The sup = max problem for the extent of generalized metric spaces

Yasushi Hirata, Yukinobu Yajima

Dedicated to the 120th birthday anniversary of Eduard Čech.

Abstract. It looks not useful to study the sup = max problem for extent, because there are simple examples refuting the condition. On the other hand, the sup = max problem for Lindelöf degree does not occur at a glance, because Lindelöf degree is usually defined by not supremum but minimum. Nevertheless, in this paper, we discuss the sup = max problem for the extent of generalized metric spaces by combining the sup = max problem for the Lindelöf degree of these spaces.

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1. Introduction

Let φ be a cardinal function and X a space. Some cardinal functions are defined in terms of

$$\varphi(X) = \sup\{|S| : S \subset X \text{ has a property } \mathcal{P}_\varphi\} + \omega.$$

The sup = max *problem for* φ is the one when $\varphi(X) = |S|$ holds for some $S \subset X$ having the property $\mathcal{P}_\varphi$. Whenever we deal with the sup = max problem for φ, note that $\varphi(X)$ should be a limit cardinal. Otherwise, this problem becomes trivial.

As a typical cardinal function for the sup = max problem, let us recall the *spread* $s(X)$ of a space X which is defined by

$$s(X) = \sup\{|D| : D \text{ is a discrete subset in } X\} + \omega.$$

First, Hajnal-Juhász [5] proved that for a Hausdorff space X with $|X| \geq \kappa$, if κ is a singular strong limit cardinal, then there is a discrete subset of size κ in X. Moreover, they also proved the following.

Theorem 1.1 (Hajnal-Juhász [6]). *Let κ be a singular cardinal with* $\mathrm{cf}(\kappa) = \omega$. *If X is a regular T_1-space with $s(X) = \kappa$, then there is a discrete subset of size κ in X.*

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The case of κ being a singular cardinal with $\mathrm{cf}(\kappa) = \omega$ seemed to be specially interesting. In fact, Roitman [14] proved that there is consistently a zero-dimensional regular T_1-space X with $s(X) = \omega_{\omega_1}$ and with no discrete subset of size ω_{ω_1} in X. And it had been naturally asked whether Theorem 1.1 holds for a Hausdorff space X. A complete answer to this problem was given by the following.

Theorem 1.2 (Kunen-Roitman [11])**.** *Let κ be a singular cardinal with $\mathrm{cf}(\kappa) = \omega$. Then there is a Hausdorff space X with $s(X) = \kappa$ and with no discrete subset of size κ if and only if there is a set $S \subset 2^\omega$ of size κ such that every subset of S of size κ is not meager.*

Thus, the sup = max problem for spread seemed to be settled before 1980. The reader might find the details of the sup = max problem in the books [9], [10]. In particular, the details of Theorems 1.1 and 1.2 are found in [10, Chapter 4].

Now, let us recall that the *extent* $e(X)$ of a space X is defined by

$$e(X) = \sup\{|D| : D \text{ is a closed discrete subset in } X\} + \omega.$$

Obviously, we have $e(X) \leq s(X)$. Since the definition of extent looks similar to that of spread, it is natural to consider the sup = max problem for extent. However, it looks vain as seen from Example 2.1 below. Due to this kind of examples, the sup = max problem for extent seems to have been never dealt with so far. Nevertheless, the situation is changed when we restrict the extent to a generalized metric space such as a Σ-space, a strict p-space or a semi-stratifiable space. Our results depend on the topological structure of a space rather than the cardinal condition of extent. In fact, for a cardinal κ, we only assume $\mathrm{cf}(\kappa) > \omega$ instead of $\mathrm{cf}(\kappa) = \omega$.

Next, let us recall that the *Lindelöf degree* $L(X)$ of a space X is defined by

$$L(X) = \min\{\kappa : \text{ every open cover of } X \text{ has a subcover of cardinality } \leq \kappa\} + \omega.$$

Then $e(X) \leq L(X)$ holds. Since Lindelöf degree is usually defined not by supremum but minimum as above, the sup = max problem for it does not seem to occur at a glance. In order to consider the sup = max problem for extent, we introduce a new type of the sup = max problem for Lindelöf degree. Indeed, the sup=max problem for Lindelöf degree is rather easier to deal with than that of extent in some cases.

Throughout this paper, all spaces are assumed to be *Hausdorff*, and κ and τ denote uncountable cardinals. For a cardinal κ, $\mathrm{cf}(\kappa)$ denotes the cofinality of κ, and the spaces κ and $\kappa + 1$ mean the spaces $[0, \kappa)$ and $[0, \kappa]$ with the usual order topology, respectively.

2. A simple example and a motivation

The following simple example seems to be a reason why the sup = max problem for extent has been never discussed so far.

Example 2.1. For every limit cardinal κ, there is a space X_κ with one non-isolated point such that $e(X_\kappa) = |X_\kappa| = \kappa$, but there is no closed discrete subset of size κ in X_κ. If $\mathrm{cf}(\kappa) = \omega$, the space X_κ is metrizable.

PROOF: Let κ be a limit cardinal. Take the subspace $X_\kappa = \{\alpha + 1 : \alpha \in \kappa\} \cup \{\kappa\}$ of $\kappa + 1$. Then X_κ has the only one non-isolated point κ with $|X_\kappa| = \kappa$.

Since $\{\alpha + 1 \in X_\kappa : \alpha < \beta\}$ is a closed discrete subset in X_κ for each $\beta \in \kappa$, we have $e(X_\kappa) = \kappa$. Let D be a closed discrete subset in X_κ. Take an open neighborhood U_0 of κ in X_κ with $|U_0 \cap D| \leq 1$. Take a $\beta_0 \in \kappa$ with $X_\kappa \cap (\beta_0, \kappa] \subset U_0$. Since $|D \setminus U_0| \leq |\beta_0| < \kappa$, we have $|D| < \kappa$.

When $\mathrm{cf}(\kappa) = \omega$, let $\{\tau_n\}$ be a sequence of cardinals with $\tau_n < \tau_{n+1}$ for each $n \in \omega$ and $\kappa = \sup_{n\in\omega} \tau_n$. Let $\mathcal{B}_n = \{\{\alpha+1\} : \alpha < \tau_n\}$ and $\mathcal{B}_{\kappa,n} = \{X_\kappa \cap (\tau_n, \kappa]\}$ for each $n \in \omega$. Then $\bigcup_{n\in\omega}(\mathcal{B}_n \cup \mathcal{B}_{\kappa,n})$ is a σ-discrete base of X_κ. Hence X_κ is metrizable. $\square$

Every metrizable space M has a σ-discrete base $\mathcal{B}$ with $|\mathcal{B}| = w(M)$, where $w(M)$ denotes the weight of M. It is well known that for a metrizable space M, we have $e(M) = s(M) = w(M) = \kappa$. So adding the assumption of $\mathrm{cf}(\kappa) > \omega$, the following is easy to see.

Proposition 2.2. *Let M be a metrizable space with $e(M) = \kappa$. Assume $\mathrm{cf}(\kappa) > \omega$. Then there is a closed discrete subset of size κ in M.*

In view of Example 2.1 and Proposition 2.2, it is natural to ask

Problem 0. Let X be a generalized metric space with $e(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$. When is there a closed discrete subset of size κ in X?

This problem is a motivation of this paper, and we will give a couple of affirmative answers to this one. As Gruenhage gave a nice survey [4] for generalized metric spaces, we sometimes quote it.

For the reader's convenience, we state the following implications for generalized metric spaces which will be dealt with.

$$\begin{tikzcd}
\text{metric space} \arrow[r] \arrow[d] & \text{Moore space} \arrow[r] \arrow[d] & \text{strict } p\text{-space} \arrow[r] & p\text{-space} \\
\text{stratifiable space} \arrow[r] & \sigma\text{-space} \arrow[r] \arrow[dr] & \text{strong } \Sigma\text{-space} \arrow[r] & \Sigma\text{-space} \\
 & & \text{semi-stratifiable} &
\end{tikzcd}$$

3. Σ-spaces

First, for the reader's convenience, we show

Fact 3.1 (folklore)**.** *Let $\mathcal{A}$ be an infinite collection of non-empty subsets in a space X. If $\mathcal{A}$ is locally finite in X, then there is a closed discrete subset D of size $|\mathcal{A}|$.*

Proof: Let $\kappa = |\mathcal{A}| \geq \omega$. We can inductively construct a subcollection $\{A_\alpha : \alpha \in \kappa\}$ of $\mathcal{A}$ and a sequence $\{x_\alpha : \alpha \in \kappa\}$ of points in X, satisfying that $x_\alpha \in A_\alpha$ and $\{x_\beta : \beta < \alpha\} \cap A_\alpha = \emptyset$ for each $\alpha \in \kappa$. Then $D := \{x_\alpha : \alpha \in \kappa\}$ is a closed discrete subset of X with $|D| = \kappa$. $\square$

Let X be a space and $\mathcal{K}$ a closed cover of X. A closed cover $\mathcal{F}$ of X is a *(mod $\mathcal{K}$)-network* for X if, whenever $K \in \mathcal{K}$ and U is open in X with $K \subset U$, there is $F \in \mathcal{F}$ with $K \subset F \subset U$ (see [13]). A space X is a *(strong) Σ-space* [12] if there is a σ-locally finite (mod $\mathcal{K}$)-network for some closed cover $\mathcal{K}$ of X by countably compact (compact) sets (cf. [4, 4.13 Definition]).

Theorem 3.2. *If X is a Σ-space with $e(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$, then there is a closed discrete subset of size κ in X.*

Proof: Let $\mathcal{K}$ be a closed cover of X by countably compact sets and $\mathcal{F}$ a σ-locally finite (mod $\mathcal{K}$)-network for X. First, we show $|\mathcal{F}| \geq \kappa$. Let D be any closed discrete subset in X. Let $\mathcal{F}_D = \{F \in \mathcal{F} : |F \cap D| < \omega\}$. Pick an $x \in D$. Take a $K_x \in \mathcal{K}$ with $x \in K_x$. Since K_x is countably compact, we have $|K_x \cap D| < \omega$. Let $U = X \setminus (D \setminus K_x)$. Then U is an open set in X with $K_x \subset U$. There is an $F_0 \in \mathcal{F}$ with $K_x \subset F_0 \subset U$. Then we have $K_x \cap D = F_0 \cap D$. We conclude that $F_0 \in \mathcal{F}_D$ and $x \in K_x \subset F_0$. Thus $\mathcal{F}_D$ covers D. This means that

$$|D| = |\bigcup\{F \cap D : F \in \mathcal{F}_D\}| \leq |\mathcal{F}_D| \cdot \omega \leq |\mathcal{F}| \cdot \omega.$$

Hence $\kappa = e(X) \leq |\mathcal{F}| \cdot \omega$ holds. By $\kappa > \omega$, we obtain $|\mathcal{F}| \geq \kappa = e(X)$.

Let $\mathcal{F} = \bigcup_{n \in \omega} \mathcal{F}_n$, where each $\mathcal{F}_n$ is locally finite in X. By $\mathrm{cf}(\kappa) > \omega$, there is $m \in \omega$ with $|\mathcal{F}_m| \geq \kappa$. It follows from Fact 3.1 that there is a closed discrete subset D^* in X with $|D^*| = |\mathcal{F}_m| \geq \kappa$. By $e(X) = \kappa$, $|D^*|$ must be equal to κ. $\square$

4. The sup = max problem for Lindelöf degree

Since Lindelöf degree is usually defined by minimum, the sup = max problem does not seem to occur. However, using another expression, Lindelöf degree can be defined by supremum.

For a collection $\mathcal{U}$ of open sets in a space X, let

$$L(\mathcal{U}) = \min\{|\mathcal{V}| : \mathcal{V} \subset \mathcal{U} \text{ with } \bigcup \mathcal{V} = \bigcup \mathcal{U}\} + \omega.$$

First, we have to check the following basic fact.

Fact 4.1. *For a space X, $L(X) = \sup\{L(\mathcal{U}) : \mathcal{U}$ is an open cover of $X\}$ holds.*

Proof: Let $\kappa = \sup\{L(\mathcal{U}) : \mathcal{U}$ is an open cover of $X\}$. Take any open cover $\mathcal{U}$ of X. Since there is a subcover $\mathcal{V}$ of $\mathcal{U}$ with $|\mathcal{V}| \leq L(X)$, we have $L(\mathcal{U}) \leq L(X)$. Hence $\kappa \leq L(X)$ holds. Take any open cover $\mathcal{U}$ of X again. By $L(\mathcal{U}) \leq \kappa$, there is a subcover $\mathcal{V}$ of $\mathcal{U}$ with $|\mathcal{V}| \leq \kappa$. Hence $L(X) \leq \kappa$ holds. $\square$

Thus, we can consider the sup = max problem for Lindelöf degree as the problem when there is an open cover $\mathcal{U}$ of a space X with $L(X) = L(\mathcal{U})$.

As a trivial case, for a Lindelöf and non-compact space X, the sup = max problems for $L(X)$ are affirmative. On the other hand, taking the space X_κ as in Example 2.1, it is easily seen that $L(X_\kappa) = \kappa$ but $L(\mathcal{U}) < \kappa$ for any open cover $\mathcal{U}$ of X_κ.

A space X is *submetalindelöf* (or *$\delta\theta$-refinable*) if for every open cover $\mathcal{U}$ of X, there is a sequence $\{\mathcal{V}_n\}$ of open refinements satisfying that for each $x \in X$ one can choose $n_x \in \omega$ such that $\mathcal{V}_{n_x}$ is point-countable at x. Related to this property, we have the following implications:

$$\begin{tikzcd}
\text{paracompact} \arrow[r] \arrow[d] & \text{metacompact} \arrow[r] \arrow[d] & \text{metalindelöf} \arrow[d] \\
\text{subparacompact} \arrow[r] & \text{submetacompact} \arrow[r] & \text{submetalindelöf}
\end{tikzcd}$$

Lemma 4.2 (Aull)**.** *If X is a submetalindelöf space, then $e(X) = L(X)$ holds.*

This can be shown by an easy modification of the proof of [1, Theorem 1].

For a space X, $A \subset X$ and a collection $\mathcal{U}$ of subsets in X, we let

$$\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}.$$

In particular, we use $\mathrm{St}(x,\mathcal{U})$ instead of $\mathrm{St}(\{x\},\mathcal{U})$, where $x \in X$.

In the proof of Lemma 4.2 (or [1, Theorem 1]), the following result is used.

Lemma 4.3 ([1, Lemmas 1 and 3])**.** *Let X be a space, $A \subset X$ and $\mathcal{U}$ be an open cover of X. Then there is a closed discrete subset D in X such that $D \subset A \subset \mathrm{St}(D,\mathcal{U})$.*

Fact 4.4 (folklore)**.** *Let X be a space with $e(X) = L(X) = \kappa$. If there is a closed discrete subset D of size κ in X, then there is an open cover $\mathcal{U}$ of X with $L(\mathcal{U}) = \kappa$.*

Proof: Let $\mathcal{U} = \{X \setminus (D \setminus \{d\}) : d \in D\}$. Since $\mathcal{U}$ has no proper subcover of X, it is an open cover of X with $L(\mathcal{U}) = |\mathcal{U}| = |D| = \kappa$. □

Theorem 4.5. *Let X be a submetalindelöf space with $e(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$. Then there is a closed discrete subset D of size κ in X if and only if there is an open cover $\mathcal{U}$ of X with $L(\mathcal{U}) = L(X) = \kappa$.*

Proof: First, by Lemma 4.2, note that $L(X) = e(X) = \kappa$ holds. It suffices by Fact 4.4 to show the "if" part. Let $\mathcal{U}$ be an open cover of X with $L(\mathcal{U}) = \kappa$. Since X is submetalindelöf, there is a sequence $\{\mathcal{V}_n\}$ of open refinements of $\mathcal{U}$ satisfying that for each $x \in X$ one can choose $n_x \in \omega$ such that $\mathcal{V}_{n_x}$ is point-countable at x. For each $n \in \omega$, let

$$X_n = \{x \in X : \mathcal{V}_n \text{ is point-countable at } x\}.$$

Then we have $X = \bigcup_{n\in\omega} X_n$. Pick an $n \in \omega$. By Lemma 4.3, there is a closed discrete subset D_n in X with $D_n \subset X_n \subset \bigcup \mathrm{St}(D_n, \mathcal{V}_n)$. Let $\mathcal{W}_n = \{V \in \mathcal{V}_n : V \cap D_n \neq \emptyset\}$. By the choice of X_n, we have $|\mathcal{W}_n| \leq |D_n| \cdot \omega \leq \kappa$.

Now, assume that $|D_n| < \kappa$ for each $n \in \omega$. Let $\tau = \sup_{n\in\omega} |D_n| \cdot \omega$. By $\mathrm{cf}(\kappa) > \omega$, we have $\tau < \kappa$. Let $\mathcal{W} = \bigcup_{n\in\omega} \mathcal{W}_n$. Since $X_n \subset \mathrm{St}(D_n, \mathcal{V}_n) = \bigcup \mathcal{W}_n$ for each $n \in \omega$ and $X = \bigcup_{n\in\omega} X_n$, $\mathcal{W}$ covers X. So $\mathcal{W}$ is an open refinement of $\mathcal{U}$. On the other hand, since

$$|\mathcal{W}| = \sup_{n\in\omega} |\mathcal{W}_n| \leq \sup_{n\in\omega} |D_n| \cdot \omega = \tau,$$

we conclude that $L(\mathcal{U}) \leq \tau < \kappa = L(X)$. This contradicts $L(\mathcal{U}) = \kappa$. Hence we obtain $|D_m| = \kappa = e(X)$ for some $m \in \omega$. □

Since a regular strong Σ-space is subparacompact, it is submetalindelöf. So the following is an immediate consequence of Lemma 4.2, Theorems 3.2 and 4.5.

Corollary 4.6. *If a regular space X is a strong Σ-space with $L(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$, then there is an open cover $\mathcal{U}$ of X with $L(\mathcal{U}) = \kappa$.*

Remark 4.7. Since every space with one non-isolated point is paracompact, it follows from Example 2.1 that the sup = max problems for $L(X)$ and $e(X)$ are both negative for a submetalindelöf space X without any additional condition.

However, we do not know the following.

Problem 1. Assume that a space X has a point-countable base with $e(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$. Is there a closed discrete subset of size κ in X?

5. Strict p-spaces

A Tychonoff space X is called a *p-space* (respectively, *strict p-space*) if there is a sequence $\{\mathcal{O}_n\}$ of collections of open sets in βX such that each $\mathcal{O}_n$ covers X and $\bigcap_{n\in\omega} \mathrm{St}(x, \mathcal{O}_n) \subset X$ (respectively, $\bigcap_{n\in\omega} \mathrm{St}(x, \mathcal{O}_n) = \bigcap_{n\in\omega} \overline{\mathrm{St}(x, \mathcal{O}_n)}^{\beta X} \subset X$) for each $x \in X$ (cf. [4, 3.15 Definition]).

Here we make use of the following characterization of p-spaces by Burke [2] (cf. [4, 3.21 Theorem]) instead of the definition.

Lemma 5.1. *A Tychonoff space X is a p-space if and only if there is a sequence $\{\mathcal{G}_n\}$ of open covers of X satisfying the following condition: If $G_n \in \mathcal{G}_n$ for each $n \in \omega$ with $\bigcap_{n\in\omega} G_n \neq \emptyset$, then*

(i) *$\bigcap_{n\in\omega} \overline{G_n}$ is compact, and*
(ii) *every open set U in X containing $\bigcap_{n\in\omega} \overline{G_n}$ contains some $\bigcap_{i\leq m} \overline{G_i}$.*

Lemma 5.2. *Let X be a space and $\mathcal{K}$ a closed cover of X by compact sets. If $\mathcal{F}$ is a (mod $\mathcal{K}$)-network for X, then $L(X) \leq |\mathcal{F}| \cdot \omega$ holds.*

Proof: Take any open cover $\mathcal{U}$ of X. Let

$$\mathcal{F}^* = \{F \in \mathcal{F} : \text{ there is a finite } \mathcal{W} \subset \mathcal{U} \text{ with } F \subset \bigcup \mathcal{W}\}.$$

For each $F \in \mathcal{F}^*$, one can assign a finite subcollection $\mathcal{V}(F)$ of $\mathcal{U}$ which covers F. Let $\mathcal{V} = \bigcup\{\mathcal{V}(F) : F \in \mathcal{F}^*\}$. Then we have $|\mathcal{V}| = |\mathcal{F}^*| \cdot \omega \leq |\mathcal{F}| \cdot \omega$. To show $L(X) \leq |\mathcal{F}| \cdot \omega$, it suffices to show that $\mathcal{V}$ covers X. Pick an $x \in X$. Take $K_x \in \mathcal{K}$ with $x \in K_x$. Since K_x is compact, there is a finite $\mathcal{W} \subset \mathcal{U}$ which covers K_x. Then there is $F_0 \in \mathcal{F}$ with $K_x \subset F_0 \subset \bigcup \mathcal{W}$. So we have $F_0 \in \mathcal{F}^*$. It follows that $x \in K_x \subset F_0 \subset \bigcup \mathcal{V}(F_0) \subset \bigcup \mathcal{V}$. Hence $\mathcal{V}$ is a subcover of $\mathcal{U}$. □

Lemma 5.3. *Let X be a p-space with $L(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$. Then there is an open cover $\mathcal{U}$ of X with $L(\mathcal{U}) = \kappa$.*

Proof: Assume that $L(\mathcal{U}) < \kappa$ for any open cover $\mathcal{U}$ of X. There is a sequence $\{\mathcal{G}_n\}$ of open covers of X, described in Lemma 5.1. For each $n \in \omega$, letting $\tau_n = L(\mathcal{G}_n)$, there is a subcover $\mathcal{H}_n$ of $\mathcal{G}_n$ with $|\mathcal{H}_n| = \tau_n$. Let $\tau = \sup_{n\in\omega} \tau_n$. Since $\tau_n < \kappa$ for each $n \in \omega$ and $\mathrm{cf}(\kappa) > \omega$, we have $\tau < \kappa$. Let

$$\mathcal{F} = \{\bigcap_{i\leq n} \overline{H_i} : H_i \in \mathcal{H}_i, i \leq n \text{ and } n \in \omega\}.$$

Since $|\bigcup_{n\in\omega} \mathcal{H}_n| \leq \tau$, note that $|\mathcal{F}| \leq \tau$. Let

$$\mathcal{K} = \{\bigcap_{n\in\omega} \overline{H_n} : H_n \in \mathcal{H}_n \text{ for each } n \in \omega \text{ with } \bigcap_{n\in\omega} H_n \neq \emptyset\}.$$

Since each $\mathcal{H}_n$ covers X, it follows from Lemma 5.1(i) that $\mathcal{K}$ is a closed cover of X by compact sets. Take any $K \in \mathcal{K}$ and any open set U in X with $K \subset U$. Then there is a sequence $\{H_n\}$ of open sets in X such that $K = \bigcap_{n\in\omega} \overline{H_n}$, where $H_n \in \mathcal{H}_n$ with $\bigcap_{n\in\omega} H_n \neq \emptyset$. By Lemma 5.1(ii), there is $m \in \omega$ with $\bigcap_{i\leq m} \overline{H_i} \subset U$. Then we have $\bigcap_{i\leq m} \overline{H_i} \in \mathcal{F}$ such that $K \subset \bigcap_{i\leq m} \overline{H_i} \subset U$. Thus $\mathcal{F}$ is a (mod $\mathcal{K}$)-network for X. It follows from Lemma 5.2 and $\kappa > \omega$ that $\kappa = L(X) \leq |\mathcal{F}| \leq \tau < \kappa$ holds. This is a contradiction. □

Theorem 5.4. *If X is a strict p-space with $e(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$, then there is a closed discrete subset of size κ in X.*

Proof: It follows from Jiang's result [7] that every strict p-space is submetacompact. Since X is submetalindelöf, it follows from Lemma 4.2 that $e(X) = L(X) = \kappa$ holds. Since X is p-space and $\mathrm{cf}(\kappa) > \omega$, it follows from Lemma 5.3 that there is an open cover $\mathcal{U}$ of X with $L(\mathcal{U}) = \kappa$. It follows from Theorem 4.5 that there is a closed discrete subset D in X with $|D| = \kappa$. □

In view of Lemma 5.3 and Theorem 5.4, it is natural to ask

Problem 2. Let X be a p-space with $e(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$. Is there a closed discrete subset of size κ in X?

Since locally compact spaces are p-spaces, Lemma 5.3 is true for a locally compact space X. However, this is somewhat generalized in what follows. A space X is *locally Lindelöf* if each point of X has an open neighborhood whose closure is Lindelöf.

Proposition 5.5. *If X is a locally Lindelöf non-compact space with $L(X) = \kappa$, then there is an open cover $\mathcal{G}$ of X with $L(\mathcal{G}) = \kappa$.*

Proof: Since the case of X being Lindelöf is obvious, we may let $\kappa > \omega$. Let $\mathcal{U}$ be any open cover of X. Take an open cover $\mathcal{G}$ of X such that $\overline{G}$ is Lindelöf for each $G \in \mathcal{G}$. Assume that $L(\mathcal{G}) < \kappa$. There is a subcover $\mathcal{H}$ of $\mathcal{G}$ with $|\mathcal{H}| = L(\mathcal{G})$. For each $G \in \mathcal{H}$, there is a countable subcollection $\mathcal{V}(G)$ of $\mathcal{U}$ covering $\overline{G}$. Let $\mathcal{V} = \bigcup\{\mathcal{V}(G) : G \in \mathcal{H}\}$. Then $\mathcal{V}$ is a subcover of $\mathcal{U}$ with $|\mathcal{V}| \leq |\mathcal{H}| = L(\mathcal{G}) < \kappa$. This implies that $L(X) \leq L(\mathcal{G}) < \kappa = L(X)$, which is a contradiction. Hence we obtain $L(\mathcal{G}) = \kappa$. □

6. Semi-stratifiable spaces

A space X is *semi-stratifiable* [3] if there is a function $g : \omega \times X \to \mathrm{Top}(X)$, where $\mathrm{Top}(X)$ denotes the topology of X, satisfying

(i) $\bigcap_{n\in\omega} g(n, x) = \{x\}$ for each $x \in X$,
(ii) $y \in \bigcap_{n\in\omega} g(n, x_n)$ implies that $\{x_n\}$ converges to y

(see also [4, 5.6 Definition]).

For a space X, $d(X)$ denotes the *density* of X, that is,

$$d(X) = \min\{|S| : S \text{ is a dense subset in } X\}.$$

Lemma 6.1 (Creed)**.** *If X is a semi-stratifiable space, then $d(X) \leq L(X)$ holds.*

This was actually showed in the proof of (1)⇒(2) of [3, Theorem 2.8]. Moreover, the following is obtained by a modification of the proof.

Lemma 6.2. *Let X be a semi-stratifiable space with $L(X) = d(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$. Then there is an open cover $\mathcal{U}$ of X with $L(\mathcal{U}) = \kappa$.*

Proof: Assume that $L(\mathcal{U}) < \kappa$ for any open cover $\mathcal{U}$ of X. Let $g : \omega \times X \to \mathrm{Top}(X)$ be a function described as above. Let $\mathcal{G}_n = \{g(n, x) : x \in X\}$ for each $n \in \omega$. Pick an $n \in \omega$. Let $\tau_n = L(\mathcal{G}_n)$. Since $\mathcal{G}_n$ is an open cover of X, we have $\tau_n < \kappa$. There is a subcover $\mathcal{H}_n$ of $\mathcal{G}_n$ with $|\mathcal{H}_n| = \tau_n$. Let $\mathcal{H}_n = \{g(n, x) : x \in T_n\}$, where $T_n \subset X$ with $|T_n| = \tau_n$. Let $\tau = \sup_{n\in\omega} \tau_n$. By $\mathrm{cf}(\kappa) > \omega$, we have $\tau < \kappa$. Let $T = \bigcup_{n\in\omega} T_n$. Pick an $x \in X$. For each $n \in \omega$, take $x_n \in T_n$ with $x \in g(n, x_n)$. By the choice of g, $\{x_n\}$ converges to x. Hence T is a dense subset in X with $|T| = \tau$. We conclude that $d(X) \leq |T| = \tau < \kappa = L(X)$. This contradicts the assumption. □

A space X is *metalindelöf* if every open cover of X has a point-countable open refinement. The following is easily seen.

Fact 6.3. *If X is a metalindelöf space, then $L(X) \leq d(X)$ holds.*

A space X is *collectionwise Hausdorff* if for every closed discrete subset D in X, there is a mutually disjoint collection $\{U_x : x \in D\}$ of open sets such that $x \in U_x$ for each $x \in D$.

Fact 6.4. *If X is a collectionwise Hausdorff space, then $e(X) \leq d(X)$ holds.*

Now, we obtain a main result in this section.

Theorem 6.5. *Let X be a semi-stratifiable space with $e(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$. If X is either metalindelöf or collectionwise Hausdorff, then there is a closed discrete subset of size κ in X.*

Proof: Since every semi-stratifiable space is subparacompact (cf. [4, 5.11 Theorem]), Lemma 4.2 assures that $e(X) = L(X) = \kappa$ holds. Moreover, it follows from Lemmas 6.1, Facts 6.3 and 6.4 that $e(X) = L(X) = d(X) = \kappa$ holds. Hence our conclusion follows from Theorem 4.5 and Lemma 6.2. □

This immediately yields

Corollary 6.6. *If X is a paracompact, semi-stratifiable space with $e(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$, then there is a closed discrete subset of size κ in X.*

The following is well known as Jones' Lemma.

Lemma 6.7 ([8])**.** *If X is a normal space, then $2^{|D|} \leq 2^{d(X)}$ holds for every closed discrete subset D in X.*

Lemma 6.8. *Let κ be a cardinal with $\mathrm{cf}(\kappa) > \omega$ such that $\{2^\tau : \tau$ is a cardinal $< \kappa\}$ has no maximum. If X is a normal space with $e(X) = \kappa$, then $e(X) \leq d(X)$ holds.*

Proof: Assume that $d(X) < \kappa$ holds. Then we have $2^{d(X)} < 2^\kappa$. By the assumption of κ, there is a cardinal $\rho < \kappa$ with $2^{d(X)} < 2^\rho < 2^\kappa$. Take a closed discrete subset D in X with $\rho < |D| < \kappa$. Then we have $2^{d(X)} < 2^\rho \leq 2^{|D|}$, which contradicts Jones' Lemma above. □

Using Lemma 6.8 instead of Facts 6.3 and 6.4, the following is obtained analogously as Theorem 6.5.

Proposition 6.9. *Let κ be a cardinal with $\mathrm{cf}(\kappa) > \omega$ such that $\{2^\tau : \tau$ is a cardinal $< \kappa\}$ has no maximum. If X is a normal, semi-stratifiable space with $e(X) = \kappa$, then there is a closed discrete subset of size κ in X.*

For a strong limit cardinal κ (i.e., $2^\tau < \kappa$ whenever $\tau < \kappa$), note that $\{2^\tau : \tau$ is a cardinal $< \kappa\}$ has no maximum.

Problem 3. Let X be a normal, semi-stratifiable space with $e(X) = \kappa$, where $\mathrm{cf}(\kappa) > \omega$. Is there a closed discrete subset of size κ in X without such an assumption of κ as above?

Remark 6.10. As stated in [4, Theorem 7.8(i)], Σ-spaces, strict p-spaces and semi-stratifiable spaces are all β-spaces. However, the sup = max equality does not hold for the extent of β-spaces, because each of the space X_κ in Example 2.1 is a paracompact β-space.

7. Subspaces of a cardinal

In general, $e(X)$ cannot bound $L(X)$. In fact, for every countably compact non-compact space X, $\omega = e(X) < \omega_1 \leq L(X)$ holds. In particular, if $X = \kappa$ for a cardinal κ with $\mathrm{cf}(\kappa) > \omega$, then X is a locally compact space with $e(X) = \omega$ and $L(X) = \mathrm{cf}(\kappa)$. Moreover, we have the following result.

Theorem 7.1. *Let κ and τ be any cardinals with $\kappa \geq \tau \geq \omega$. Then there is a subspace X of κ such that $L(X) = \kappa$ and $e(X) = \tau$.*

Proof: Let $X = \kappa \setminus (R \cup L)$, where R is the set of all regular cardinals with $< \kappa$ and L is the set of all limit ordinals with $\leq \tau$. Obviously, $L(X) \leq |X| \leq \kappa$ holds. Let D be a closed discrete subset in X. Let $D \setminus \tau = \{\alpha_\xi : \xi \in \mu\}$ and $\alpha_\zeta < \alpha_\xi$ for every $\zeta < \xi < \mu$. Then, $\mu \leq \omega$ holds. Actually, assume that $\mu \geq \omega$. Taking $\{\alpha_n : n \in \omega\} \subset D \setminus \tau$, let $\beta = \sup\{\alpha_n : n \in \omega\}$. Then we have $\beta \notin X$ since D is closed discrete, and $\beta \notin R \cup L$ since $\mathrm{cf}(\beta) = \omega \leq \tau \leq \alpha_0 < \alpha_1 \leq \beta$. Therefore, $\beta = \kappa$ holds. If $\mu > \omega$, we have $\alpha_\omega \geq \beta = \kappa$, which contradicts $\alpha_\omega \in D \setminus \tau \subset \kappa$. So we obtain $\mu \leq \omega$. It follows from $|D \cap \tau| \leq \tau$ and $|D \setminus \tau| \leq \mu \leq \omega$ that $|D| \leq \tau$ holds. Hence we have $e(X) \leq \tau$.

Let λ be a regular cardinal. If $\lambda \leq \tau$, then $D_\lambda = \{\alpha + 1 : \alpha \in \lambda\}$ is a closed discrete subset of X, so $\lambda = |D_\lambda| \leq e(X)$. Therefore $\tau \leq e(X)$ holds. If $\lambda \leq \kappa$, then $\mathcal{U}_\lambda = \{X \cap [0, \alpha] : \alpha < \lambda\} \cup \{(\lambda, \kappa)\}$ is an open cover of X, so we have $\lambda = L(\mathcal{U}_\lambda) \leq L(X)$. Therefore $\kappa \leq L(X)$ holds. Thus, we conclude that $e(X) = \tau$ and $L(X) = \kappa$. □

Next, we construct a space X with $L(X) = e(X)$ such that Theorem 4.5 does not hold. Of course, such a space X must not be submetalindelöf. A typical example of non-submetalindelöf spaces is a stationary subspace of κ with $\mathrm{cf}(\kappa) > \omega$ (it is easily checked by the Pressing Down Lemma). So we try to find such a space in the class of subspaces of a cardinal κ.

For a subset S of κ, we denote by $\mathrm{Lim}(S)$ the set of all limit points of S in κ, that is, $\mathrm{Lim}(S) = \{\alpha \in \kappa : \alpha = \sup(S \cap \alpha)\}$.

Theorem 7.2. *Let κ be a regular limit cardinal $> \omega$. Then there is a subspace X of κ, satisfying the following;*

(i) $L(X) = e(X) = \kappa$,
(ii) *there is an open cover $\mathcal{U}$ of X with $L(\mathcal{U}) = \kappa$ and*
(iii) *there is no closed discrete subset of size κ in X.*

Proof: Define a subspace X of κ by putting

$$X = \kappa \setminus \bigcup\{(\lambda, \lambda + \lambda] \cap \mathrm{Lim}(\kappa) : \lambda \text{ is a cardinal with } \lambda < \kappa\}.$$

Obviously, $e(X) \leq L(X) \leq |X| \leq \kappa$ holds. For each infinite cardinal $\lambda < \kappa$,

$$D_\lambda := \{\lambda + \alpha + 1 : \alpha \in \lambda\} \subset X \cap (\lambda, \lambda + \lambda)$$

is a closed discrete subset in X, and so we have $\lambda = |D_\lambda| \le e(X)$. Therefore $\kappa \le e(X)$ holds, thus $L(X) = e(X) = \kappa$. Since κ is regular, $\mathcal{U} := \{X \cap [0,\alpha] : \alpha \in \kappa\}$ is an open cover of X with $L(\mathcal{U}) = \kappa$. Let Z be a subset in X with $|Z| = \kappa$. Then Z is unbounded in κ. Take a sequence $\{\lambda_n : n \in \omega\}$ of infinite cardinals with $< \kappa$ and a sequence $\{\zeta_n : n \in \omega\}$ of members of Z inductively such that $\lambda_n \le \zeta_n < \lambda_{n+1}$ for each $n \in \omega$. Let $\lambda = \sup\{\lambda_n : n \in \omega\} (= \sup\{\zeta_n : n \in \omega\})$. Since λ is a cardinal and X contains all cardinals less than κ, we have $\lambda \in X \cap \operatorname{Lim}(Z)$. So Z is not a closed discrete subset in X. Hence, there is no closed discrete subset of size κ in X. $\square$

Theorem 7.3. *Let κ be a singular limit cardinal. Then there is a subspace X of $\kappa + \kappa$ satisfying the following;*

(i) $L(X) = e(X) = \kappa$,

(ii) *there is an open cover $\mathcal{U}$ of X with $L(\mathcal{U}) = \kappa$ and*

(iii) *there is no closed discrete subset of size κ in X.*

However, there is no subspace of κ satisfying these three conditions.

Proof: Take a strictly increasing sequence $\{\kappa_\xi : \xi \in \operatorname{cf}(\kappa) + 1\}$ in $\kappa + 1$ with $\kappa_{\operatorname{cf}(\kappa)} = \kappa$ such that for each $\xi \le \operatorname{cf}(\kappa)$, we have

- if ξ is not a limit ordinal, then κ_ξ is a regular uncountable cardinal,
- if ξ is a limit ordinal, then $\kappa_\xi = \sup\{\kappa_\eta : \eta \in \xi\}$.

We define a subspace X of $\kappa + \kappa$ by putting

$$X = (\kappa + \kappa) \setminus \big((\kappa \cap \operatorname{Lim}(\kappa)) \cup \{\kappa + \kappa_\xi : \xi \in \operatorname{cf}(\kappa)\}\big).$$

Obviously, $e(X) \le L(X) \le |X| \le |\kappa + \kappa| = \kappa$ holds. For each infinite cardinal $\lambda < \kappa$, $D_\lambda := \{\alpha + 1 : \alpha \in \lambda\} \subset X \cap \lambda$ is a closed discrete subset in X, and so we have $\lambda = |D_\lambda| \le e(X)$. Therefore $\kappa \le e(X)$ holds, thus $L(X) = e(X) = \kappa$.

Put $X_\xi = (\kappa + \kappa_\xi, \kappa + \kappa_{\xi+1})$ for each $\xi \in \operatorname{cf}(\kappa)$, $X_{-1} = (\kappa, \kappa + \kappa_0)$, and $X_{-2} = (\kappa \setminus \operatorname{Lim}(\kappa)) \cup \{\kappa\}$. Then we have $X = \bigoplus_{-2 \le \xi < \operatorname{cf}(\kappa)} X_\xi$. For each $\xi \in \operatorname{cf}(\kappa)$, let

$$\mathcal{U}_\xi := \{(\kappa + \kappa_\xi, \kappa + \kappa_\xi + \alpha] : 0 < \alpha < \kappa_{\xi+1}\}.$$

Then each $\mathcal{U}_\xi$ is an open cover of X_ξ with $L(\mathcal{U}_\xi) = \kappa_{\xi+1}$, since $\kappa_{\xi+1}$ is a regular cardinal. Therefore $\mathcal{U} := \{X_{-2}, X_{-1}\} \cup \bigcup_{\xi \in \operatorname{cf}(\kappa)} \mathcal{U}_\xi$ is an open cover of X with $L(\mathcal{U}) = \kappa$.

Let D be a closed discrete subset in X. For $\kappa \in X$, $D \cap \kappa$ is bounded in κ, and so $|D \cap X_{-2}| < \kappa$. Obviously, $|D \cap X_{-1}| \le \kappa_0 < \kappa$ holds. Pick a $\xi \in \operatorname{cf}(\kappa)$. Note that X_ξ is homeomorphic to $\kappa_{\xi+1} \setminus \kappa_\xi$. Since $\kappa_{\xi+1}$ is countably compact, so is X_ξ. Since X_ξ is clopen in X, $D \cap X_\xi$ must be finite. Hence we have

$$\begin{aligned}|D| &= |(D \cap X_{-2}) \cup (D \cap X_{-1}) \cup \bigcup_{\xi \in \operatorname{cf}(\kappa)} (D \cap X_\xi)| \\ &\le \max\{|D \cap X_{-2}|, \kappa_0, \operatorname{cf}(\kappa)\} < \kappa.\end{aligned}$$

Next, let us assume that there is a subspace X of κ with $e(X) = \kappa$. If $\mathrm{cf}(\kappa) > \omega$ and X is stationary in κ, then (ii) fails. Actually, if $\mathcal{U}$ is an open cover of X, then by the Pressing Down Lemma, there is $\gamma < \mathrm{cf}(\kappa)$ such that $\{X \cap (\kappa_\gamma, \kappa_\xi] : \xi \in (\gamma, \mathrm{cf}(\kappa))\}$ partially refines $\mathcal{U}$, so $L(\mathcal{U}) \leq \max\{\kappa_\gamma, \mathrm{cf}(\kappa)\} < \kappa$. If $\mathrm{cf}(\kappa) = \omega$ or X is non-stationary in κ with $\mathrm{cf}(\kappa) > \omega$, then (iii) fails. To see this, take an unbounded subset C of κ such that $X \cap \mathrm{Lim}(C) = \emptyset$. By induction, we can take a strictly increasing sequence $\{c(\xi) : \xi \in \mathrm{cf}(\kappa)\}$ by members of C and a sequence $\{D_\xi : \xi \in \mathrm{cf}(\kappa)\}$ of closed discrete subsets in X such that $D_\xi \subset (c(\xi), c(\xi+1))$ and $|D_\xi| \geq \kappa_\xi$ for each $\xi \in \mathrm{cf}(\kappa)$. Actually, if $c(\xi) \in C$ is taken for $\xi \in \kappa$, then by $e(X) = \kappa$, we can take a closed discrete subset D'_ξ in X such that $|D'_\xi| = \max\{|c(\xi)|, \kappa_\xi, \mathrm{cf}(\kappa)\}^+ < \kappa$. By $D'_\xi = \bigcup_{\zeta \in \mathrm{cf}(\kappa)} (D'_\xi \cap \kappa_\zeta)$ and $\mathrm{cf}(\kappa) < |D'_\xi| = \mathrm{cf}(|D'_\xi|)$, there is $D''_\xi \subset D'_\xi$ which is bounded in κ and $|D''_\xi| = |D'_\xi|$. Take $c(\xi+1) \in C$ with $D''_\xi \subset c(\xi+1)$ and let $D_\xi = D''_\xi \cap (c(\xi), c(\xi+1))$. By $D''_\xi = (D''_\xi \cap [0, c(\xi)]) \cup D_\xi$ and $|D''_\xi \cap [0, c(\xi)]| \leq \max\{|c(\xi)|, \omega\} < |D'_\xi| = |D''_\xi|$, we have $|D_\xi| = |D'_\xi| \geq \kappa_\xi$. So we can take the required sequences $\{c(\xi) : \xi \in \mathrm{cf}(\kappa)\}$ and $\{D_\xi : \xi \in \mathrm{cf}(\kappa)\}$. Let $D = \bigcup_{\xi \in \mathrm{cf}(\kappa)} D_\xi$. For $X \cap \mathrm{Lim}(C) = \emptyset$, $\{X \cap (c(\xi), c(\xi+1)) : \xi \in \mathrm{cf}(\kappa)\}$ is discrete in X. So $\{D_\xi : \xi \in \mathrm{cf}(\kappa)\}$ is also discrete in X, hence D is a closed discrete subset in X. For each $\xi \in \mathrm{cf}(\kappa)$, we have $\kappa_\xi \leq |D_\xi| \leq |D|$. Hence $|D| = \kappa$ holds, and so (iii) fails. □

As stated in Corollary 4.6, the sup = max equality holds for the Lindelöf degree of strong Σ-spaces. However, the following result shows that the sup = max equality does not hold for the Lindelöf degree of Σ-spaces.

Proposition 7.4. *Let κ be a limit cardinal. Then there is a countably compact subspace X of $\kappa + 1$ with $L(X) = \kappa$ such that $L(\mathcal{U}) < \kappa$ for any open cover $\mathcal{U}$ of X.*

Proof: Let $X = (\kappa + 1) \setminus \{\xi \in \kappa : \mathrm{cf}(\xi) > \omega\}$. Since X contains $\{\xi \in \mathrm{Lim}(X) : \mathrm{cf}(\xi) = \omega\}$, it is countably compact. Pick a regular uncountable cardinal $\lambda < \kappa$. Letting $\mathcal{U}_\lambda = \{X \setminus (\alpha, \lambda) : \alpha \in \lambda\}$, it is an open cover of X. Moreover, we have $L(\mathcal{U}_\lambda) = \lambda \leq L(X)$. By $L(X) \leq |X| = \kappa$, we obtain $L(X) = \kappa$.

Let $\mathcal{U}$ be an open cover of X. For each $\alpha \in X$, take $U_\alpha \in \mathcal{U}$ with $\alpha \in U_\alpha$. By $\kappa \in U_\kappa$, there is $\gamma \in \kappa$ with $X \cap (\gamma, \kappa] \subset U_\kappa$. Then $\mathcal{V} := \{U_\alpha : \alpha \in (X \cap [0, \gamma]) \cup \{\kappa\}\}$ is a subcover of $\mathcal{U}$ with $|\mathcal{V}| \leq |[0, \gamma] \cup \{\kappa\}| < \kappa$. Hence we have $L(\mathcal{U}) < \kappa$. □

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Department of Mathematics, Kanagawa University, Yokohama,
221-8686 Japan

E-mail: yhira@jb3.so-net.ne.jp,
yajimy01@kanagawa-u.ac.jp

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