a tame family that the union $U_n = \bigcup \mathcal{T}_n$ is a tame open set and $U_n \subset U_{n-1}$, where $U_{-1} = X$. In particular, the union $\bigcup \mathcal{T} = U_0$ is tame open set in $X$ and the set $T = \bigcup^\infty \mathcal{T} = \bigcap_{n\in\omega} U_n$ is a tame $G_\delta$-set in $X$. □

The classes of dense tame open sets and dense tame $G_\delta$-sets have the following cofinality property.

**Proposition 3.** (1) *Each open subset of a manifold $X$ contains a dense tame open set.*
(2) *Each $G_\delta$-subset of a manifold contains a dense tame $G_\delta$-set.*

Proof: Let $X$ be a manifold and $d$ be a metric generating the topology of $X$.

1. Given an open set $V \subset X$ and an open cover $\mathcal{U}$ of $V$ we shall construct a tame open set $W \subset X$ such that $W$ is dense in $V$ and the family $\bar{\mathcal{C}}(W)$ refines the cover $\mathcal{U}$. Replacing $V$ by $V \setminus \partial X$, we can assume that the set $V$ does not intersect the boundary $\partial X$ of $X$. Replacing the set $V$ by $V \setminus \{v\}$ for some point $v \in V$, we can additionally assume that the set $V$ is not compact. We can also assume that $V = \bigcup \mathcal{U}$. Without loss of generality, the manifold $X$ is connected and hence separable. So, we can fix a countable dense subset $\{x_n\}_{n\in\omega}$ in $V$. By induction we can construct an increasing number sequence $(n_k)_{k\in\omega}$ and a sequence $B_k$ of tame open balls in $X$ such that for each $k \in \omega$ the following conditions hold:

(1) $n_k$ is the smallest number $n$ such that $x_n \notin \bigcup_{i<k} \bar{B}_k$;
(2) $B_k$ is a tame open ball such that $x_{n_k} \in B_k$, the closure $\bar{B}_k$ of $B_k$ in $X$ has diameter $< 2^{-k}$ and is contained in $U \setminus \bigcup_{i<k} \bar{B}_k$ for some set $U \in \mathcal{U}$.

It is easy to check that $W = \bigcup_{k\in\omega} B_k$ is a required dense tame open set in $V$ with $\bar{\mathcal{C}}(W) = \{\bar{B}_k\}_{k\in\omega} \prec \mathcal{U}$.

2. Fix an arbitrary $G_\delta$-set $G$ in $X$ and write it as the intersection $G = \bigcap_{n\in\omega} U_n$ of a decreasing sequence $(U_n)_{n\in\omega}$ of open sets in $X$. By the (proof of the) preceding item, we can construct inductively a decreasing sequence $(V_n)_{n\in\omega}$ of tame open sets in $X$ such that for every $n \in \omega$ we get

- $\operatorname{mesh}\bar{\mathcal{C}}(V_n) < 2^{-n}$,
- $\bigcup\bar{\mathcal{C}}(V_n) \subset V_{n-1} \cap U_n$, and
- $V_n$ is dense in $V_{n-1} \cap U_n$.

Here we assume that $V_{-1} = X$. It follows that $\mathcal{V} = \bigcup_{n\in\omega} \mathcal{C}(V_n)$ is a tame family of tame open balls whose limit set $\bigcup^\infty \mathcal{V} = \bigcap_{n\in\omega} V_n$ is a required dense tame $G_\delta$-set in $G$. □

It is easy to see that any two tame open balls in a connected $\mathbb{I}^n$-manifold are ambiently homeomorphic. A similar fact holds also for dense tame open sets. Generalizing earlier results of Whyburn [17] and Cannon [5], Banakh and Repovš in [3, Corollary 2.8] proved the following Uniqueness Theorem for dense tame open sets.

**Theorem 2** (Uniqueness Theorem for Dense Tame Open Sets in Manifolds)**.** *Any two dense tame open sets $U, U' \subset X$ of a manifold $X$ are ambiently homeomorphic. Moreover, for each open cover $\mathcal{U}$ of $X$ there is a homeomorphism $h : (X, U) \to (X, U')$ such that $(h, \mathrm{id}) \prec \mathcal{S}t(\bar{\mathcal{C}}(U), \mathcal{U}) \vee \mathcal{S}t(\bar{\mathcal{C}}(U'), \mathcal{U})$.*

This theorem will be our main tool in the proof of the following Uniqueness Theorem for dense tame $G_\delta$-sets.

**Theorem 3** (Uniqueness Theorem for Dense Tame $G_\delta$-Sets in Manifolds)**.** *Any two dense tame $G_\delta$-sets $G, G'$ in a manifold $X$ are ambiently homeomorphic. Moreover, for each open cover $\mathcal{U}$ of $X$ there is a homeomorphism $h : (X, G) \to (X, G')$ such that $(h, \mathrm{id}) \prec \mathcal{U}$.*

Proof: Fix a bounded complete metric $d$ generating the topology of the manifold $X$. By [9, 8.1.10], the metric $d$ can be chosen so that the cover $\{\bar{B}(x, 1) : x \in X\}$ by closed balls of radius 1 refines the cover $\mathcal{U}$. In this case any two functions $f, g : X \to X$ with $d(f, g) = \sup_{x\in X} d(f(x), g(x)) \le 1$ are $\mathcal{U}$-near.

Represent the tame $G_\delta$-sets $G$ and $G'$ as the limit sets $G = \bigcup^\infty \mathcal{G}$ and $G' = \bigcup^\infty \mathcal{G}'$ of suitable tame families $\mathcal{G}$ and $\mathcal{G}'$ of tame open balls in $X$. For every $n \in \omega$ let $\mathcal{G}_n = \{U \in \mathcal{G} : |\{V \in \mathcal{G} : V \supset \bar{U}\}| \ge n\}$ and $\mathcal{G}'_n = \{U \in \mathcal{G}' : |\{V \in \mathcal{G}' : V \supset \bar{U}\}| \ge n\}$. It follows that $G = \bigcap_{n\in\omega} \bigcup \mathcal{G}_n$ and $G' = \bigcap_{n\in\omega} \bigcup \mathcal{G}'_n$.

Let $U_{-1} = U'_{-1} = X$ and $h_{-1} : X \to X$ be the identity homeomorphism of $X$. Let also $\mathcal{U}_{-1} = \mathcal{U}'_{-1}$ be a cover of $X$ by open subsets of diameter $\le \frac{1}{8}$.

For every $n \in \omega$ we shall construct a homeomorphism $h_n : X \to X$, two tame open sets $U_n, U'_n \subset X$, and open covers $\mathcal{U}_n$, $\mathcal{U}'_n$ of the sets $U_n$, $U'_n$, respectively, such that

(1) $G \subset U_n \subset U_{n-1} \cap \bigcup \mathcal{G}_n$ and $\bar{\mathcal{C}}(U_n) \prec \mathcal{U}_{n-1}$;
(2) $G' \subset U'_n \subset U'_{n-1} \cap \bigcup \mathcal{G}'_n$ and $\bar{\mathcal{C}}(U'_n) \prec \mathcal{U}'_{n-1} \wedge h_{n-1}(\mathcal{U}_{n-1})$;
(3) $h_n(U_n) = U'_n$;

(4) $h_n|X \setminus U_{n-1} = h_{n-1}|X \setminus U_{n-1}$;
(5) $d(h_n, h_{n-1}) \le 2^{-n-1}$ and $d(h_n^{-1}, h_{n-1}^{-1}) \le 2^{-n-1}$;
(6) $\mathrm{mesh}(\mathcal{U}'_n) < 2^{-n-3}$, $\mathrm{mesh}(\mathcal{U}_n) < 2^{-n-3}$, and $\mathcal{S}t^2(\mathcal{U}_n) \prec \{B(x, d(x, X \setminus U_n)/2) : x \in U_n\}$.

Assume that for some $n \in \omega$ the open sets $U_{n-1}, U'_{n-1}$, open covers $\mathcal{U}_{n-1}, \mathcal{U}'_{n-1}$ and a homeomorphism $h_{n-1} : (X, U_{n-1}) \to (X, U'_{n-1})$ satisfying the conditions (1)–(6) have been constructed. Consider the subfamilies $\mathcal{F}_n = \{U \in \mathcal{G}_n : \{\bar{U}\} \prec \mathcal{U}_{n-1}\}$ and $\mathcal{F}'_n = \{U \in \mathcal{G}'_n : \{\bar{U}\} \prec \mathcal{U}'_{n-1} \wedge h_{n-1}(\mathcal{U}_{n-1})\}$. The vanishing property of the tame families $\mathcal{G}$ and $\mathcal{G}'$ implies that the sets $U_n = \bigcup \mathcal{F}_n$ and $U'_n = \bigcup \mathcal{F}'_n$ satisfy the conditions (1), (2) of the inductive construction. The sets $U_n$ and $U'_n$ are tame open, being unions of the tame families $\mathcal{F}_n$ and $\mathcal{F}'_n$, respectively. Moreover, $\bar{\mathcal{C}}(U_n) \prec \mathcal{U}_{n-1}$ and $\bar{\mathcal{C}}(U'_n) \prec \mathcal{U}'_{n-1} \wedge h_{n-1}(\mathcal{U}_{n-1})$.

Now we shall construct a homeomorphism $h_n : (X, U_n) \to (X, U'_n)$. Since $h_{n-1}(U_{n-1}) = U'_{n-1}$, each connected component $C \in \mathcal{C}(U_{n-1})$ of the open set $U_{n-1}$ maps onto the connected component $C' = h_{n-1}(C) \in \mathcal{C}(U'_{n-1})$ of the set $U'_{n-1}$. Taking into account that each set $\bar{B} \in \bar{\mathcal{C}}(U_n)$ is a compact connected subset of the open set $\bigcup \mathcal{U}'_{n-1} = U'_{n-1}$, we see that the intersection $U'_n \cap C'$ is a dense tame open set in the open set $C'$. Consequently, its image $h_{n-1}^{-1}(U'_n \cap C')$ is a dense tame open set in the open set $C = h_{n-1}^{-1}(C')$. By Theorem 2, there is a homeomorphism of pairs $g_C : (C, C \cap U_n) \to (C, h_{n-1}^{-1}(C' \cap U'_n))$ which is $\mathcal{W}_C$-near to the identity map $\mathrm{id}_C : C \to C$ for the cover $\mathcal{W}_C = \mathcal{S}t(\bar{\mathcal{C}}(C \cap U_n), \mathcal{U}_{n-1}) \vee \mathcal{S}t(\bar{\mathcal{C}}(h_{n-1}^{-1}(C' \cap U'_n)), \mathcal{U}_{n-1})$.

Taking into account that

$$\begin{aligned}\bar{\mathcal{C}}(C \cap U_n) \prec \bar{\mathcal{C}}(U_n) \prec \mathcal{U}_{n-1} \text{ and } &\bar{\mathcal{C}}(h_{n-1}^{-1}(U'_n \cap C')) \prec \bar{\mathcal{C}}(h_{n-1}^{-1}(U'_n)) \\ &= h_{n-1}^{-1}(\bar{\mathcal{C}}(U'_n)) \prec h_{n-1}^{-1}(h_{n-1}(\mathcal{U}_{n-1})) = \mathcal{U}_{n-1},\end{aligned}$$

we conclude that

$$\begin{aligned}\mathcal{W}_C &= \mathcal{S}t(\bar{\mathcal{C}}(C \cap U_n), \mathcal{U}_{n-1}) \vee \mathcal{S}t(\bar{\mathcal{C}}(h_{n-1}^{-1}(C' \cap U'_n)), \mathcal{U}_{n-1}) \\ &\quad \prec \mathcal{S}t(\mathcal{U}_{n-1}, \mathcal{U}_{n-1}) \vee \mathcal{S}t(\mathcal{U}_{n-1}, \mathcal{U}_{n-1}) \\ &= \mathcal{S}t(\mathcal{U}_{n-1}) \vee \mathcal{S}t(\mathcal{U}_{n-1}) \prec \mathcal{S}t^2(\mathcal{U}_{n-1}) \prec \{B(x, d(X \setminus U_{n-1})/2) : x \in U_{n-1}\}.\end{aligned}$$

Now the vanishing property of the family $\mathcal{C}(U_{n-1})$ implies that the map $g_n : X \to X$ defined by

$$g_n(x) = \begin{cases} x & \text{if } x \notin U_{n-1}, \\ g_C & \text{if } x \in C \in \mathcal{C}(U_{n-1}) \end{cases}$$

is a homeomorphism of $X$ such that $(g_n, \mathrm{id}) \prec \mathcal{S}t^2(\mathcal{U}_{n-1})$ and $(g_n, \mathrm{id}) \prec \mathcal{C}(U_{n-1})$. Then $h_n = h_{n-1} \circ g_n$ is a homeomorphism of $X$ satisfying the conditions (3) and (4) of the inductive construction.

To prove the condition (5) we shall consider separately the cases of $n = 0$ and $n > 0$. If $n = 0$, then $h_0 = g_0$ and hence $(h_0, h_{-1}) = (g_0, \mathrm{id}) \prec \mathcal{S}t^2(\mathcal{U}_{-1})$. It follows from $\mathrm{mesh}(\mathcal{U}_{-1}) \le 1/8$ that $d(h_0^{-1}, h_{-1}^{-1}) = d(h_0, h_{-1}) \le \mathrm{mesh}(\mathcal{S}t^2(\mathcal{U}_{-1})) \le \frac{1}{2}$.

If $n > 0$, then $(h_n, h_{n-1}) = (h_{n-1} \circ g_n, h_{n-1} \circ \mathrm{id}) \prec h_{n-1}(\mathcal{C}(U_{n-1})) = \mathcal{C}(U'_{n-1}) \prec \mathcal{U}'_{n-2}$ implies $d(h_n, h_{n-1}) \le \mathrm{mesh}(\mathcal{U}'_{n-2}) \le 2^{-n-1}$. By analogy, $(h_n^{-1}, h_{n-1}^{-1}) = (g_n^{-1} \circ h_{n-1}^{-1}, h_{n-1}^{-1}) = (g_n^{-1}, \mathrm{id}) = (g_n, \mathrm{id}) \prec \mathcal{C}(U_{n-1}) \prec \mathcal{U}_{n-2}$ implies $d(h_n^{-1}, h_{n-1}^{-1}) \le \mathrm{mesh}(\mathcal{U}_{n-2}) \le 2^{-n-1}$. So, the condition (5) holds.

Finally, using the paracompactness of the metrizable spaces $U_n$ and $U'_n$ choose two open covers $\mathcal{U}_n$ and $\mathcal{U}'_n$ of $U_n$ and $U'_n$ satisfying the condition (6).

After completing the inductive construction, we obtain a sequence of homeomorphisms $h_n : (X, U_n) \to (X, U'_n)$, $n \in \omega$. The condition (5) guarantees that the limit map $h = \lim_{n\to\infty} h_n$ is a well-defined homeomorphism of $X$ such that $d(h, \mathrm{id}) \le 1$. Moreover, the conditions (1) and (3) imply

$$h(G) = h\Big(\bigcap_{n\in\omega} U_n\Big) = \bigcap_{n\in\omega} h(U_n) = \bigcap_{n\in\omega} U'_n = G'.$$

By the choice of the metric $d$, the inequality $d(h, \mathrm{id}) \le 1$ implies $(h, \mathrm{id}) \prec \mathcal{U}$. So, $h : (X, G) \to (X, G')$ is a required homeomorphism of pairs with $(h, \mathrm{id}) \prec \mathcal{U}$. $\square$

Now we are able to prove a characterization of $\sigma\mathcal{Z}_0$-universal sets in manifolds.

**Theorem 4** (Characterization of $\sigma\mathcal{Z}_0$-Universal Sets in Manifolds)**.** *For a subset $A$ of a manifold $X$ the following conditions are equivalent:*

(1) *$A$ is $\sigma\mathcal{Z}_0$-universal in $X$;*
(2) *$A$ is $\mathcal{Z}_0$-absorptive in $X$;*
(3) *the complement $X \setminus A$ is a dense tame $G_\delta$-set in $X$.*

Proof: We shall prove the equivalences (3) $\Leftrightarrow$ (2) $\Leftrightarrow$ (1). Let $d$ be a metric generating the topology of the manifold $X$.

To prove that (3) $\Rightarrow$ (2), assume that the complement $X \setminus A$ is a dense tame $G_\delta$-set in $X$. To prove that $A$ is $\mathcal{Z}_0$-absorptive, fix any open set $V \subset X$, an open cover $\mathcal{U}$ of $V$ and a closed nowhere dense subset $K \subset X$. We lose no generality assuming that $\mathcal{U} \prec \{B(x, d(x, X \setminus V)/2) : x \in V\}$. Since $V \setminus (A \cup K)$ is a dense $G_\delta$-set in $V$, we can apply Proposition 3 and find a dense tame $G_\delta$-set $G \subset V \setminus (A \cup K)$. The characterization of tame $G_\delta$-sets given in Proposition 2 implies that the intersection $V \cap (X \setminus A) = V \setminus A$ is a dense tame $G_\delta$-set in $V$. By Theorem 3, there is a homeomorphism of pairs $h : (V, G) \to (V, V \setminus A)$ such that $(h, \mathrm{id}) \prec \mathcal{U}$. Since $\mathcal{U} \prec \{B(x, d(x, X \setminus V)/2) : x \in V\}$, the homeomorphism $h$ of $V$ extends to a homeomorphism $\bar{h} : X \to X$ such that $\bar{h}|X \setminus V = \mathrm{id}$. Observing that $\bar{h}(V \cap K) \subset \bar{h}(V \setminus G) = V \cap A$, we see that the set $A$ is $\mathcal{Z}_0$-absorptive.

To prove that (2) $\Rightarrow$ (3), assume that the set $A$ is $\mathcal{Z}_0$-absorptive. By Proposition 3, the dense $G_\delta$-set $X \setminus A$ contains a dense tame $G_\delta$-set $G$ in $X$. Since $A \subset X \setminus G$, the set $X \setminus G \in \sigma\mathcal{Z}_0$ is $\mathcal{Z}_0$-absorptive. By the Uniqueness Theorem 3, there is a homeomorphism of pairs $h : (X, A) \to (X, X \setminus G)$. Then $X \setminus A = h(G)$ is a dense tame $G_\delta$-set in $X$, which completes the proof of the implication (2) $\Rightarrow$ (3).

By Proposition 3, $X$ contains a dense tame $G_\delta$-set $G$ and by the implication (3) $\Rightarrow$ (2) proved above the complement $X \setminus G$ is $\mathcal{Z}_0$-absorptive. Now Corollary 1 yields the equivalence (2) $\Leftrightarrow$ (1). □

Theorem 4 implies:

**Corollary 2.** *Each dense $G_\delta$-subset of a dense tame $G_\delta$-set in a manifold is tame.*

We finish this paper by some open problems. It is clear that each tame $G_\delta$-set in a manifold is zero-dimensional. However, not each zero-dimensional dense $G_\delta$-subset of the Hilbert cube $\mathbb{I}^\omega$ is tame.

**Proposition 4.** *For any dense $G_\delta$-set $G \subset \mathbb{I}$ the countable product $G^\omega$ is not a tame $G_\delta$-set in $\mathbb{I}^\omega$.*

Proof: Assuming that $G^\omega$ is tame, we can find a dense tame open set $T \subset \mathbb{I}^\omega$ containing $G^\omega$. By Theorem 1.4 of [3], the complement $S = \mathbb{I}^\omega \setminus T$ is homeomorphic to the Hilbert cube and the boundary $\bar{B} \setminus B$ of each tame open ball $B \in \mathcal{C}(T)$ in $\mathbb{I}^\omega$ is a $Z_\omega$-set in $S$. Let $\mathrm{pr}_n : \mathbb{I}^\omega \to \mathbb{I}$, $n \in \omega$, denote the projection of the Hilbert cube $\mathbb{I}^\omega$ onto the $n$th coordinate. Since $\mathbb{I}^\omega \setminus T \subset \bigcup_{n\in\omega} \mathrm{pr}_n^{-1}(\mathbb{I} \setminus G)$, Baire Theorem yields a non-empty open subset $W \subset S$ such that $W \subset \mathrm{pr}_n^{-1}(\mathbb{I} \setminus G)$ for some $n \in \omega$. Since $S$ is homeomorphic to the Hilbert cube, we can assume that the set $W$ is connected and hence is contained in $\mathrm{pr}_n^{-1}(t)$ for some point $t \in \mathbb{I} \setminus G$. Since the union $\Delta = \bigcup_{B\in\mathcal{C}(U)} \bar{B} \setminus B$ is a $\sigma Z_\omega$-set in $S$, we can choose a point $x_0 \in W \setminus \Delta$. Choose an open neighborhood $U$ of $x_0$ in $\mathbb{I}^\omega$ such that $U \cap S \subset W$ and $U \setminus \mathrm{pr}_n^{-1}(t)$ has at most two connected components.

Since the family $\mathcal{C}(T)$ is vanishing and $T = \bigcup \mathcal{C}(T)$ is dense in $\mathbb{I}^\omega$, there are three pairwise distinct tame open balls $B_1, B_2, B_3 \in \mathcal{C}(T)$ such that $\bar{B}_1 \cup \bar{B}_2 \cup \bar{B}_2 \subset U$. Since the set $U \setminus \mathrm{pr}_n^{-1}(t)$ has at most two connected components, there are two distinct indices $1 \le i, j \le 3$ such that the balls $B_i$ and $B_j$ meet the same connected component $V$ of $U \setminus \mathrm{pr}_n^{-1}(t)$. Since $\bar{B}_i \setminus B_i \subset U \cap S \subset \mathrm{pr}_n^{-1}(t)$, the set $V \cap B_i$ is closed-and-open in the connected set $V$ and hence coincides with $V$. So, $V \subset B_i$. By the same reason, $V \subset B_j$, which is not possible as the balls $B_i$ and $B_j$ are disjoint. □

**Problem 1.** *Can the countable power $G^\omega$ of a dense $G_\delta$-set $G \subset \mathbb{I}$ be covered by countably many dense tame $G_\delta$-sets?*

By Smirnov's result [9, 5.2.B], the Hilbert cube $\mathbb{I}^\omega$ can be covered by $\aleph_1$ zero-dimensional $G_\delta$-sets.

**Problem 2.** *What is the smallest cardinality of a cover of the Hilbert cube $\mathbb{I}^\omega$ by tame $G_\delta$-sets? Is it equal to $\aleph_1$?* (By Theorem 1.6 of [2] this cardinality does not exceed $\mathrm{add}(\mathcal{M})$, the additivity of the ideal $\mathcal{M}$ of meager subsets on the real line.)

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