Connected LCA groups are sequentially connected

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Dedicated to the 120th birthday anniversary of Eduard Čech.

Abstract. We prove that every connected locally compact Abelian topological group is sequentially connected, i.e., it cannot be the union of two proper disjoint sequentially closed subsets. This fact is then applied to the study of extensions of topological groups. We show, in particular, that if H is a connected locally compact Abelian subgroup of a Hausdorff topological group G and the quotient space G/H is sequentially connected, then so is G.

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1. Introduction

Connectedness is a subtle topological property which appears in a great number of important results in General Topology and Topological Algebra. The influence of connectedness on the algebraic structure of locally compact Abelian (LCA, for short) topological groups is especially strong — see [5, Sections 24 and 25].

Sequential connectedness is a natural strengthening of connectedness. A space is called sequentially connected if it cannot be represented as the union of two non-empty disjoint sequentially closed subsets. As usual, a subset Y of a space is sequentially closed in X if Y contains the limits of all convergent sequences lying in Y. Unlike connectedness, the notion of sequential connectedness is much less understood [4], [6], [9].

Our aim is to partially fill in this gap and show that every connected LCA group is sequentially connected (Theorem 2.5). We also prove in Theorem 3.5 that if H is a closed, sequentially connected, feathered subgroup of a Hausdorff topological group and the quotient space G/H is sequentially connected, then so is G. In particular, if H is a connected, LCA subgroup of a Hausdorff topological group G, then G is sequentially connected iff so is the quotient space G/H (Corollary 3.6).

2. Sequential connectedness in LCA groups

The following result will be used in the proof of Theorem 2.4:

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Lemma 2.1. *Let K be a metrizable topological group and τ be a cardinal. Suppose that an element $x \in K^\tau$ and a set $S \subseteq K^\tau$ satisfy the following conditions:*

(1) *the set $C(x) = \{x(\alpha) : \alpha \in \tau\}$ is countable;*
(2) *if $\beta, \beta' \in \tau$ and $x(\beta) = x(\beta')$, then $y(\beta) = y(\beta')$ for each $y \in S$.*

Then the subgroup $\langle S\rangle$ of K^τ generated by S is metrizable.

Proof: For every $z \in C(x)$, choose $\alpha_z \in \tau$ such that $x(\alpha_z) = z$. Then $A = \{\alpha_z : z \in C(x)\}$ is a countable subset of τ. It is easy to verify, using (2), that the restriction of the projection $\pi_A \colon K^\tau \to K^A$ to $\langle S\rangle$ is a topological isomorphism of $\langle S\rangle$ onto $\langle \pi_A(S)\rangle$. Since A is countable and K is metrizable, the groups $\langle \pi_A(S)\rangle$ and $\langle S\rangle$ are metrizable. $\square$

Let us call an element x of a topological group G *metrizable* if the cyclic subgroup of G generated by x is metrizable. Notice that by [1, Proposition 3.6.20], an element $x \in G$ is metrizable if and only if the closure of $\langle x\rangle$ in G is metrizable.

Corollary 2.2. *Let K be a metrizable topological group and τ be a cardinal. If $x \in K^\tau$ and the set $C(x) = \{x(\alpha) : \alpha \in \tau\}$ is countable, then the element x is metrizable.*

Proof: The conclusion follows from Lemma 2.1 if one takes $S = \{x\}$. $\square$

Lemma 2.3. *Let K be a compact connected metrizable group and τ be an arbitrary cardinal number. If $x \in K^\tau$ is a metrizable element of K^τ and the set $C(x)$ is countable, then x is contained in a closed connected metrizable subgroup of K^τ.*

Proof: For every $z \in C(x)$, take $\alpha_z \in \tau$ such that $z = x(\alpha_z)$. Then $A = \{\alpha_z : z \in C(x)\}$ is a non-empty countable subset of τ. Let π_A be the projection of K^τ onto K^A. Since the group K is divisible (see [1, Theorem 9.6.15]) we can define, for every $\alpha \in A$, a sequence $\{b(\alpha, n) : n \in \mathbb{N}^+\}$ in K such that $b(\alpha, 1) = x(\alpha)$ and $nb(\alpha, n) = b(\alpha, n-1)$ for each integer $n \geq 2$. For every $n \in \mathbb{N}$, denote by y_n the element of K^A such that $y_n(\alpha) = b(\alpha, n)$ for each $\alpha \in A$. It is clear from our definitions that $y_1 = \pi_A(x)$ and $ny_n = y_{n-1}$ for each integer $n \geq 2$. We now define a sequence $\{x_n : n \in \mathbb{N}^+\}$ in K^τ as follows. First, let $x_1 = x$. If $n \geq 2$, let $x_n(\alpha) = b(\alpha, n)$ if $\alpha \in A$, and $x_n(\beta) = x_n(\alpha)$ provided that $\beta \in \tau \setminus A$, $\alpha \in A$, and $x(\beta) = x(\alpha)$. Then the elements x_n's satisfy $\pi_A(x_n) = y_n$ and $nx_n = x_{n-1}$ for $n \geq 2$.

Furthermore, we claim that the element x and the set $\{x_n : n \in \mathbb{N}^+\}$ satisfy conditions (1) and (2) of Lemma 2.1. Since $C(x)$ is countable, it suffices to verify that if $n \in \mathbb{N}^+$, $\beta, \beta' \in \tau$, and $x(\beta) = x(\beta')$, then $x_n(\beta) = x_n(\beta')$. Indeed, our claim is immediate if either $\beta \in A$ or $\beta' \in A$. Suppose therefore that $\beta, \beta' \in \tau \setminus A$. Since $z = x(\beta) \in C(x)$, we see that $\alpha = \alpha_z \in A$ and $x(\alpha) = z = x(\beta)$. Again, the equalities $x_n(\beta) = x_n(\beta') = x_n(\alpha)$ follow from the definition of x_n. Applying Lemma 2.1 we conclude, therefore, that the subgroup D of K^τ generated by the set $\{x_n : n \in \omega\}$ is metrizable. Hence the closure of D in K^τ, say, $\overline{D}$ is also metrizable. Notice that $x \in D \subseteq \overline{D}$.

It remains to verify that the subgroup $\overline{D}$ of K^τ is connected. First, it follows from the equalities $nx_n = x_{n-1}$ for $n \geq 2$ that the group D is divisible. Since the closure of a divisible subgroup of a compact group is again divisible, the group $\overline{D}$ is connected by [1, Theorem 9.6.15]. This completes the proof of the lemma. $\square$

Theorem 2.4. *Every compact connected Abelian group is sequentially connected.*

Proof: Let G be a compact connected Abelian group. The Pontryagin dual $G^\wedge$ of G is a discrete torsion-free Abelian group [1, Theorem 9.6.11]. By [12, 4.1.6], the group $G^\wedge$ can be embedded into a divisible Abelian group, say, D. Let T be the torsion part of D. Clearly, $T \cap G^\wedge = \{e_D\}$. Denote by p the quotient homomorphism of D onto D/T. Then the group D/T is divisible, torsion-free, and $p(G^\wedge)$ is an isomorphic copy of $G^\wedge$ in D/T. Therefore, we can assume without loss of generality that the group D is torsion-free.

Since D is divisible and torsion-free, it is isomorphic to a direct sum of copies of the group of rationals, $\mathbb{Q}$. In other words, $D \cong \mathbb{Q}^{(\kappa)}$, for some cardinal κ (see [12, 4.1.5]). In what follows D carries the discrete topology. Let $r\colon D^\wedge \to G^{\wedge\wedge}$ be the mapping defined by $r(\chi) = \chi \restriction G^\wedge$, for each $\chi \in D^\wedge$. By [1, Proposition 9.6.2], r is a continuous homomorphism of $D^\wedge$ onto $G^{\wedge\wedge}$. It follows from [1, Proposition 9.6.25] that $D^\wedge \cong K^\tau$, where $K = (\mathbb{Q}_d)^\wedge$ and $\mathbb{Q}_d$ is the discrete group of rationals. By the Pontryagin duality theorem (see [1, Theorem 9.5.20]), the groups G and $G^{\wedge\wedge}$ are topologically isomorphic. Therefore, r is a continuous homomorphism of the compact group K^τ onto G. Since $K^\wedge \cong \mathbb{Q}_d$ is torsion-free, the group K is connected by [1, Theorem 9.6.11]. In addition, the group K is divisible according to [1, Theorem 9.6.15].

Let M be the union of all metrizable connected subgroups of K^τ. We claim that M is a dense subgroup of K^τ. First, M contains the neutral element of K^τ. It is also clear that $M^{-1} = M$. If $x, y \in M$, then there are metrizable connected subgroups C_x and C_y of K^τ such that $x \in C_x$ and $y \in C_y$. Let F_x and F_y be the closures of C_x and C_y, respectively, in K^τ. Then both F_x and F_y are compact connected metrizable groups, and so is the subgroup $F_x + F_y$ of K^τ as a continuous homomorphic image of the product group $F_x \times F_y$. Thus the element $x + y$ is contained in a connected metrizable group $F_x + F_y$, whence it follows that $M + M \subseteq M$. Hence M is a subgroup of K^τ. The density of M in K^τ is almost evident. Indeed, every element of the Σ-product ΣK^τ (see [1, Section 1.6]) is contained in a connected metrizable subgroup, so $\Sigma K^\tau \subseteq M$. Since ΣK^τ is dense in K^τ, so is M.

Let us show that M is *sequentially dense* in K^τ, i.e., for every point $y \in K^\tau$, there exists a sequence in M converging to y. Clearly, it suffices to take $y \in K^\tau \setminus M$. Let d be an invariant metric on K which generates the topology of K. For every $x \in K$ and every $\varepsilon > 0$, denote by $B(x, \varepsilon)$ the open ball $\{x' \in K : d(x, x') < \varepsilon\}$. Given a positive integer n, we can find a finite set $B_n \subseteq K$ such that K is covered by the open balls $B(x, 1/n)$ with $x \in B_n$.

For every $\alpha \in \tau$ and every integer $n > 0$, take an element $b(\alpha, n) \in B_n$ such that $d(y(\alpha), b(\alpha, n)) < 1/n$ and define a point $x_n \in K^\tau$ by $x_n(\alpha) = b(\alpha, n)$ for each $\alpha \in \tau$. Our definition of the elements x_n's implies that $x_n(\alpha) \to y(\alpha)$ for each $\alpha \in \tau$. Hence the sequence $\{x_n : n \in \mathbb{N}^+\}$ converges to y.

Further, each coordinate of x_n belongs to the finite set B_n, so the element $x_n \in K^\tau$ is metrizable by Corollary 2.2. Since the set $C(x_n)$ is finite, Lemma 2.3 implies that x_n is contained in a connected metrizable subgroup of K^τ. We have thus proved that $\{x_n : n \in \omega\} \subseteq M$ and that y is in the sequential closure of M. Since y is an arbitrary element of K^τ, the sequential closure of M is the whole group K^τ.

Suppose for a contradiction that the space K^τ fails to be sequentially closed. Then there exist non-empty disjoint sequentially closed subsets R and T of K^τ such that $R \cup T = K^\tau$. We can assume that R contains the neutral element e of K^τ. Take an arbitrary connected metrizable subgroup L of K^τ. Then L is covered by the disjoint sets $L \cap R$ and $L \cap T$, where $L \cap R \neq \emptyset$. Since L is metrizable, the sets $L \cap R$ and $L \cap T$ are closed in L. Hence the connectedness of L implies that the set $L \cap T$ is empty, i.e., $L \subseteq R$. It now follows that $M \subseteq R$. Since R is sequentially closed in K^τ and the sequential closure of M is K^τ, we conclude that $R = K^\tau$. This contradiction shows that the group K^τ is sequentially connected.

Finally, the group G is the image of K^τ under a continuous homomorphism r. Since sequential connectedness is preserved by continuous onto mappings, G is also sequentially connected. □

Let us show that the conclusion of Theorem 2.4 can be extended to the wider class of locally compact Abelian groups.

Theorem 2.5. *Every connected locally compact Abelian group G is sequentially connected.*

PROOF: Since G is a LCA group, it follows from [5, Theorem 24.30] that it has the form $\mathbb{R}^n \times C$, where $\mathbb{R}$ is the group of reals with the usual topology, n is a non-negative integer, and C is a locally compact group containing an open compact subgroup. The group C is a continuous homomorphic image of G under the natural projection of $\mathbb{R}^n \times C$ onto C, so C is connected. Therefore every open subgroup of C coincides with C and, hence, C is compact. By Theorem 2.4, C is sequentially connected. Since the space $\mathbb{R}^n$ is connected and metrizable, it is sequentially connected. Hence the product space $G = \mathbb{R}^n \times C$ is sequentially connected by [6, Theorem 4.2]. □

It is worth noting that the sequential connectedness of the space $\mathbb{R}^n \times C$ at the end of the proof of Theorem 2.5 can also be deduced from [4, Theorem 2.2]: A space is sequentially connected if and only if it is a continuous image of a connected metrizable space.

Although our proof of Theorem 2.4 makes use of the Pontryagin–van Kampen duality theorem, we hope that the following problem has a good chance to be solved affirmatively:

Problem 2.6. *Does Theorem* 2.4 *remain valid in the non-Abelian case, i.e., can one drop "Abelian" in the theorem?*

The referee noted that if the answer to the above problem is affirmative, then Theorem 2.5 is also valid in the non-Abelian case. Indeed, by a theorem of Davis [2], every locally compact topological group G is *homeomorphic* to the product space $C\times\mathbb{R}^n\times D$, where C is a compact subgroup of G, n is a non-negative integer, and D is a discrete space. If G is connected, then so is C and D is a singleton. Hence the sequential connectedness of C would imply the same property of G.

3. Sequential connectedness in extensions of topological groups

It was shown in [10, Theorem 4.7] that if a topological group G contains a locally compact, metrizable, connected, invariant subgroup H such that the quotient group G/H is sequentially connected, then G is also sequentially connected. We prove in Theorem 3.5 below that if H is a closed, sequentially connected, feathered subgroup of a topological group G and the quotient space G/H is sequentially connected, then so is G. Since every first countable connected space is sequentially connected, our result implies Theorem 4.7 of [10].

The proof of Theorem 3.5 requires several auxiliary results. Let us note that Lemma 3.1 is a more general version of [1, Theorem 1.5.20], while Lemma 3.3 itself is a generalization of [10, Theorem 4.7].

Lemma 3.1. *Let H and K be closed subgroups of a topological group G such that $K\subseteq H$. Suppose that X is a subspace of G/K and that the spaces H/K and $Y=\pi^K_H(X)\subseteq G/H$ are first countable, where π^K_H is the natural quotient mapping of G/K onto G/H. Then X is first countable as well.*

Proof: We assume that both G/K and G/H are left quotient spaces. Denote by π_K and π_H the quotient mappings of G onto G/K and G/H, respectively. Then $\pi_H=\pi^K_H\circ\pi_K$. Let e be the neutral element of G. Since the space G/K is homogeneous, we can assume that $e_K=\pi_K(e)\in X$. Clearly, it suffices to verify that X is first countable at e_K.

Let $\{W_n : m\in\omega\}$ and $\{U_n : n\in\omega\}$ be families of symmetric open neighborhoods of e in G such that $\{\pi_K(W_n)\cap\pi_K(H) : n\in\omega\}$ is a base of $\pi_K(H)\cong H/K$ at the point e_K, $\{\pi_H(U_n)\cap Y : n\in\omega\}$ is a base for Y at $\pi_H(e)$, and $W^2_{n+1}\subseteq W_n$ for each $n\in\omega$. We claim that the family

$$\gamma=\{\pi_K(U_i\cap W_j)\cap X : i,j\in\omega\}$$

is a base for X at e_K.

Indeed, take an arbitrary neighborhood O of e_K in G/K and let V be an open neighborhood of e in G such that $\pi_K(V^2)\subseteq O$. It follows from our choice of the family $\{W_n : n\in\omega\}$ that there exists $m\in\omega$ such that $\pi_K(W_m)\cap\pi_K(H)\subseteq\pi_K(V)$. Similarly, we can find an integer $k\in\omega$ such that $\pi_H(U_k)\cap Y\subseteq\pi_H(V\cap W_{m+1})$. Let us show that $\pi_K(U_k\cap W_{m+1})\cap X\subseteq O$.

Take an arbitrary element $x \in \pi_K(U_k \cap W_{m+1}) \cap X$ and choose an element $v \in U_k \cap W_{m+1}$ such that $\pi_K(v) = x$. Then

$$\pi_H(v) = \pi_H^K(x) \in \pi_H^K(\pi_K(U_k) \cap X) \subseteq \pi_H(U_k) \cap Y \subseteq \pi_H(V \cap W_{m+1}).$$

Therefore, $v \in (V \cap W_{m+1})H$. Further, since $v \in W_{m+1} = W_{m+1}^{-1}$ and $W_{m+1}^2 \subseteq W_m$, we see that $v \notin W_{m+1} \cdot (G \setminus W_m)$. Hence we conclude that $v \in (V \cap W_{m+1}) \cdot (H \cap W_m)$. Notice that $\pi_K(H \cap W_m) \subseteq \pi_K(H) \cap \pi_K(W_m) \subseteq \pi_K(V)$, whence it follows that $H \cap W_m \subseteq VK$. Therefore,

$$x = \pi_K(v) \in \pi_K\left[(V \cap W_{m+1}) \cdot (H \cap W_m)\right] \subseteq \pi_K(VVK) = \pi_K(V^2) \subseteq O.$$

This completes the proof of the inclusion $\pi_K(U_k \cap W_{m+1}) \cap X \subseteq O$. Thus γ is a countable local base for X at e_K and the space X is first countable. □

The following result is a step towards the proof of Proposition 3.4.

Lemma 3.2. *Let H and K be closed subgroups of a Hausdorff topological group G such that $K \subseteq H$ and the quotient space H/K is metrizable. If a sequence $\{y_n : n \in \omega\}$ in G/H converges to a point $y^* \in G/H$, then there exists a sequence $\{x_n : n \in \omega\}$ in G/K converging to a point $x^* \in G/K$ such that $\pi_H^K(x^*) = y^*$ and $\pi_H^K(x_n) = y_n$, for each $n \in \omega$.*

Proof: Again we assume that G/H and G/K are left quotient spaces. As in the proof of Lemma 3.1, let π_H and π_K be the quotient mappings of G onto G/H and G/K, respectively. Let also π_H^K be the natural quotient mapping of G/K onto G/H. Then $\pi_H = \pi_H^K \circ \pi_K$, whence it follows that the mapping π_H^K is continuous and open.

Suppose that a sequence $\{y_n : n \in \omega\} \subseteq G/H$ converges to a point y^*. Then $Y = \{y^*\} \cup \{y_n : n \in \omega\}$ is a compact metrizable subspace of G/H. By Lemma 3.1, the subspace $X = (\pi_H^K)^{-1}(Y)$ of G/K is first countable. It is also clear that the restriction of π_H^K to X is a continuous open mapping of X onto Y. Take a point $x^* \in G/K$ such that $\pi_H^K(x^*) = y^*$. Then $x^* \in X$, and there exists a countable base $\{V_k : k \in \omega\}$ for X at x^*. Since $y_n \to y^*$ in G/H, each V_k meets *almost all* fibers $(\pi_H^K)^{-1}(y_n)$, $n \in \omega$. Using this property of V_k's, one defines elements $x_n \in (\pi_H^K)^{-1}(y_n)$ by induction on k (*not* on n) such that each V_k contains almost all x_n's. Hence the sequence $\{x_n : n \in \omega\} \subseteq X$ converges to x^*, $\pi_H^K(x^*) = y^*$, and $\pi_H^K(x_n) = y_n$ for each $n \in \omega$. □

Lemma 3.3. *Let H and K be closed subgroups of a Hausdorff topological group G such that $K \subseteq H$. If the quotient space G/H is sequentially connected and the quotient space H/K is metrizable and connected, then G/K is sequentially connected.*

Proof: Again we assume that G/H and G/K are left quotient spaces. As in the proof of Lemma 3.1, let π_H and π_K be the quotient mappings of G onto G/H and G/K, respectively. Let also π_H^K be the natural quotient mapping of G/K onto G/H. Then $\pi_H = \pi_H^K \circ \pi_K$, whence it follows that the mapping π_H^K is open.

Suppose for a contradiction that G/K fails to be sequentially connected. Then we can find non-empty disjoint sequentially closed sets A and B in G/K such that $G/K = A \cup B$. Notice that each fiber $(\pi^K_H)^{-1}(x)$, with $x \in G/H$, is homeomorphic to the metrizable connected space H/K which is clearly sequentially connected. Hence each fiber of the mapping π^K_H lies either in A or in B and the sets $C = \pi^K_H(A)$ and $D = \pi^K_H(B)$ are disjoint, non-empty, and cover G/H. Notice that $A = (\pi^K_H)^{-1}(C)$ and $D = (\pi^K_H)^{-1}(B)$. Since the space G/H is sequentially connected, one of the sets C, D must contain a sequence converging to a point outside of the set. Suppose that this set is C and that a sequence $\{y_n : n \in \omega\} \subseteq C$ converges to a point $y^* \in D$. Applying Lemma 3.2, we find a sequence $\{x_n : n \in \omega\} \subseteq G/K$ converging to an element $x^* \in G/K$ such that $\pi^K_H(x^*) = y^*$ and $\pi^K_H(x_n) = y_n$ for each $n \in \omega$. It is clear that $\{x^*\} \cup \{x_n : n \in \omega\} \subseteq X$. Since $\{y_n : n \in \omega\} \subseteq C$ and $y^* \in D$, we see that $\{x_n : n \in \omega\} \subseteq A$ and $x^* \in B$. The latter contradicts our choice of the set A. Thus the space G/K is sequentially connected. □

We recall that a Hausdorff topological group H is called *feathered* (see [1, Section 4.3]) if H contains a non-empty compact subset with a countable neighborhood base in H. By [1, Theorem 4.3.20], H is feathered if and only if it contains a compact subgroup K such that the quotient space H/K is metrizable.

Proposition 3.4. *Let H be a closed feathered subgroup of a Hausdorff topological group G and $\pi\colon G \to G/H$ be the canonical quotient mapping. If a sequence $\{y_n : n \in \omega\}$ in G/H converges to an element $y^* \in G/H$, then there exists a sequence $\{x_n : n \in \omega\} \subseteq G$ converging to an element $x^* \in G$ such that $\pi(x^*) = y^*$ and $\pi(x_n) = y_n$, for each $n \in \omega$.*

Proof: Take a compact subgroup K of H such that the quotient space H/K is metrizable. Given a sequence $\{y_n : n \in \omega\}$ in G/H converging to an element $y^* \in G/H$, we can apply Lemma 3.2 to find a sequence $\{z_n : n \in \omega\}$ in G/K converging to an element $z^* \in G/K$ such that $\pi^K_H(z^*) = y^*$ and $\pi^K_H(z_n) = y_n$ for each $n \in \omega$, where $\pi^K_H\colon G/K \to G/H$ is the canonical quotient mapping. It remains to "lift" the sequence $\{z_n : n \in \omega\}$ with its limit z^* to a convergent sequence in G.

Since the group K is compact, there exists a cardinal $\tau \geq \omega$ and a family $\{K_\alpha : \alpha < \tau\}$ of closed subgroups of K satisfying the following conditions:

(i) $K_0 = K$;
(ii) $K_\beta \subseteq K_\alpha$ if $\alpha < \beta < \tau$;
(iii) $K_\alpha/K_{\alpha+1}$ is metrizable;
(iv) if $\beta < \tau$ is a limit ordinal, then $K_\beta = \bigcap_{\alpha<\beta} K_\alpha$;
(v) $\{e\} = \bigcap_{\alpha<\tau} K_\alpha$, where e is the neutral element of G.

For every $\alpha < \tau$, denote by π_α the quotient mapping of G onto G/K_α. If $\alpha < \beta < \tau$, then there exists a mapping $\pi_{\beta,\alpha}\colon G/K_\beta \to G/K_\alpha$ satisfying $\pi_\alpha = \pi_{\beta,\alpha} \circ \pi_\beta$. It is clear that the mappings π_α, π_β, and $\pi_{\beta,\alpha}$ are perfect and open.

It follows from (i)–(v) that the space G is the limit of the inverse system $\mathcal{S} = \{G/K_\alpha,\ \pi_{\beta,\alpha} : \alpha < \beta < \tau\}$ and that this system is *continuous* in the sense that for every limit ordinal $\gamma \geq \omega$, the space G/K_γ is the limit of the inverse system $\mathcal{S}_\gamma = \{G/K_\alpha,\ \pi_{\beta,\alpha} : \alpha < \beta < \gamma\}$.

Let $C_0 = \{z^*\} \cup \{z_n : n \in \omega\}$. Clearly C_0 is a countable compact subset of G/K_0. Suppose that $\gamma < \tau$ and, for each $\alpha < \gamma$, we have defined compact subsets (in fact, convergent sequences with the corresponding limit points) $C_\alpha \subseteq G/K_\alpha$ satisfying the following condition for all α, β with $\alpha < \beta < \gamma$:

($*$) $\pi_{\beta,\alpha} \restriction C_\beta$ is a one-to-one mapping of C_β onto C_α.

Let z_α^* be the unique non-isolated point of C_α, where $\alpha < \gamma$.

If γ is a limit ordinal, then the continuity of the system $\mathcal{S}_\gamma$ implies that

$$C_\gamma = \bigcap_{\alpha<\gamma} \pi_{\gamma,\alpha}^{-1}(C_\alpha)$$

is a compact subset of G/K_γ and for each $\alpha < \gamma$, the restriction of $\pi_{\gamma,\alpha}$ to C_γ is a one-to-one mapping of C_γ onto C_α. Hence C_γ contains a unique non-isolated point, say, z_γ^* and the set $C_\gamma \setminus \{z_\gamma^*\}$ converges to z_γ^*. It is clear that $\pi_{\gamma,\alpha}(z_\gamma^*) = z_\alpha^*$ for each $\alpha < \gamma$.

Finally, suppose that $\gamma = \alpha + 1$. By ($*$), K_α is a convergent sequence with the limit point z_α^*. By (iii), the quotient space K_α/K_γ is metrizable, so we apply Lemma 3.2 to find a convergent sequence C_γ with the limit point z_γ^* such that $\pi_{\gamma,\alpha}$ maps C_γ onto C_α in a one-to-one way. This finishes our construction of the sets C_α's.

Let $C = \bigcap_{\alpha<\tau} \pi_\alpha^{-1}(C_\alpha)$. Again, the continuity of the system $\mathcal{S}$ implies that C_α is a compact subset of G and for each $\alpha < \tau$, π_α maps C onto C_α in a one-to-one way. Therefore C is a convergent sequence in G with a limit point x^*. Evidently, $\pi_0(x^*) = z^*$, and we can enumerate the set $C \setminus \{x^*\}$ as $\{x_n : n \in \omega\}$ such that $\pi_0(x_n) = z_n$ for each $n \in \omega$. Since $\pi = \pi_H^K \circ \pi_0$, we conclude that $\pi(x^*) = y^*$ and $\pi(x_n) = y_n$ for each $n \in \omega$. This completes the proof of the theorem. □

The following result answers Question 4.8 of [10] affirmatively.

Theorem 3.5. *Let H be a closed, sequentially connected, feathered subgroup of a Hausdorff topological group G. If the quotient space G/H is sequentially connected, then so is G.*

Proof: Our argument is close to the one in the proof of Lemma 3.3. Suppose for a contradiction that G is not sequentially connected. Then G is the union of disjoint non-empty sequentially closed sets A and B. Denote by π the quotient mapping of G onto G/H. As in the proof of Lemma 3.3, we see that $C = \pi(A)$ and $D = \pi(B)$ are disjoint subsets of G/H. It is also clear that $A = \pi^{-1}(C)$ and $B = \pi^{-1}(D)$. Since the space G/H is sequentially connected, one of the sets C or D cannot be sequentially closed in G/H. Let it be C. Then C contains a sequence $\{y_n : n \in \omega\}$ converging to a point $y^* \in D$. Applying Proposition 3.4, we can "lift" the sequence $\{y_n : n \in \omega\}$ to a sequence $\{x_n : n \in \omega\}$ in G converging to some element $x^* \in G$. In other words, we have $\pi(x^*) = y^*$ and $\pi(x_n) = y_n$, for each $n \in \omega$. Then $\{x_n : n \in \omega\} \subseteq A$ and $x^* \in B$, whence it follows that A

fails to be sequentially closed in G. This contradiction completes the proof of the theorem. □

Since a locally compact Hausdorff topological group is Raĭkov complete and feathered, the following fact is now almost immediate.

Corollary 3.6. *Let H be a connected LCA subgroup of a Hausdorff topological group G. Then G is sequentially connected iff so is the quotient space G/H.*

Proof: It is clear that sequential connectedness is preserved by (sequentially) continuous onto mappings. Since G/H is a continuous image of G, the necessity is evident. The sufficiency follows from Theorems 2.5 and 3.5. □

4. Comments and conjectures

Similarly to sequential compactness, sequential connectedness is countably productive [6, Theorem 4.2]. We conjecture that sequential connectedness becomes productive if we take first countable factors:

Conjecture 4.1. *Let $\{X_i : i \in I\}$ be a family of first countable connected spaces. Then the product space $\prod_{i\in I} X_i$ is sequentially connected.*

By the Ivanovsky–Kuz'minov theorem (see [7] and [8]), every compact topological group is dyadic. Hence one can try to generalize Theorem 2.5 as follows:

Conjecture 4.2. *Every connected compact dyadic space is sequentially connected.*

Let us repeat Problem 2.6 in the form of a conjecture:

Conjecture 4.3. *Every locally compact connected (not necessarily Abelian) topological group is sequentially connected.*

According to [1, Corollary 4.2.2], every compact topological group of countable tightness is metrizable. Using this fact, we deduce the following more general result:

Proposition 4.4. *Every feathered topological group G of countable tightness is metrizable.*

Proof: There exists a compact subgroup K of G such that the quotient space G/K is metrizable [1, Theorem 4.3.20]. Since $t(K) \le t(G) \le \omega$, the group K is metrizable. Thus, both spaces K and G/K are metrizable, and so is G by [1, Corollary 1.5.21]. □

Corollary 4.5. *Let H be a locally compact sequential subgroup of a Hausdorff topological group G. If the quotient space G/H is sequential, then so is G.*

Proof: Since every locally compact topological group is feathered, Proposition 4.4 implies that H is metrizable. Hence the group G is sequential by [10, Theorem 4.3]. □

Since the group H in Corollary 4.5 is metrizable by a complete metric, we can try to generalize the corollary as follows:

Conjecture 4.6. *Let H be a (completely) metrizable subgroup of a Hausdorff topological group G. If the quotient space G/H is sequential, so is G.*

Modifying Example 2.4.20 in [3], one can show that the product of a normal second countable space with a normal Fréchet space can fail to be sequential. This indicates that, probably, the strong form of Conjecture 4.6 is false.

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