Comment.Math.Univ.Carolin. 54,2 (2013) 197–217

Extremal pseudocompact Abelian groups: A unified treatment

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Dedicated to the 120th birthday anniversary of Eduard Čech.

Abstract. The authors have shown [Proc. Amer. Math. Soc. 135 (2007), 4039–4044] that every nonmetrizable, pseudocompact abelian group has both a proper dense pseudocompact subgroup and a strictly finer pseudocompact group topology. Here they give a comprehensive, direct and self-contained proof of this result.

Keywords: pseudocompact topological group, extremal topological group, proper dense pseudocompact subgroup, abelian

Classification: 22A05, 22B05

1. Introduction

The aim of this paper is to present a proof of Theorem 1.1, obtained by the authors in [9].

Theorem 1.1. *Let G be an abelian, nonmetrizable pseudocompact group. Then G is neither r- nor s-extremal.*

With this paper, we hope that we have made good on the promise given in [9], namely "to present a polished, complete and self-contained proof of Theorem 1.1" — but subject to these boundary conditions: We omit proofs of theorems available in the familiar monographs [19], [20] and [25], and we omit also proofs of the following two basic results, given respectively by Weil [31] in 1938 and by Comfort and Ross [13] in 1966.

Theorem 1.2. *A topological group is totally bounded if and only if it is a dense subgroup of a compact group.*

Theorem 1.3. (a) *Every pseudocompact group is totally bounded.*

(b) *A totally bounded group is pseudocompact if and only if it is G_δ-dense in its Weil completion.*

Remarks 1.4. (a) *À propos* of Theorem 1.2, it is shown by Weil [31] that the compact group in which a totally bounded topological group G is dense is unique (in the obvious sense). That fact justifies the convention, which already we implemented in stating Theorem 1.3, of referring to this compactification of a totally bounded group G as *the* Weil completion of G; we denote this by the symbol $\overline{G}$.

(b) It follows from (a) that for H a subgroup of a totally bounded group G, one may identify $\overline{H}$ with $\mathrm{cl}_{\overline{G}} H$. Often in what follows, we make that identification.

The plan of this paper is to give the proof of Theorem 1.1 in as brief and succinct a form as is compatible with the inclusion of all proofs; we postpone discursive remarks, commentary and historical perspective until Section 5. Since the *raison d'être* of this paper may not be immediately apparent to the reader — it contains, finally, a single theorem for which we gave in 2007 a detailed proof — a word is in order concerning its motivation and purpose.

As presented in [9], the proof of Theorem 1.1 has two independent parts which in their essentials are disjoint. First is the case, already treated in 1988 in [12], that G is a torsion group. The second step, where G is not a torsion group, reduced after some effort to the case that G is connected. For this reason, most published efforts to prove Theorem 1.1 following the publication of [12], including our own efforts [8], [7], concentrated on the connected case; indeed it had been speculated, but not proved, that if Theorem 1.1 were false then there was a connected counterexample. Once Theorem 1.1 was known to be true, it became appropriate to strive for a unified approach treating simultaneously the totally disconnected/torsion case and the connected/not-torsion case. That is the role of the present paper. We show that the main new idea in [8] can also be used to deal with all cases, resulting in the present coherent proof that we promised in [9]. It turns out that the most efficient bifurcation is not between the torsion and the connected cases, but between the cases in which the given (pseudocompact, abelian) group does or does not contain a closed G_δ-subgroup which is torsion.

2. Definitions and other preliminaries

We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the set of integers, rational numbers, real numbers, and complex numbers, respectively, and

$$\mathbb{T} := \{t \in \mathbb{C} : |t| = 1\};$$

$\mathbb{P}$ is the set of primes. Each of these sets is given, as needed, its usual algebraic and topological properties.

Except for the circle group $\mathbb{T}$, written multiplicatively as usual, we use additive notation for groups G known to be abelian (with identity $0 = 0_G$), multiplicative notation for general groups G (with identity $1 = 1_G$). For a group G and $A \subseteq G$, the symbol $\langle\langle A\rangle\rangle$ denotes the subgroup of G generated by A. If A is a subgroup of G, then a subset X of G is called *independent over* A if for every $x \in X$ we have $\langle\langle \{x\}\rangle\rangle \cap \langle\langle X \setminus \{x\}\rangle\rangle \cap A = \{1\}$. A set X independent over G itself is said simply to be *independent*. The cardinality of a maximal independent set of non-torsion elements of an abelian group G is called the *torsion-free rank* of G, here denoted $r_0(G)$. It is known that $r_0(G)$ is an invariant of G, well-defined in the sense that all such maximal independent subsets of G have the same cardinality ([20, 16.3], [25, A.11]). It is clear that if $h : G \twoheadrightarrow H$ is a surjective homomorphism, then $r_0(H) \le r_0(G)$. See [20, pp. 85–86] for additional details.

Already in Section 1 we have implicitly invoked the following convention, which pertains throughout this paper (and also in [9]): *All hypothesized topological spaces, including topological groups, are assumed to be completely regular, Hausdorff spaces, i.e., to be Tychonoff spaces.*

For a space $X = (X, \mathcal{T})$ and $p \in X$, we denote by $\mathcal{N}_X(p)$, or simply by $\mathcal{N}(p)$ if ambiguity is impossible, the set

$$\mathcal{N}_X(p) = \{U \subseteq X : p \in U \in \mathcal{T}\}.$$

When $X = (X, \mathcal{T})$ is a space and $Y \subseteq X$, we denote by $(Y, \mathcal{T})$ the set Y with the topology inherited from $(X, \mathcal{T})$.

The symbol wX denotes the weight of a topological space X. A subspace of a space X is *G_δ-dense* in X if it meets every nonempty G_δ-subset of X. If X is a set and κ a cardinal number, then $[X]^{\leq\kappa}$ denotes $\{A \subseteq X : |A| \leq \kappa\}$.

For spaces X and Y with $Y \subseteq X$, we denote as usual by $\mathrm{cl}_X Y$ the closure of Y in X.

For a topological group G, we denote by $\widehat{G}$ the set of continuous homomorphisms from G into $\mathbb{T}$.

As indicated above, we avoid excess clutter and "noise" in the main body of this paper by giving explicit citations to the literature only to those needed results which are available in [19], [20] and [25]. Other credits and discursive comments are deferred until Section 5.

Although Theorem 1.1 is stated (and is known to be true) only for abelian groups, many of the preliminary lemmas and theorems hold without that algebraic restriction. To avoid unnecessary restrictions, throughout this paper we omit the abelian hypothesis whenever we know that we can legitimately do so.

Definition 2.1. A space X is *pseudocompact* if each continuous real-valued function on X is bounded.

Definition 2.2. A pseudocompact group $G = (G, \mathcal{T})$ is

(a) *r-extremal* if no topology on G strictly finer than $\mathcal{T}$ makes G a pseudocompact topological group;

(b) *s-extremal* if G admits no proper dense pseudocompact subgroup.

The symbols r and s were chosen (in [3]) to evoke the words <u>r</u>efinement and <u>s</u>ubgroup.

Theorem 2.3. (a) *A dense subgroup H of a topological group G is pseudocompact if and only if* (i) *G is pseudocompact and* (ii) *H is G_δ-dense in G; and*

(b) *the product of any set of pseudocompact groups is pseudocompact.*

Proof: Both statements follow from Theorem 1.3 and the uniqueness of the Weil completion. In (a) we have $\overline{H} = \overline{G}$, so 1.3(b) applies; and in (b) with $G = \Pi_{i \in I} G_i$ with each G_i a pseudocompact group, we have that G is G_δ-dense in $\Pi_{i \in I} \overline{G_i} = \overline{G}$. □

First we establish the fact that metrizable pseudocompact groups are both r- and s-extremal.

Theorem 2.4. *For a pseudocompact group G, these conditions are equivalent.*

(a) $w(G) \le \omega$.
(b) $\{1\}$ *is a* G_δ*-subset of* G.
(c) G *is compact and metrizable.*

Proof: That (a) $\Rightarrow$ (b) is clear, and (c) $\Rightarrow$ (a) is familiar ([19, 4.2.9 and 4.2.8]); for those assertions, the pseudocompactness hypothesis is redundant. To see that (b) $\Rightarrow$ (c), let $\{1\} = (\bigcap_{n<\omega} \widetilde{U_n}) \cap G$ with each $\widetilde{U_n} \in \mathcal{N}_{\overline{G}}(1)$. Then $\bigcap_{n<\omega} \widetilde{U_n} = \{1\}$, for otherwise $\bigcap_{n<\omega} (\widetilde{U_n} \backslash \{1\})$ is a nonempty G_δ-set in $\overline{G}$ which misses G, contrary to Theorem 2.3(a) (with G and $\overline{G}$ replacing H and G, respectively). Thus $\{1\}$ is a G_δ-set in the compact group $\overline{G}$, so $\overline{G}$ is first-countable ([19, 3.3.4]) and hence metrizable ([25, 8.5]8.5). Like every metrizable space, $\overline{G}$ has no proper G_δ-dense subspace, so $G = \overline{G}$ and G is compact and metrizable. □

Corollary 2.5. *Let $G = (G, \mathcal{T})$ be a pseudocompact group of countable weight. Then*

(a) G *is r-extremal; and*
(b) G *is s-extremal.*

Proof: (a) Let $\mathcal{U} \supseteq \mathcal{T}$ be a pseudocompact group topology on G. The set $\{1\}$ is a G_δ-set in $(G, \mathcal{T})$, hence also in $(G, \mathcal{U})$, so by Theorem 2.4 both $(G, \mathcal{T})$ and $(G, \mathcal{U})$ are compact and metrizable. Then the function id : $(G, \mathcal{U}) \twoheadrightarrow (G, \mathcal{T})$ is a homeomorphism ([19, 3.1.13]), so $\mathcal{U} = \mathcal{T}$.

(b) Again by Theorem 2.4 the group $(G, \mathcal{T})$ is metrizable, so it has no proper G_δ-dense subspace. □

With Corollary 2.5 behind us, we turn to our principal assignment: showing that pseudocompact abelian groups of uncountable weight are neither r- nor s-extremal. We proceed *via* a sequence of lemmas.

The following notation, suggested in [12] and used in several of our references [7], [8], [21], [5], [9], continues to be useful. Here and throughout this paper, the adjective *normal* (as applied to subgroups of a topological group) is intended in the algebraic sense.

Notation 2.6. Let G be a topological group. The symbol $\Lambda(G)$ denotes the set of closed, normal, G_δ-subgroups of G.

The following useful statement, to be used frequently in what follows, is (a special case of) the *Kakutani-Kodaira Theorem*. See [25, 8.7] and its proof.

Lemma 2.7. *Let K be a compact group and let $\{U_n : n < \omega\} \subseteq \mathcal{N}_K(1)$. Then there is $N \in \Lambda(K)$ such that $N \subseteq \bigcap_{n<\omega} U_n$ and $w(K/N) \le \omega$.*

Lemma 2.8. *Let G be a totally bounded topological group and let $\{U_n : n < \omega\} \subseteq \mathcal{N}_G(1)$. Then there is $N \in \Lambda(G)$ such that $N \subseteq \bigcap_{n<\omega} U_n$ and $\{N\}$ is a G_δ-set in G/N.*

PROOF: When G is compact the statement is immediate from Lemma 2.7. In the general case choose $\widetilde{U_n} \in \mathcal{N}(1_{\overline{G}})$ such that $\widetilde{U_n} \cap G = U_n$, and using Lemma 2.7 choose $\widetilde{N} \in \Lambda(\overline{G})$ so that $\widetilde{N} \subseteq \bigcap_{n<\omega} \widetilde{U_n}$ and $\{\widetilde{N}\}$ is a G_δ-set in $\overline{G}/\widetilde{N}$, say $\{\widetilde{N}\} = \bigcap_{n<\omega} \widetilde{V_n}$ with $\widetilde{V_n}$ open in $\overline{G}/\widetilde{N}$ and $\widetilde{V_n} \subseteq \widetilde{\phi}[\widetilde{U_n}]$ where $\widetilde{\phi} : \overline{G} \twoheadrightarrow \overline{G}/\widetilde{N}$ is the usual map. Set $N := \widetilde{N} \cap G$ and $W_n := (\widetilde{\phi}^{-1}(V_n)) \cap G$. Then $N \in \Lambda(G)$ and it suffices, since the usual map $\phi : G \twoheadrightarrow G/N$ is an open map ([25, 5.17]), to show that $\{N\} = \bigcap_{n<\omega} \phi[W_n]$. If $aN \in \bigcap_{n<\omega} \phi[W_n]$ with $a \in G \subseteq \overline{G}$ then for each $n < \omega$ there is $w_n \in W_n \subseteq \widetilde{\phi}^{-1}(\widetilde{V_n})$ such that $w_nN = \phi(w_n) = \phi(a) = aN$, and then $w_n\widetilde{N} = a\widetilde{N}$ and $a\widetilde{N} \in \bigcap_{n<\omega} \widetilde{V_n} = \{\widetilde{N}\}$. Then $a \in \widetilde{N}$, so $a \in N$ and $aN = N$, as required. □

Lemma 2.9. *For a totally bounded topological group G, these conditions are equivalent.*

(a) *G is pseudocompact;*
(b) *$N \in \Lambda(G) \Rightarrow G/N$ is compact metric; and*
(c) *$N \in \Lambda(G) \Rightarrow G/N$ is compact.*

PROOF: (a) $\Rightarrow$ (b). Choose $\{\widetilde{U_n} : n < \omega\} \subseteq \mathcal{N}_{\overline{G}}(1_{\overline{G}})$ such that $N = (\bigcap_{n<\omega} \widetilde{U_n}) \cap G$, and use Lemma 2.7 to choose $\widetilde{M} \in \Lambda(\overline{G})$ such that $\widetilde{M} \subseteq \bigcap_{n<\omega} \widetilde{U_n}$ and $w(\overline{G}/\widetilde{M}) \leq \omega$. Then $\overline{G}/\widetilde{M}$ is compact and metrizable by Theorem 2.4((a) $\Rightarrow$ (c)). Let $\widetilde{\phi} : \overline{G} \twoheadrightarrow \overline{G}/\widetilde{M}$ be the natural homomorphism and set $\phi := \widetilde{\phi}|G$. Since G is G_δ-dense in $\overline{G}$ the set $\overline{\phi}[G] = \phi[G]$ is G_δ-dense in the metrizable space $\overline{G}/\widetilde{M}$, so $\phi[G] = \widetilde{G}/\overline{M}$. With $M := \widetilde{M} \cap G$ we have $\phi[G] \simeq G/\ker(\phi) = G/M$ ([25, 5.34]), so G/M is compact metric. Then G/N, the continuous image of G/M ([25, 5.35]), is also compact and metrizable [19, 3.1.22].

(b) $\Rightarrow$ (c). This is obvious.

(c) $\Rightarrow$ (a). By Theorem 2.3(a) and Lemma 2.7, it suffices to show for each $\widetilde{N} \in \Lambda(\overline{G})$ and $a \in \overline{G}$ that $a\widetilde{N} \cap G \neq \emptyset$. Again with the natural homomorphism $\widetilde{\phi} : \overline{G} \twoheadrightarrow \overline{G}/\widetilde{N}$, $N := \widetilde{N} \cap G$ and $\phi := \overline{\phi}|G$, we have $\phi[G] = \overline{G}/\widetilde{N}$ (since G is dense in $\overline{G}$ and $\phi[G] \simeq G/\ker(\phi) = G/N$ is compact by (c)). Thus there is $x \in G$ such that $\phi(x) = a\widetilde{N}$, and then $x \in a\widetilde{N} \cap G$. □

Corollary 2.10. *Let G be a pseudocompact topological group and let $N \in \Lambda(G)$. Then N is pseudocompact.*

PROOF: By Lemma 2.9((c) $\Rightarrow$(a)) it suffices to show for each $M \in \Lambda(N)$ that N/M is compact. Since M is a G_δ-set of G with $1 \in M$, there is by Lemma 2.7 $E \in \Lambda(G)$ such that $E \subseteq M$. Since G/E is compact by Lemma 2.9, its closed subgroup N/E (that is, the kernel of the canonical homomorphism $G/E \twoheadrightarrow G/N$) is compact. Then N/M, the continuous image of N/E [25, 5.35], is compact. □

Lemma 2.11. *Let G be a pseudocompact topological group such that $w(G) > \omega$. Then*

(a) $|G| \geq \mathfrak{c}$;

(b) $N \in \Lambda(G) \Rightarrow \mathrm{cl}_{\overline{G}} N = \overline{N} \in \Lambda(\overline{G})$; *and*
(c) $N \in \Lambda(G) \Rightarrow w(N) = w(G)$.

Proof: (a) We have $\{1\} \notin \Lambda(G)$ since otherwise the space $G \simeq G/\{1\}$ is compact metrizable, and then $w(G) \leq \omega$. Thus there is a sequence $\{N_n : n < \omega\} \subseteq \Lambda(G)$ such that $G = N_0 \supsetneqq \ldots \supsetneqq N_n \supsetneqq N_{n+1} \supsetneqq \ldots$. Then $N := \bigcap_{n<\omega} N_n$ satisfies $N \in \Lambda(G)$ and $|G/N| \geq \omega$. Since G is pseudocompact the set N is not open in G, so G/N is not discrete [25, 5.21], hence is dense-in-itself. A compact dense-in-itself metrizable space contains homeomorphically a copy of the usual Cantor set ([19, 4.5.5]), so $|G| \geq |G/N| \geq \mathfrak{c}$.

(b) Let $\widetilde{\phi} : \overline{G} \twoheadrightarrow \overline{G}/\overline{N}$ be the canonical homomorphism and set $\phi := \widetilde{\phi}|G$. From the topological isomorphism $\phi[G] \simeq G/\ker(\phi) = G/N$ ([25, 5.34]) and Lemma 2.9((a) $\Rightarrow$ (b)) it follows that $\phi[G]$ is compact and metrizable. Since $\phi[G]$ is dense in $\widetilde{\phi}[\overline{G}]$, we have $\phi[G] = \widetilde{\phi}[\overline{G}]$, so $\phi[G] = \widetilde{\phi}[\overline{G}] = \overline{G}/\overline{N}$ is compact and metrizable. Then $\{\overline{N}\}$ is a G_δ-set in $\overline{G}/\overline{N}$, so $\overline{N} = \widetilde{\phi}^{-1}(\{\overline{N}\}) \in \Lambda(\overline{G})$.

(c) We recall three familiar facts. (1) For compact spaces X we have $\psi(X) = \chi(X)$ ([19, 3.1.F(a)]); (2) for compact topological groups K we have $\chi(K) = w(K)$ (to see that, note that if $\{U_\eta : \eta < \chi(K)\}$ is a base at 1 and finite $F_\eta \subseteq K$ is chosen so that $K = F_\eta \cdot U_\eta$, then $\{xU_\eta : \eta < \chi(K),\, x \in F_\eta\}$ is a base for K); (3) weight does not change upon passage from a (Tychonoff) space to a dense subspace ([19, 2.1.C(a)]). It is clear further that for a closed subgroup A of a group K we have $\psi(K) \leq \psi(A) + \psi(K/A)$. From that data we conclude in the present case that

$$\omega < w(G) = w(\overline{G}) = \psi(\overline{G}) \leq \psi(\overline{N}) + \psi(\overline{G}/\overline{N}) = w(\overline{N}) + \omega = w(\overline{N}) = w(N),$$

so $w(G) \leq w(N) \leq w(G)$, as required. $\square$

Lemma 2.12. *Let $G = (G, \mathcal{T})$ be a totally bounded topological group and H a closed, normal subgroup of G. If H and G/H are pseudocompact, then G is pseudocompact.*

Proof: Suppose first that H is compact. It is enough to show that

(1) no family $\mathcal{U} \in [\mathcal{T}\backslash\{\emptyset\}]^\omega$ is locally finite in G

— that is, each such family $\mathcal{U}$ accumulates at some $x \in G$ in the sense that each neighborhood of x in G meets infinitely many of the sets $U \in \mathcal{U}$ ([19, 3.10.22]). Given such a family $\mathcal{U}$ there is, since G/H is pseudocompact and the usual map $\phi : G \twoheadrightarrow G/H$ is open, a point $p \in G$ such that $\{\phi[U] : U \in \mathcal{U}\}$ accumulates in G/H at pH. We claim then that $\mathcal{U}$ accumulates at some point $x \in pH$. If that fails then since pH is compact there is a set V, open in G, such that $pH \subseteq V$ and V meets only finitely many $U \in \mathcal{U}$. Then there is $W \in \mathcal{N}_G(1)$ such that $pH \subseteq pWH = pHW \subseteq V$ ([25, 4.10]), and then $\phi[pW]$ is a neighborhood in G/H of pH which meets $\phi[U]$ for only finitely many $U \in \mathcal{U}$. Thus statement (1) is proved.

We turn to the general case: H is assumed pseudocompact but not necessarily compact. According to Lemma 2.9((c) $\Rightarrow$ (a)) it suffices to show that G/N is compact for each $N \in \Lambda(G)$. We begin with a weaker statement:

(2) $$N \in \Lambda(G) \Rightarrow G/N \text{ is pseudocompact.}$$

Given such N, the groups G/H and $(G/(H \cap N))/(H/(H \cap N))$ are topologically isomorphic ([25, 5.35]); further, since H is pseudocompact and $H \cap N \in \Lambda(H)$, Lemma 2.9((a) $\Rightarrow$ (c)) applies to show $H/(H \cap N)$ is compact. Then by the preceding paragraph, with $G/(H \cap N)$ and $H/(H \cap N)$ replacing G and H respectively, we have that $G/(H \cap N)$ is pseudocompact. Then G/N, the continuous image of $G/(H \cap N)$ ([25, 5.35]), is pseudocompact. So (2) is proved.

Since N is a G_δ-set of G, by Lemma 2.8 there is $M \in \Lambda(G)$ such that $M \subseteq N$ and $\{M\}$ is a G_δ-set in G/M. Then (2) implies (with M replacing N) that G/M is pseudocompact, so Theorem 2.4((b) $\Rightarrow$ (c)) implies that G/M is compact (and metrizable). Then G/N, the continuous image of G/M ([25, 5.35]), is compact, as required. $\square$

Remark 2.13. Often in this paper the project of finding or constructing a proper G_δ-dense subgroup H of a hypothesized pseudocompact abelian group G proceeds in two stages: First, giving a proper G_δ-dense subgroup E of some $N \in \Lambda(G)$; then, using N and E to construct H as required. The next two lemmas fit this two-step pattern. For another instance in a different context, see steps (B) and (C) in the proof of Theorem 4.1.

Lemma 2.14. *Let G be a pseudocompact abelian group and let $G = \bigcup_{n<\omega} A_n$. Then*

(a) *there exist $N \in \Lambda(G)$, $n < \omega$, and $p \in G$ such that $A_n \cap (p+N)$ is G_δ-dense in $p + N$; and*

(b) *if $\{A_n : n < \omega\}$ is an increasing sequence of subgroups of G, the choice $p = 0$ is possible.*

Proof: (a) We assume that (a) fails, and recursively for $n < \omega$ we will define $p_n \in G$ and $N_n \in \Lambda(G)$.

Since A_0 is not G_δ-dense in G there are $p_0 \in G$ and $N_0 \in \Lambda(G)$ such that $A_0 \cap (p_0 + N_0) = \emptyset$. Suppose now that $n < \omega$ and that $p_k \in G$ and $N_k \in \Lambda(G)$ have been chosen for all $k < n$ such that

$A_k \cap (p_k + N_k) = \emptyset$ for all $k < n$, and

$(p_k + N_k) \subseteq (p_{k-1} + N_{k-1})$ when $1 \leq k < n$.

Since (by assumption) $A_n \cap (p_{n-1} + N_{n-1})$ is not G_δ-dense in $p_{n-1} + N_{n-1}$, there are $p_n \in (p_{n-1} + N_{n-1})$ and $N_n \in \Lambda(G)$ with $N_n \subseteq N_{n-1}$ such that

$(p_n + N_n) \cap A_n \cap (p_{n-1} + N_{n-1}) = \emptyset$

and hence

$$A_n \cap (p_n + N_n) = \emptyset. \tag{3}$$

The recursive construction is complete.

Each set N_n is closed in G, so from (3) and the relation $\mathrm{cl}_{\overline{G}} N_n = \overline{N_n} \in \Lambda(\overline{G})$ (Lemma 2.11(b)) it follows that

$$A_n \cap (p_n + \overline{N_n}) = \emptyset \text{ for each } n < \omega. \tag{4}$$

The family $\{p_n + \overline{N_n} : n < \omega\}$ is a descending sequence of nonempty compact G_δ-subsets of $\overline{G}$. Thus $S := \bigcap_{n<\omega} (p_n + \overline{N_n})$ is a nonempty G_δ-subset of $\overline{G}$. Then from (4) we have

$\emptyset = S \cap (\bigcup_{n<\omega} A_n) = S \cap G,$

contradicting the fact that G is G_δ-dense in $\overline{G}$ (Theorem 2.3(a)). Thus the assumption that (a) fails cannot hold, and (a) is proved.

(b) We assume without loss of generality, with p, N and n as given by (a) and replacing n if necessary by a larger integer, that $p \in A_n$. Then

$(A_n - p) \cap N = A_n \cap N$

is G_δ-dense in N, as required. □

Lemma 2.15. *Let G be a pseudocompact abelian group such that some $N \in \Lambda(G)$ has a G_δ-dense subgroup E such that $r_0(N/E) \geq \mathfrak{c}$. Then G has a G_δ-dense subgroup H such that $r_0(G/H) \geq \mathfrak{c}$.*

Proof: Let $X \in [N]^{\mathfrak{c}}$ be independent over E, let $\{X_0, X_1\}$ be a partition of X with $|X_0| = |X_1| = \mathfrak{c}$, and let $\{x_\eta + N : \eta < \lambda\}$ enumerate G/N, say with $x_0 = 0$. From Lemma 2.9(b) we have $\lambda \leq \mathfrak{c}$. By recursion on $\eta < \lambda$ we will choose $y_\eta \in X_0 \cup \{0\}$ such that

$\langle\langle X_1 \rangle\rangle \cap (\langle\langle \{x_\xi + y_\xi : \xi \leq \eta\} \rangle\rangle + E) = \{0\}.$

Let $y_0 = 0$, for $\eta < \lambda$ suppose that y_ξ has been defined for all $\xi < \eta$, set

$S_\eta := \langle\langle x_\xi + y_\xi : \xi < \eta \rangle\rangle$

and note that $|S_\eta| < \mathfrak{c}$ and (from the inductive hypothesis) that

$\langle\langle X_1 \rangle\rangle \cap (S_\eta + E) = \{0\}.$

We claim that there is $y_\eta \in X_0$ such that

$\langle\langle X_1 \rangle\rangle \cap (\langle\langle S_\eta \cup \{x_\eta + y_\eta\} \rangle\rangle + E) = \{0\}.$

If the claim fails then for each $y \in X_0$ there are $r_y \in \langle\langle X_1 \rangle\rangle$, $s_y \in S_\eta$, $n_y \in \mathbb{Z}\backslash\{0\}$, and $e_y \in E$ such that

$$r_y = s_y + n_y(x_\eta + y) + e_y \neq 0, \tag{5}$$

and then since $|X_0| = \mathfrak{c}$ there are distinct $y, y' \in X_0$, $n \in \mathbb{Z}\backslash\{0\}$, and $s \in S_\eta$ such that

$$n = n_y = n_{y'} \text{ and } s = s_y = s_{y'}.$$

From the independence of X_0 over E (and $0 \in E$) we have $n(y - y') \neq 0$, so (5) gives

$$\emptyset \neq n(y - y') = r_y - r_{y'} - e_y + e_{y'} \in \langle\langle X_0 \rangle\rangle \cap (\langle\langle X_1 \rangle\rangle + E),$$

contradicting the hypothesis that X is independent over E.

The claim is proved and y_η is defined for all $\eta < \lambda$. Writing

$$S := \bigcup_{\eta<\lambda} S_\eta \text{ and then } H := S + E$$

we have

$$\langle\langle X_1 \rangle\rangle \cap H = \{0\}$$

and hence

$$r_0(G/H) \geq |X_1| = \mathfrak{c}.$$

To see that H is G_δ-dense in G, note that each nonempty G_δ-set $A \subseteq G$ meets some coset of the form $x_\eta + N$; then A meets $x_\eta + E$ and we have

$$\emptyset \neq A \cap (x_\eta + E) \subseteq A \cap H,$$

as required. □

Theorem 2.16. *Let $G = (G, \mathcal{T})$ be a pseudocompact abelian group. If G contains a dense pseudocompact subgroup H such that G/H can be mapped homomorphically onto some non-degenerate compact group, then G is not r-extremal.*

Proof: We are to find a pseudocompact group topology on G which strictly refines $\mathcal{T}$.

Let $\pi : G \twoheadrightarrow G/H$ be the natural homomorphism, let $h : G/H \twoheadrightarrow K$ be a homomorphism with K a non-degenerate compact (abelian) group, and using [25, 22.12, 22.17] let $1 \neq \chi \in \widehat{K}$. Write $\psi := \chi \circ h \circ \pi : G \twoheadrightarrow S \subseteq \mathbb{T}$ (with S a compact subgroup of $\mathbb{T}$, $|S| > 1$) and define $\phi : G \to G \times S$ by $\phi(x) := (x, \psi(x))$. From Theorem 2.3(b) the topological group $(G, \mathcal{T}) \times S$ is pseudocompact. We claim that $\widetilde{G} := \mathrm{graph}(\phi)$ is G_δ-dense in $(G, \mathcal{T}) \times S$. Indeed each nonempty G_δ-set $A \subseteq (G, \mathcal{T}) \times S$ contains a set of the form $B \times \{t\}$ with B a nonempty G_δ set in $(G, \mathcal{T})$ and with $t \in S$, and since $H \subseteq \ker(\psi) = \psi^{-1}(\{1_\mathbb{T}\})$ is G_δ-dense in $(G, \mathcal{T})$ by Theorem 2.3(a), also $\psi^{-1}(\{t\})$ is G_δ-dense in $(G, \mathcal{T})$ and hence $\emptyset \neq \widetilde{G} \cap B \subseteq \widetilde{G} \cap A$. It follows from Theorem 2.3(a) that $\widetilde{G}$ is pseudocompact in the topology inherited from $(G, \mathcal{T}) \times S$. Then the topology $\mathcal{U}$ on G, given by the requirement that $\phi : (G, \mathcal{U}) \twoheadrightarrow \widetilde{G} \subseteq (G, \mathcal{T}) \times S$ is a homeomorphism, is a pseudocompact group topology on G. Since $\widetilde{G}$ is a proper dense subgroup of $(G, \mathcal{T}) \times S$, it is not closed there, so ϕ is not continuous and the containment $\mathcal{U} \supseteq \mathcal{T}$ is proper, as required. □

We conclude this section of preliminaries with four lemmas: two of algebraic flavor, one combinatorial, one topological.

Lemma 2.17. *Let G be a pseudocompact abelian group. If $r_0(G) < \mathfrak{c}$ then G is a torsion group, i.e., $r_0(G) = 0$.*

Proof: Suppose first that G is compact. If some connected subspace of G contains distinct points x and y, then since $\widehat{G}$ separates points ([25, 22.17]) there is $\chi \in \widehat{G}$ such that $\chi(x-y) \neq 1$; then $\chi[G]$, a nondegenerate connected subgroup of $\mathbb{T}$, satisfies $\chi[G] = \mathbb{T}$ and we have $r_0(G) \geq r_0(\mathbb{T}) = \mathfrak{c}$. If G contains no such connected subspace then, being (locally) compact, G is *zero-dimensional* in the sense that its topology has a basis of clopen sets (see [25, 3.5] or [19, 6.2.9]) and it follows from the structure theorem for zero-dimensional compact abelian groups ([25, 25.22]) that G, since it satisfies $r_0(G) > 0$, contains (both algebraically and topologically) either (1) for some prime p a copy of the set Δ_p of p-adic integers or (2) a product of the form $K = \Pi_{n<\omega}\mathbb{Z}(p_n)$ with $(p_n)_n$ a sequence of distinct primes; since Δ_p is a torsion group (see [25, 10.2] or [20, pp. 17-18] for the definition of addition in Δ_p), we have then $r_0(G) \geq |\Delta_p| = \mathfrak{c}$ or $r_0(G) \geq |r_0(K)| = \mathfrak{c}$.

Now let G be arbitrary (not necessarily compact), and suppose that G contains algebraically a copy Z of $\mathbb{Z}$. Choosing $N \in \Lambda(G)$ such that $Z \cap N = \{0\}$, we have with the standard quotient map $\pi : G \twoheadrightarrow G/N$ that $\pi(Z)$ is isomorphic to Z, so the group G/N, which is compact by Theorem 2.9, satisfies $0 < r_0(G/N) < \mathfrak{c}$. This contradicts the 'compact' case already treated. □

Lemma 2.18. *Let G be a pseudocompact abelian torsion group. Then G is of bounded order.*

Proof: For $0 < n < \omega$ let $G(n) := \{x \in G : nx = 0\}$. Since $G = \bigcup_{0<n<\omega} G(n)$, G is pseudocompact and each $G(n)$ is closed, some $G(n)$ has nonempty interior in G ([19, 3.10.F(e)]). Then $G(n)$ is open in G ([25, 5.5]), hence closed, so $G/G(n)$ is pseudocompact and discrete ([25, 5.26]), hence finite. If $|G/G(n)| = m < \omega$ then $nmx = 0$ for each $x \in G$. □

Lemma 2.19. *Let $\kappa \geq \omega$ and let $\mathcal{A} \subseteq \mathcal{P}(2^\kappa)$ satisfy*

(i) *$\mathcal{B} \in [\mathcal{A}]^{\leq\kappa} \Rightarrow \bigcap\mathcal{B} \in \mathcal{A}$; and*

(ii) *$A \in \mathcal{A} \Rightarrow |A| = 2^\kappa$.*

Then there is a countable partition $\{I_n : n < \omega\}$ of 2^κ such that $|A \cap I_n| = 2^\kappa$ for each $A \in \mathcal{A}$, $n < \omega$.

Proof: Let $2^\kappa = \{0,1\}^\kappa$ have the usual (compact) Tychonoff topology, let $\mathcal{U}$ be the set of its clopen subsets, and let

$$\mathcal{V} := \{U \in \mathcal{U} : \text{ there is } A(U) \in \mathcal{A} \text{ such that } |U \cap A(U)| < 2^\kappa\}.$$

Let $\mathcal{B} := \{A(U) : U \in \mathcal{V}\}$. Then $\mathcal{B} \subseteq \mathcal{A}$ and $|\mathcal{B}| \leq |\mathcal{V}| \leq |\mathcal{U}| = \kappa$, so $\mathcal{B} \in [\mathcal{A}]^{\leq\kappa}$ and hence $\bigcap\mathcal{B} \in \mathcal{A}$ and $|\bigcap\mathcal{B}| = 2^\kappa$ by (i) and (ii).

Since $|(\bigcap\mathcal{B}) \cap U| \leq |A(U) \cap U| < 2^\kappa$ for each $U \in \mathcal{V}$ and $|\mathcal{V}| \leq \kappa < \mathrm{cf}(2^\kappa)$, we have

$|(\bigcap \mathcal{B}) \cap (\bigcup \mathcal{V})| < 2^\kappa$,

hence with $X := (\bigcap \mathcal{B})\backslash(\bigcup \mathcal{V})$ we have $|X| = 2^\kappa$.

As is the case for every subset of 2^κ of cardinality 2^κ, there is (for X) a pairwise disjoint sequence $\{U_n : n < \omega\} \subseteq \mathcal{U}$ such that $X \cap U_n \neq \emptyset$ for each $n < \omega$. (In detail: Let $U_0 \in \mathcal{U}$ satisfy $U_0 \cap X \neq \emptyset$ and $|X\backslash U_0| = 2^\kappa$. Then recursively for $n < \omega$ if pairwise disjoint $U_k \in \mathcal{U}$ have been defined for $k < n$, choose $U_n \in \mathcal{U}$ so that $X \cap U_n \neq \emptyset$, $U_n \cap (\bigcup_{k<n} U_k) = \emptyset$ and $|X\backslash(\bigcup_{k\leq n} U_k)| = 2^\kappa$.)

With $\{U_n : n < \omega\}$ so defined, it is clear that $|A \cap U_n| = 2^\kappa$ for each $A \in \mathcal{A}$, $n < \omega$. Indeed if that fails for some A and n, then $U_n \in \mathcal{V}$ and we have the contradiction

$$\emptyset \neq X \cap U_n = ((\bigcap \mathcal{B})\backslash(\bigcup \mathcal{V})) \cap U_n \subseteq U_n\backslash \bigcup \mathcal{V} \subseteq U_n\backslash U_n = \emptyset.$$

To complete the proof, set

$$I_n := U_n \text{ for } 0 < n < \omega,\ I_0 := 2^\kappa\backslash \bigcup\{U_n : 0 < n < \omega\}. \qquad \square$$

Notation 2.20. For $n \in \mathbb{Z}$ and an abelian group G, let
$G^{(n)} := \{nx : x \in G\}$.

Remark 2.21. It is worth remarking that for $N \in \Lambda(G)$ and $0 \neq n \in \mathbb{Z}$ the relation $N^{(n)} \in \Lambda(G)$ can fail — indeed even with $N = G$ and $N^{(n)} = G^{(n)}$ compact. For an example, take $G = (\mathbb{Z}(n))^\kappa$ with $\kappa > \omega$; then $G^{(n)} = \{0\}$, but $\{0\}$ is not a G_δ-set in G.

Lemma 2.22. *Let G be a pseudocompact abelian group, and let $N \in \Lambda(G)$ and $n \in \mathbb{Z}$. Then*

(a) $\overline{N}^{(n)} \in \Lambda(\overline{G}^{(n)})$;
(b) $\mathrm{cl}_G(N^{(n)}) = \overline{N}^{(n)} \cap G$; *and*
(c) $\mathrm{cl}_G(N^{(n)}) \in \Lambda(\mathrm{cl}_G(G^{(n)}))$.

PROOF: (a) The map $f : \overline{G} \twoheadrightarrow \overline{G}^{(n)}$ given by $x \mapsto nx$ is a continuous surjective homomorphism between compact groups, hence is open ([25, 5.29]). This implies, again by compactness, that the image under f of a closed G_δ-subset of $\overline{G}$ is a closed G_δ-subset of $\overline{G}^{(n)}$. Then since $\overline{N} \in \Lambda(\overline{G})$ by Lemma 2.11(b), we have $\overline{N}^{(n)} \in \Lambda(\overline{G}^{(n)})$.

(b) Since $N^{(n)}$ is dense in $\overline{N}^{(n)}$, we have $\overline{N^{(n)}} = \overline{N}^{(n)}$, so using Remark 1.4(b)

$$\overline{N}^{(n)} \cap G = \overline{N^{(n)}} \cap G = (\mathrm{cl}_{\overline{G}}\, N^{(n)}) \cap G = \mathrm{cl}_G\, N^{(n)}.$$

(c) From (a) we have $\overline{N}^{(n)} \cap G \in \Lambda(\overline{G}^{(n)} \cap G)$,
and (b) then gives

$$\mathrm{cl}_G(N^{(n)}) \in \Lambda(\overline{G}^{(n)} \cap G). \tag{6}$$

Obviously, we have

$$\mathrm{cl}_G(N^{(n)}) \subseteq \mathrm{cl}_G(G^{(n)}) \subseteq \overline{G}^{(n)} \cap G, \tag{7}$$

and (c) is immediate from (6) and (7). □

Corollary 2.23. *Let G be a pseudocompact abelian group and let $n_0, n_1 \in \mathbb{Z}\backslash\{0\}$. If $\mathrm{cl}_G(G^{(n_0)}) \in \Lambda(G)$ and $\mathrm{cl}_G(G^{(n_1)}) \in \Lambda(G)$ then $\mathrm{cl}_G(G^{(n_0 n_1)}) \in \Lambda(G)$.*

Proof: From Lemma 2.11(b) we have $\overline{G^{(n_0)}} \in \Lambda(\overline{G})$ and $\overline{G^{(n_1)}} \in \Lambda(\overline{G})$, and then from Lemma 2.22(a) it follows that

$$\overline{G^{(n_0)}}^{(n_1)} \in \Lambda(\overline{G^{(n_1)}}) \subseteq \Lambda(\overline{G}). \tag{8}$$

Since $G^{(n_0 n_1)} = (G^{(n_0)})^{(n_1)}$ is dense in the compact space $\overline{G^{(n_0)}}^{(n_1)}$ we have

$$\overline{G^{(n_0)}}^{(n_1)} = \overline{G^{(n_0 n_1)}},$$

so (8) gives $\overline{G^{(n_0 n_1)}} \in \Lambda(\overline{G})$; then from Lemma 2.22(b) (taking $N = G$ and $n = n_0 n_1$ there) it follows that

$$\mathrm{cl}_G(G^{(n_0 n_1)}) = \overline{G^{(n_0 n_1)}} \cap G \in \Lambda(G),$$

as required. □

3. The construction: preliminaries

In the interest of efficiency, we will use some helpful *ad hoc* notation. We describe this now.

Notation and Discussion 3.1. Throughout this section, G will be a pseudocompact abelian group, $\{H_i : i \in I\}$ a set of nontrivial abelian groups, and T an abelian group. We set

$$H := \bigoplus_{i\in I} H_i \oplus T.$$

For $i \in I$ we denote by $\pi_i : H \twoheadrightarrow H_i$ the projection, similarly by $\pi_T : H \twoheadrightarrow T$ the projection.

For $x \in H$ we denote by $s(x)$ the *I-support* of x, i.e., $s(x) := \{i \in I : x_i \neq 0_i\}$; and for $E \subseteq H$ we set $s(E) := \bigcup_{x\in E} s(x)$.

With those conventions, we assume

(1) $G \subseteq H$;
(2) $\pi_i[G] = H_i$ for each $i \in I$; and,
(3) $|s(N)| \geq \mathfrak{c}$ for each $N \in \Lambda(G)$.

Finally for $\emptyset \neq A \subseteq I$ we set

$$G(A) := G \cap (\bigoplus_{i\in A} H_i \oplus \bigoplus_{i\in I\backslash A} \{0_i\} \oplus T) = \{x \in G : s(x) \subseteq A\},$$

and then

$$\mathcal{A} := \{A \subseteq I : G(A) \text{ contains an element of } \Lambda(G)\}.$$

Lemma 3.2. *Assume the conventions of 3.1, and let $A \in \mathcal{A}$. Then*

(a) $|A| \geq \mathfrak{c}$*; and*
(b) $|I \setminus A| \leq \mathfrak{c}$.

Proof: Given A, fix $N \in \Lambda(G)$ such that $N \subseteq G(A)$.

(a) $s(N) \subseteq A$, so $|A| \geq \mathfrak{c}$ by (3).

(b) Since $|G/N| \leq \mathfrak{c}$ by Lemma 2.9(b), there is $E \subseteq G$ such that $|E| \leq \mathfrak{c}$ and $G = E + N$. For each $i \in I \backslash s(N)$, by (2) of 3.1 there is $x(i) \in G$ such that $x(i)_i \neq 0_i$, say $x(i) = e(i) + y(i)$ with $e(i) \in E$ and $y(i) \in N$. Since $y(i)_i = 0_i$ we have $x(i)_i = e(i)_i \neq 0_i$, so $i \in s(E)$. This shows

$$I \backslash s(N) \subseteq s(E) = \bigcup_{e \in E} s(e)$$

with $|E| \leq \mathfrak{c}$ and each $|s(e)| < \omega$, so

$$|I \backslash s(N)| \leq |s(E)| \leq \mathfrak{c},$$

as required. □

Corollary 3.3. *There is a countable partition $\mathcal{I} = \{I_n : n < \omega\}$ of I such that $|A \cap I_n| \geq \mathfrak{c}$ for each $A \in \mathcal{A}$ and $n < \omega$.*

Proof: Since $\Lambda(G)$ is closed under countable intersections, the same is true of the family $\mathcal{A}$. Furthermore $\mathcal{A} \neq \emptyset$ since $G \in \Lambda(G)$, and with $A \in \mathcal{A}$ we have $|I| \geq |A| \geq \mathfrak{c}$ by Lemma 3.2(a).

If $|I| = \mathfrak{c}$ the required statement is then immediate from (the case $\kappa = \omega$ of) Lemma 2.19. If $|I| > \mathfrak{c}$ let $\mathcal{I} = \{I_n : n < \omega\}$ be an arbitrary partition of I with each $|I_n| > \mathfrak{c}$, and note that if some $A \in \mathcal{A}$ and $n < \omega$ satisfy $|A \cap I_n| < \mathfrak{c}$ then

$$|I \backslash A| \geq |I_n \backslash A| = |I_n| > \mathfrak{c},$$

contrary to Lemma 3.2(b). □

Lemma 3.4. *Continue the notation of* (3.1)–(3.3), *and for $n < \omega$ set*

$$J_n := \bigcup_{i \leq n} I_i \text{ and } G_n := G(J_n).$$

Then there exist $N \in \Lambda(G)$ and $n < \omega$ such that $G_n \cap N$ is a proper G_δ-dense subgroup of N.

Proof: The sequence $\{G_n : n < \omega\}$ is an increasing sequence of subgroups of G such that $G = \bigcup_{n<\omega} G_n$, so by Lemma 2.14 there exist $N \in \Lambda(G)$ and $n < \omega$ such that $G_n \cap N$ is G_δ-dense in N. The inclusion $G_n \cap N \subseteq N$ is proper, since otherwise the relation

$$N \subseteq G_n = G(J_n)$$

would give $J_n \in \mathcal{A}$ and hence $|J_n \cap I_{n+1}| \geq \mathfrak{c}$, contradicting the fact that $\mathcal{I}$ is a partition of I. □

4. Proof of Theorem 1.1

With the requisite tools assembled, we proceed to the proof of Theorem 1.1. This partitions into two cases, depending on whether or not some $N \in \Lambda(G)$ is a torsion group. We treat these separately, in each case referring crucially to the context created in Discussion 3.1.

Theorem 4.1. *Let $G = (G, \mathcal{T})$ be an abelian, nonmetrizable pseudocompact group. If some $N \in \Lambda(G)$ is a torsion group, then G is neither r- nor s-extremal.*

Proof: We proceed *via* three statements, here called (A), (B) and (C).

(A) There is $p \in \mathbb{P}$ such that the group $\widetilde{G} := G/(\mathrm{cl}_G(G^{(p)}))$ (with the usual quotient topology) is a nonmetrizable pseudocompact group.

(B) With p and $\widetilde{G}$ as in (A), the group $\widetilde{G}$ has a dense, pseudocompact subgroup K such that $|\widetilde{G}/K| = p$.

(C) With p as in (A), G has a dense pseudocompact subgroup H such that $|G/H| = p$.

We prove (A). As with every pseudocompact abelian torsion group, N is of bounded order (Lemma 2.18), so there is an integer $n \neq 0$ such that $\mathrm{cl}_N(N^{(n)}) = N^{(n)} = \{0_N\}$. Since $w(N) = w(G) > \omega$ (Lemma 2.11(c)), the set $\{0\}$ is not a G_δ-set in N (Theorem 2.4), so $\mathrm{cl}_N(N^{(n)}) = N^{(n)} \notin \Lambda(N)$. It then follows from Theorem 2.23 that there is $p \in \mathbb{P}$ (with p a prime divisor of n) such that $\mathrm{cl}_N(N^{(p)}) \notin \Lambda(N)$. Then $\mathrm{cl}_G(G^{(p)}) \notin \Lambda(G)$, for otherwise from Lemma 2.22(c) we would have

$$\mathrm{cl}_G(N^{(p)}) \in \Lambda(\mathrm{cl}_G(G^{(p)})) \subseteq \Lambda(G)$$

and then the contradiction

$$\mathrm{cl}_N(N^{(p)}) = \mathrm{cl}_G(N^{(p)}) = \mathrm{cl}_G(N^{(p)}) \cap N \in \Lambda(N).$$

Note that since $\mathrm{cl}_G(G^{(p)})$ is closed in G, the group $\widetilde{G} := G/(\mathrm{cl}_G(G^{(p)}))$ is indeed a (Hausdorff) topological group ([25, 5.26]). Since $\mathrm{cl}_G(G^{(p)})$ is not a G_δ-set in G, the group $\widetilde{G}$, being pseudocompact, is nonmetrizable (Theorem 2.4).

We prove (B). As with every abelian group G such that $G^{(p)} = \{0\}$, $\widetilde{G}$ has algebraically the form $\widetilde{G} = \bigoplus_{i \in I} \mathbb{Z}(p)_i$ with each $\mathbb{Z}(p)_i = \mathbb{Z}(p) = \mathbb{Z}/pZ$ ([20, 8.5]). Then conditions (1) and (2) of 3.1 are satisfied (with $\widetilde{G}$ here in the role of G there and $H = \widetilde{G} + T$ with $T = \{0\}$); condition (3) also is satisfied since each $N \in \Lambda(G)$ is pseudocompact (Corollary 2.10), hence satisfies $|N| \geq \mathfrak{c}$ by Lemma 2.11(a). Thus by Lemma 3.4 some $M \in \Lambda(\widetilde{G})$ has a proper G_δ-dense subgroup, say E. Replacing E if necessary by a larger maximal proper subgroup of M, we may assume $|M/E| = p$. Fixing $x \in M \backslash E$, we note that

$$\{E, x + E, 2x + E, \ldots, (p-1)x + E\}$$

enumerates the cosets of E in M; and that each such coset $kx + E$ is G_δ-dense in M. Now choose a (necessarily discontinuous) homomorphism $h : \widetilde{G} \to \mathbb{T}$ such that $h \equiv 1$ on E, $h(x) \neq 1$, say $h(x) = t \in \mathbb{Z}(p) \subseteq \mathbb{T}$. Continuing our algebraic convention (additive notation for generic abelian groups, multiplicative notation for $\mathbb{T}$ and its subgroups) we write $K := \ker(h) = h^{-1}(\{1\})$ and we claim that K is G_δ-dense in $\widetilde{G}$.

Let A be a nonempty G_δ-subset of $\widetilde{G}$, choose $a \in A$, say with $h(a) = t^k \in \mathbb{Z}(p)$, and note that since $a \in A \cap (a+M)$, the set $A \cap (a+M)$ is a nonempty G_δ-subset of $a+M$. Now $-kx+E$ is G_δ-dense in M, so $a-kx+E$ is G_δ-dense in $a+M$; hence

$$(A \cap (a+M)) \cap (a-kx+E) \neq \emptyset.$$

Since $h(a-kx) = t^k \cdot t^{-k} = 1$ and $h \equiv 1$ on E, we have $a-kx+E \subseteq K$ and therefore

$$\emptyset \neq (A \cap (a+M)) \cap (a-kx+E) \subseteq (A \cap (a-kx+E)) \subseteq A \cap K,$$

as asserted. It follows from Theorem 2.3(a) that K, a G_δ-dense subgroup of the pseudocompact group G, is itself pseudocompact.

We prove (C). With p, $\widetilde{G}$ and K as in (B), let $\phi : G \twoheadrightarrow \widetilde{G}$ be the natural projection, recall that ϕ is both a continuous and an open map ([25, 5.16 and 5.17]), and set $H := \phi^{-1}(K)$. If H were closed in G it would be open (since $|G/H| < \omega$), and then $\phi[H] = K$ would be open in $\widetilde{G}$ and hence closed in $\widetilde{G}$, contrary to the fact that K is a proper, dense subgroup of $\widetilde{G}$. So H is not closed in G and therefore, since no subgroup of G lies properly between G and H, H is dense in G.

To see that H is pseudocompact, let $\phi|H$ denote the restriction of ϕ to H, and using the relation $H \supseteq \mathrm{cl}_G(G^{(p)})$ write

$$K = \phi[H] = (\phi|H)[H] \simeq H/\ker(\phi|H) = H/\ker(\phi) = H/\mathrm{cl}_G(G^{(p)});$$

here the topological isomorphism indicated by the symbol $\simeq$ is given by [25, 5.27].

Clearly the group $G^{(p)}$, and hence $\mathrm{cl}_G(G^{(p)})$, is pseudocompact; and K is pseudocompact by (B). Identifying the topological groups K and $H/\mathrm{cl}_G(G^{(p)})$, we deduce then from Lemma 2.12, replacing there G by H and H by $\mathrm{cl}_G(G^{(p)})$, that H is pseudocompact.

Thus (C) is proved.

As is clear from (C), G is not s-extremal.

Let $\mathcal{U}$ be the smallest topology on G containing the hypothesized pseudocompact topology $\mathcal{T}$ such that H and each of its cosets in G is $\mathcal{U}$-clopen. (In detail: For $U \subseteq G$ we have $U \in \mathcal{U}$ if and only if

$$x \in G \Rightarrow U \cap (x+H) \text{ is open in } (x+H, \mathcal{T}).)$$

Then $(G, \mathcal{U})$, the union of finitely many (pairwise disjoint) pseudocompact subspaces (that is, the cosets of H in G), is itself pseudocompact. The inclusion $\mathcal{T} \subseteq \mathcal{U}$ is proper since $H \in \mathcal{U} \setminus \mathcal{T}$, and we conclude that $G = (G, \mathcal{T})$ is not r-extremal. □

Theorem 4.2. *Let G be an abelian, nonmetrizable pseudocompact group. If no $N \in \Lambda(G)$ is a torsion group, then G is neither r- nor s-extremal.*

Proof: Let H denote the divisible hull of G ([25, A.15-A.17], [20, 24.1–24.4]). We have then

$G \subseteq H := \bigoplus_{i\in I} \mathbb{Q}_i \oplus tH$,

with $|I| = r_0(G)$, each $\mathbb{Q}_i$ being a copy of the group $\mathbb{Q}$ of rational numbers, and tH the torsion subgroup of H ([25, A.14] or [20, 23.1]). Again it is clear that conditions (1) and (2) of 3.1 hold. Condition (3) also holds: each $N \in \Lambda(G)$ is pseudocompact (Theorem 2.10) and is not a torsion group, so $r_0(N) \geq \mathfrak{c}$ by Theorem 2.17 and then necessarily (in the notation of 3.1) one has $|s(N)| \geq \mathfrak{c}$.

Let $\{I_n : n < \omega\}$, N and n be as constructed in §3. We assume without loss of generality that $n = 0$, and we set $E := G_n \cap N$.

We claim that $r_0(N/E) \geq \mathfrak{c}$. To prove that, we show that there is $X \in [N]^{\mathfrak{c}}$, independent over E, such that each $x \in X$ has infinite order.

To construct X, continuing the notation of §3, we use transfinite induction to choose for $\eta < \mathfrak{c}$ a point $x_\eta \in N$ such that

$$x_\eta \notin G\Big(I_0 \cup \bigcup_{\xi<\eta} s(x_\xi)\Big).$$

Let $0 \leq \eta < \mathfrak{c}$, assume that x_ξ has been defined for all $\xi < \eta$, and set

$$W_\eta := I_0 \cup \bigcup_{\xi<\eta} s(x_\xi).$$

Then

$$|I_1 \cap W_\eta| = \Big|I_1 \cap \bigcup_{\xi<\eta} s(x_\xi)\Big| < \mathfrak{c}.$$

Since $|I_1 \cap A| \geq \mathfrak{c}$ for each $A \in \mathcal{A}$ we have $W_\eta \notin \mathcal{A}$, so the containment $N \subseteq G(W_\eta)$ fails and there is $x_\eta \in N \setminus G(W_\eta)$. This completes the transfinite construction.

To see that $X = \{x_\eta : \eta < \mathfrak{c}\}$ is as required, let $\eta_1 < \eta_2 < \cdots < \eta_m < \mathfrak{c}$, and $n_1, n_2, \cdots n_m \in \mathbb{Z} \setminus \{0\}$ (repetitions allowed, possibly $m = 1$). By construction we have

$$x_{\eta_m} \notin G\Big(I_0 \cup \bigcup_{\xi<\eta_m} s(x_\xi)\Big),$$

so the inclusion $s(x_{\eta_m}) \subseteq I_0 \cup \bigcup_{\xi<\eta_m} s(x_\xi)$ fails; let $\bar{\imath} \in s(x_\eta)$ witness that failure. Then $(x_{\eta_m})_{\bar{\imath}} \neq 0_{\bar{\imath}}$ and $(x_\xi + e)_{\bar{\imath}} = 0_{\bar{\imath}}$ for every $\xi < \eta_m$ and $e \in E$. Then clearly x_{η_m} has infinite order, and

$$n_m x_{\eta_m} \notin \langle\!\langle \{x_\xi : \xi < \eta\} \rangle\!\rangle + E,$$

as required.

Since $r_0(N/E) \geq \mathfrak{c}$, we have from Lemma 2.15 that there is a (proper) G_δ-dense pseudocompact subgroup M of G such that $r_0(G/M) \geq \mathfrak{c}$.

That proves that G is not s-extremal. To see that G is not r-extremal, note from the relation $r_0(G/M) \geq \mathfrak{c}$ that G/M contains a subgroup F which is isomorphic to $\bigoplus_{\eta<\mathfrak{c}} \mathbb{Z}_\eta$. There is a surjective homomorphism $\phi : F \twoheadrightarrow \mathbb{T}$, and since $\mathbb{T}$ is divisible the map ϕ extends to a homomorphism $h : G/M \twoheadrightarrow \mathbb{T}$ ([25, Theorem A.7]). Theorem 2.16 then applies to show that G is not r-extremal. □

Finally in the interest of completeness we restate Theorem 1.1.

Theorem 4.3. *Let G be an abelian, nonmetrizable pseudocompact group. Then G is neither r- nor s-extremal.*

Proof: The statement is immediate from Theorems 4.1 and 4.2. □

5. References to the literature

Pseudocompact spaces were introduced and defined as in Definition 2.1 by Hewitt [24]. He showed *inter alia* that a (Tychonoff) space is pseudocompact if and only if it is G_δ-dense in its Stone-Čech compactification, hence in every Tychonoff space in which it is densely embedded. Pseudocompact spaces were subsequently characterized by Glicksberg [22] as those in which each locally finite family of open subsets is finite. (See also [19, 3.10.22] for a verification of this equivalence.) The authors of [13] used Glicksberg's characterization *en route* to the characterization given in Theorem 1.3(b). For topological groups G, this latter criterion has proved so useful that for practical purposes many authors have adopted it as the definition of pseudocompactness, suppressing completely all mention of the function ring $C(G, \mathbb{R})$ and locally finite families.

So far as the authors can determine, the terms *r-extremal* and *s-extremal* were introduced and defined explicitly (as in Definition 2.2) in [3], but the term *extremal* had been used in this general context already in [12]. The authors there conjectured Theorem 1.1 in full generality ([12, p. 25]), and they gave the proof for pseudocompact abelian torsion groups. (Indeed, as indicated in [12, 7.3], the "torsion" hypothesis may be weakened to "zero-dimensional".)

Together with Gladdines [7, 4.5], we showed subsequently that no pseudocompact abelian group G such that $r_0(G) > \mathfrak{c}$ is s-extremal; extending the analysis of [7], Comfort and Galindo showed in addition that such groups G are not r-extremal [5, 5.10], [21, 7.3]. In our initial proof [9] of Theorem 1.1 we simply cited these results, thus achieving license to restrict attention to the case of groups with nontrivial connected subspaces. (For clarity, we reiterate the principal contribution of the present paper: To give a unified, self-contained proof of Theorem 1.1 which is not dependent upon treating separately *ab initio* the zero-dimensional and the connected cases.)

Theorem 2.3 was established in [13] in company with the additional result (not needed here) that a totally bounded topological group G is pseudocompact if and only if $\overline{G} = \beta G$. The proof of these results in [13] proceeds by using work of Kakutani and Kodaira [27], Halmos [23, §64], and Ross and Stromberg [29] to show that for every Baire set E in a compact topological group K there is $N \in \Lambda(K)$ such that $E = EN$. A more direct approach to these theorems of [13], avoiding any appeal to the works [27], [23] and [29], was given subsequently by de Vries [30]; the treatment of these matters in the expository paper [2] invokes and duplicates the essentials of de Vries' argument.

The literature contains generalizations of Theorems 1.3 and 2.3 which reach far beyond the boundaries anticipated in [13]; some of these more recent developments are treated by Arhangel'skii and Tkachenko [1, Chapter 6].

It may be remarked concerning Theorem 2.3 that only (a) requires serious proof, since when (a) is known then (b) follows from the uniqueness of the Weil completion of a totally bounded group.

The fact that a pseudocompact metrizable space is compact, as in Theorem 2.4, follows from [25, 8.3] and [19, 8.5.13(c)]. Alternatively one may argue, as in the proof of [12, 2.3], in the wider context of topological spaces which are not necessarily topological groups, that such a space, being metrizable, is normal [19, 4.1.13]; and then, being pseudocompact and normal, it is countably compact [19, 3.10.21]; and a countably compact metrizable space is compact [19, 3.11.1 and 3.11.3].

The fact that a pseudocompact group in which $\{0\}$ is a G_δ-set must be (compact and) metrizable was apparently first noted in [14, 3.1]; the proof there invokes from [13] the fact that $\overline{G} = \beta G$ for such G. The present proof that (b) $\Rightarrow$ (c) in Theorem 2.4 seems more direct.

Corollary 2.5, reducing the study of r- and s-extremal pseudocompact abelian groups G to the case $w(G) > 0$, is given in [12, 2.4 and 3.6, respectively].

Lemma 2.7 is established in [27]; see also [25, 8.7].

Lemma 2.8 is given in [12, 1.6(b)].

The conclusion of Lemma 2.9 is given in [14, 3.3] under weaker hypotheses. See also [12, 6.1].

Corollary 2.10 is from [12, 6.2].

Van Douwen [18] proved that every infinite pseudocompact space X without isolated points satisfies $|X| \geq \mathfrak{c}$. Lemma 2.11(a), a weaker result, was established in [11, 2.5(a)] using Lemma 2.7 and elementary properties of Haar measure on the group $\overline{G}$. The present proof of Lemma 2.11(a) is taken from [8, 2.12(a)]; part (b) is from [8, 2.7(c)], given here with a more direct proof; part (c) is from [8, 2.7(e)].

Lemma 2.12 is given in [12, Theorem 6.3(c)]. This proof is new.

Lemma 2.14 is from [8, 2.13(b),(c)]. The proof given there depended on the fact that not only every pseudocompact group G, but also G in its so-called P-space modification (that is, G in the smallest group topology in which each $N \in \Lambda(G)$ is open) has the property that the intersection of countably many dense open subsets remains dense ([11, 2.4]).

We proved Lemma 2.15 in [9, 3.1]. A more general result, developed independently, is given by Dikranjan, Giordano Bruno and Milan [17, 4.11]. That paper takes as its point of departure the familiar result given here as Corollary 2.5, namely that pseudocompact metrizable groups are both r- and s-extremal. That fact presents metrizability as a strong form of extremality, thus motivating the authors of [17] to define and investigate weak forms of metrizability and their relation to new extremality concepts.

Theorem 2.16 is proved in [5, 4.4(a)], [21, 3.7.1], where the authors resolved this question ([5, 3.10], [21, 3.4, 3.6]): Given a pseudocompact abelian group $(G, \mathcal{T})$, for which (discontinuous) $h \in \mathrm{Hom}(G, \mathbb{T})$ is the smallest topology $\mathcal{U}$ refining $\mathcal{T}$ and making h continuous also pseudocompact? The use of an isomorphism $\phi : G \to (G, \mathcal{T}) \times K$ as in the proof of Theorem 2.16 to define a strict pseudocompact refinement $\mathcal{U}$ of $\mathcal{T}$ goes back (at least) to [10, 3.3].

Lemma 2.17 is familiar for compact abelian groups [15, 2.3], [6, 4.1]. The extension to pseudocompact groups was noted in [8, 2.17], with a brief indication of proof. See also [17, 2.14], and see [16, 2.17 and 3.17] for a more comprehensive statement.

So far as the authors are aware, Lemma 2.18, a simple application of the fact that every pseudocompact space satisfies the conclusion of the Baire category theorem ([19, 3.10(f)(e)]), was first noted in [12, 7.4].

Lemma 2.19 is [9, 3.2]. An alternative proof of an equivalent statement, based on the fact that $\mathfrak{c}$ is not a measurable cardinal, is given in [4, Proof of **(9)**].

En route to partial results about r- and s-extremality, several authors have considered groups of the form nG and nN (with $N \in \Lambda(G)$) and their closures in G, N, $\overline{G}$ or $\overline{N}$; see for example [12, §7], [8, §5], [7, §4], [21, §6] and see in particular [5, 5.6], where Lemma 2.22 for groups G assumed to be either r- or s-extremal is proved. The key to Lemma 2.22 is the simple observation that a continuous open surjection between compact spaces preserves closed G_δ-sets, a fact we noted with proof in [8, 3.2].

The essentials of Lemma 3.2(a) appear already in [9, 4.3], but it should be noted that Lemma 3.2(b) gives a crucially stronger conclusion than we achieved in §3 of [9] (which treated only the "connected case"): indeed, now using Lemma 3.2(b) we find in Lemma 3.4, much as in [9, §4], a G_δ-dense subgroup of some $N \in \Lambda(G)$ — but now we see that this may be chosen proper, a condition invoked in the proof of statement (B) in Theorem 4.1. The essentials of the argument used to prove statement (C) in Theorem 4.1 are given (in the restricted context treated there) in [12, p. 39].

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(*Received* December 10, 2012)