Continua with unique symmetric product

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Abstract. Let X be a metric continuum. Let $F_n(X)$ denote the hyperspace of nonempty subsets of X with at most n elements. We say that the continuum X has *unique hyperspace* $F_n(X)$ provided that the following implication holds: if Y is a continuum and $F_n(X)$ is homeomorphic to $F_n(Y)$, then X is homeomorphic to Y. In this paper we prove the following results: (1) if X is an indecomposable continuum such that each nondegenerate proper subcontinuum of X is an arc, then X has unique hyperspace $F_2(X)$, and (2) let X be an arcwise connected continuum for which there exists a unique point $v \in X$ such that v is the vertex of a simple triod. Then X has unique hyperspace $F_2(X)$.

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1. Introduction

A *continuum* is a compact connected metric space with more that one point. Given a continuum X and a positive integer n, we consider the following hyperspaces of X:

$$\begin{aligned} 2^X &= \{A \subset X : A \text{ is closed and nonempty}\}, \\ C(X) &= \{A \in 2^X : A \text{ is connected}\}, \\ C_n(X) &= \{A \in 2^X : A \text{ has at most } n \text{ components}\}, \\ F_n(X) &= \{A \in 2^X : A \text{ has at most } n \text{ points}\}. \end{aligned}$$

All these hyperspaces are considered with the Hausdorff metric H_X [13, Theorem 2.2].

The hyperspace $F_n(X)$ is called the n^{th}-*symmetric product of* X.

Let $\mathcal{H}(X)$ denote one of the hyperspaces 2^X, $C(X)$, $C_n(X)$ or $F_n(X)$. We say that the continuum X has *unique hyperspace* $\mathcal{H}(X)$ provided that the following implication holds: if Y is a continuum and $\mathcal{H}(X)$ is homeomorphic to $\mathcal{H}(Y)$, then X is homeomorphic to Y.

The topic of this paper is inserted in the following general problem.

Problem 1. Find conditions on the continuum X in order that X has unique hyperspace $\mathcal{H}(X)$.

Problem 1 has been widely studied by a number of authors. The recently published paper [10] contains a detailed survey on this subject. For $n \geq 4$, the most general result on uniqueness of symmetric products is Theorem 5 of [5] that says that if $n \geq 4$ and X is a wired continuum, then X has unique hyperspace $F_n(X)$. Since the class of wired continua includes finite graphs, dendrites with closed set of end points, fans, compactifications of the ray $[0, \infty)$, compactifications of the real line and indecomposable arc continua, we have that Theorem 5 of [5] generalizes previous results in [1], [3], [6], [7], [9] and [12].

A continuum X is *indecomposable* provided that X is not the union of two proper subcontinua. An *arc continuum* is a continuum such that all its nondegenerate proper subcontinua are arcs. A *simple triod* is a continuum Y such that $Y = L_1 \cup L_2 \cup L_3$ where each L_i is an arc, and there exists a point v in Y such that v is an end point of each L_i and if $i \neq j$, then $L_i \cap L_j = \{v\}$. The point v is called the *vertex* of Y.

In [10, Question 42] it is asked if indecomposable arc continua have unique hyperspaces $F_2(X)$ and $F_3(X)$ (see also [5, Question 27]).

In this paper we prove the following theorems.

Theorem. *Let X be an indecomposable arc continuum. Then X has unique hyperspace $F_2(X)$.*

Theorem. *Let X be an arcwise connected continuum for which there exists a unique point $v \in X$ which is the vertex of a simple triod. Then X has unique hyperspace $F_2(X)$.*

The following questions remain open [10, Question 42].

Question 2. *Let X be an indecomposable arc continuum. Does X have unique hyperspace $F_3(X)$?*

Question 3. *Let X be an arcwise connected continuum for which there exists a unique point $v \in X$ which is the vertex of a simple triod. Does X have unique hyperspace $F_3(X)$?*

2. Indecomposable arc continua

We need the following conventions.

Given a continuum X, $A \subset X$, $x \in X$ and $\varepsilon > 0$, let $B(x, \varepsilon)$ be the open ε-ball around x in X and $N(A, \varepsilon) = \bigcup\{B(x, \varepsilon) \subset X : x \in A\}$.

Given a topological space Y and a point $e \in Y$. The point e is an *end point* of Y provided that e is an end point of each arc in Y containing e. Let

$$\begin{aligned} F_1(Y) &= \{\{y\} : y \in Y\}, \\ E(Y) &= \{e \in Y : e \text{ is an end point of } Y\} \text{ and} \\ \mathcal{AC}(Y) &= \{K \subset Y : K \text{ is an arc component of Y}\}. \end{aligned}$$

We consider the following families of continua:

$$\begin{aligned}\mathfrak{A} = \{X : X &\text{ is a continuum and there exists } \varepsilon > 0 \text{ such that if } A \text{ is} \\ &\text{a subcontinuum of } X \text{ and } 0 < \text{diameter}(A) < \varepsilon, \text{ then } A \text{ is an arc}\},\\ \mathfrak{DC} = \{X : X &\text{ is a continuum and for each } K \in \mathcal{AC}(X),\ K \text{ is dense in } X\},\\ \mathfrak{W} = \{X \in \mathfrak{A} \cap \mathfrak{DC} : X &\text{ has uncountably many arc components}\}.\end{aligned}$$

Given $X \in \mathfrak{W}$, let

$$\mathcal{N}\partial F_2(X) = \{A \in F_2(X) : A \text{ does not belong to the manifold interior of a 2-cell } \mathcal{M} \text{ that is contained in } F_2(X)\}.$$

By [14, Theorem 11.15], it follows that each indecomposable arc continuum belongs to $\mathfrak{W}$.

Given a continuum X and subsets J, L of X, let

$$\langle J, L\rangle = \{A \in F_2(X) : A \subset J \cup L,\ A \cap J \neq \emptyset \text{ and } A \cap L \neq \emptyset\}.$$

Lemma 4. *Let X be a continuum. Then $\mathcal{AC}(F_2(X)) = \{\langle J, L\rangle : J, L \in \mathcal{AC}(X)\}$.*

Proof: (Compare with [5, Lemma 22]). Let $J, L \in \mathcal{AC}(X)$. We take $\{x, y\}, \{u, v\} \in \langle J, L\rangle$, where $x, u \in J$ and $y, v \in L$ (here, it is possible that $J = L$, $x = y$ or $u = v$). Since J and L are arc components of X, there exist arcs α and β (possibly degenerate) such that $x, u \in \alpha \subset J$ and $y, v \in \beta \subset L$. Notice that $\{\{x, r\} : r \in \beta\} \subset \langle J, L\rangle$ and $\{\{s, v\} : s \in \alpha\} \subset \langle J, L\rangle$ are homeomorphic to β and α, respectively and the element $\{x, v\}$ belongs to both sets. Thus, $\{\{x, r\} : r \in \beta\} \cup \{\{s, v\} : s \in \alpha\}$ is an arcwise connected subset of $\langle J, L\rangle$ that contains $\{x, y\}$ and $\{u, v\}$. We have shown that $\langle J, L\rangle$ is arcwise connected.

Let $\mathcal{D}$ be the arc component of $F_2(X)$ containing $\langle J, L\rangle$. Fix an element $\{x, y\} \in \langle J, L\rangle$, where $x \in J$ and $y \in L$; and let $\{u, v\} \in \mathcal{D}$. Let Δ be an (possibly degenerate) arc in $F_2(X)$ such that $\{x, y\}, \{u, v\} \in \Delta$. Let $B = \bigcup\{A : A \in \Delta\}$. By [5, Lemma 1], B has at most two components. Moreover, if $B = B_1 \cup B_2$, where B_1 and B_2 are the components of B (possibly $B_1 = B_2$), we may assume that $x, u \in B_1$ and $y, v \in B_2$. By [4, Lemma 2.2], B_1 and B_2 are locally connected continua and then they are arcwise connected. Thus, $B_1 \subset J$ and $B_2 \subset L$. This proves that $\{u, v\} \in \langle J, L\rangle$. We have shown that $\mathcal{D} = \langle J, L\rangle$. Therefore, $\langle J, L\rangle$ is an arc component of $F_2(X)$.

Clearly, $F_2(X) = \bigcup\{\langle J, L\rangle : J, L \in \mathcal{AC}(X)\}$. Therefore, $\mathcal{AC}(F_2(X)) = \{\langle J, L\rangle : J, L \in \mathcal{AC}(X)\}$. □

Corollary 5. *Let X be a continuum. Then X is arcwise connected if and only if $F_2(X)$ is arcwise connected.*

Lemma 6. *Let $X \in \mathfrak{W}$ and $A \in F_2(X)$. Then $A \in \mathcal{N}\partial F_2(X)$ if and only if $A \in F_1(X)$ or $A \cap E(X) \neq \emptyset$.*

Proof: (Necessity). Suppose that $A = \{x, y\}$, where $x \neq y$ and $A \cap E(X) = \emptyset$. Then there exist arcs E and F in X such that $x \in E$, $y \in F$, x is not an end point of E and y is not an end point of F. Shortening E and F, if necessary, we may assume that $E \cap F = \emptyset$. It is easy to show that the map $\varphi : E \times F \to \langle E, F\rangle$ given by $\varphi(u, v) = \{u, v\}$ is a homeomorphism. Thus $\langle E, F\rangle$ is a 2-cell containing $\{x, y\}$ in its interior as manifold. This proves that $A \notin \mathcal{N}\partial F_2(X)$ and completes the proof of the necessity.

(Sufficiency). Let d be a metric for X. Since $X \in \mathfrak{A}$, there exists $\varepsilon > 0$ such that for each subcontinuum B of X with $0 < \text{diameter}(B) < \varepsilon$, we have that B is an arc. Suppose that $A \notin \mathcal{N}\partial F_2(X)$. Then there exists a 2-cell $\mathcal{M}$ in $F_2(X)$ such that A is in the manifold interior of $\mathcal{M}$. Taking a smaller 2-cell contained in $\mathcal{M}$, if necessary, we may assume that diameter$(\mathcal{M}) <\frac{\varepsilon}{2}$. Let $B = \bigcup\{D \in F_2(X) : D \in \mathcal{M}\}$. We need to show that $A \notin F_1(X)$ and $A \cap E(X) = \emptyset$.

Suppose, first, that $A \in F_1(X)$. Then $A = \{x\}$ for some $x \in X$. By [5, Lemma 1], B is a subcontinuum of X. Since diameter$(\mathcal{M}) < \frac{\varepsilon}{2}$, $H_X(D, \{x\}) < \frac{\varepsilon}{2}$ for each $D \in \mathcal{M}$. Thus, $B \subset N(\{x\}, \frac{\varepsilon}{2})$ and diameter$(B) < \varepsilon$. Hence, B is an arc. Thus, (see [11, Section 13]) there exists a homeomorphism $h : F_2(B) \to [0, 1] \times [0, 1]$ such that $h(\{x\}) = (0, 0)$. Notice that $\mathcal{M} \subset F_2(B)$, so $h(\mathcal{M}) \subset [0, 1] \times [0, 1]$ and $h(\mathcal{M})$ is a 2-cell containing $(0, 0)$ in its manifold interior. This contradicts the Invariance of Domain Theorem [8, Theorem VI 9] and completes the proof that $A \notin F_1(X)$.

Now, suppose that $A \cap E(X) \neq \emptyset$. Then $A = \{v, x\}$ for some $v \in E(X)$ and $v \neq x$. Let $\delta = \min\{\frac{d(v,x)}{2}, \frac{\varepsilon}{2}\} > 0$. In this case, we may assume that diameter$(\mathcal{M}) < \delta$. Since $B \subset N(A, \delta) = B(v, \delta) \cup B(x, \delta)$ and $v, x \in B$, by [5, Lemma 1], we have that B has exactly two components E and F, where $E \subset B(v, \delta)$ and $F \subset B(x, \delta)$ and, for each $D \in \mathcal{M}$, $D\cap E \neq \emptyset$ and $D\cap F \neq \emptyset$. This implies that $\mathcal{M} \subset \langle E, F\rangle$. By the choice of ε, E and F are (possibly degenerate) arcs. Since $A \in \mathcal{M}$, $A \in \langle E, F\rangle$, so $v \in E$ and $x \in F$. Thus, v is an end point of E. Let $g : E \times F \to \langle E, F\rangle$ be given by $g(s, t) = \{s, t\}$. Clearly, g is a homeomorphism. Hence, $\langle E, F\rangle$ is a 2-cell (or an arc) having $A = \{v, x\}$ in its manifold boundary and $\mathcal{M}$ is a 2-cell such that A belongs to the manifold interior of $\mathcal{M}$ and $\mathcal{M} \subset \langle E, F\rangle$. This contradicts again the Invariance of Domain Theorem [8, Theorem VI 9] and completes the proof of the sufficiency. □

Lemma 7. *Let $X \in \mathfrak{W}$ and $\mathcal{K} \in \mathcal{AC}(\mathcal{N}\partial F_2(X))$. Then $\mathcal{K}$ is one of the forms described in* (1)–(4).

(1) $\mathcal{K} = F_1(L)$ *for some* $L \in \mathcal{AC}(X)$ *such that* $L \cap E(X) = \emptyset$.
(2) $\mathcal{K} = F_1(L) \cup \{\{v, x\} \in F_2(X) : x \in L\}$ *for some* $L \in \mathcal{AC}(X)$ *and* $v \in E(X) \cap L$.
(3) $\mathcal{K} = \{\{v, x\} \in F_2(X) : x \in L\}$ *for some* $L \in \mathcal{AC}(X)$ *and* $v \in E(X) - L$.
(4) $\mathcal{K} = \{\{v, x\} \in F_2(X) : x \in J\} \cup \{\{w, x\} \in F_2(X) : x \in L\}$ *for some* $L, J \in \mathcal{AC}(X)$, $v \in E(X) \cap L$, $w \in E(X) \cap J$ *and* $L \neq J$.

Proof: First, we will see that for each $J \in \mathcal{AC}(X)$, $|J \cap E(X)| \leq 1$. Suppose to the contrary that there exist two different elements $v, w \in J \cap E(X)$. Since

J is arcwise connected, there exists an arc $\alpha \subset J$ with end points v and w. We claim that $J = \alpha$. Suppose that there exists a point $x \in J - \alpha$. Since J is arcwise connected, there exists an arc β such that $\beta \cap \alpha = \{y\}$ for some $y \in J$. If $y = v$, then $\alpha \cup \beta$ is an arc in X, $v \in \alpha \cup \beta$ and v is not an end point of $\alpha \cup \beta$. This contradicts the fact that $v \in E(X)$. Thus, $y \neq v$. Similarly, $y \neq w$. Hence, $y \in \alpha - \{v, w\}$ and y is the vertex of the simple triod $\alpha \cup \beta$ in X. Since $X \in \mathfrak{W}$, there exists $\varepsilon > 0$ such that if $B \in C(X)$ and $0 < \text{diameter}(B) < \varepsilon$, then B is an arc. Since $\alpha \cup \beta$ contains simple triods of diameter less than ε, we obtain a contradiction. This completes the proof that $J = \alpha$. Since $X \in \mathfrak{W}$, J is dense in X, so $X = \alpha$. This contradicts the fact that X has uncountably many arc components ($X \in \mathfrak{W}$). We have shown that $|J \cap E(X)| \leq 1$.

By Lemma 4, $\mathcal{AC}(F_2(X)) = \{\langle J, L\rangle \subset F_2(X) : J, L \in \mathcal{AC}(X)\}$. Thus, there exist $J, L \in \mathcal{AC}(X)$ such that $\mathcal{K} \subset \langle J, L\rangle$. We consider five cases.

Case 1. $J = L$ and $J \cap E(X) = \emptyset$.

In this case, $\mathcal{K} \subset \langle J, L\rangle = F_2(J)$. By Lemma 6, $F_2(J) \cap \mathcal{N}\partial F_2(X) = F_1(J)$. Since $F_1(J)$ is homeomorphic to J, $F_1(J)$ is arcwise connected. Since $\mathcal{K} \subset F_2(J) \cap \mathcal{N}\partial F_2(X)$, we conclude that $\mathcal{K} = F_1(J)$ and $\mathcal{K}$ is as in (1).

Case 2. $J = L$ and $J \cap E(X) \neq \emptyset$.

In this case, $\mathcal{K} \subset \langle J, L\rangle = F_2(J)$. Let $v \in J \cap E(X)$. Then $J \cap E(X) = \{v\}$. By Lemma 6, $F_2(J) \cap \mathcal{N}\partial F_2(X) = F_1(J) \cup \{\{v, x\} : x \in J\}$. Since the function $x \to \{v, x\}$, from J onto $\{\{v, x\} : x \in J\}$ is continuous, $\{\{v, x\} : x \in J\}$ is arcwise connected and intersects $F_1(J)$ in the element $\{v\}$. Thus, $F_1(J) \cup \{\{v, x\} : x \in J\}$ is arcwise connected and $\mathcal{K} = F_1(J) \cup \{\{v, x\} : x \in J\}$. Hence, $\mathcal{K}$ is as in (2).

Case 3. $J \neq L$ and $(J \cup L) \cap E(X) = \emptyset$.

By Lemma 6, $\langle J, L\rangle \cap \mathcal{N}\partial F_2(X) = \emptyset$. Since $\mathcal{K} \subset \langle J, L\rangle \cap \mathcal{N}\partial F_2(X)$, we conclude that this case is impossible.

Case 4. $J \neq L$, $J \cap E(X) \neq \emptyset$ and $L \cap E(X) = \emptyset$.

Let $v \in J$ be such that $J \cap E(X) = \{v\}$. By Lemma 6, $\langle J, L\rangle \cap \mathcal{N}\partial F_2(X) = \{\{v, x\} : x \in L\}$. Since this set is arcwise connected, we obtain that $\mathcal{K} = \{\{v, x\} : x \in L\}$ and $\mathcal{K}$ is as in (3).

Case 5. $J \neq L$, $J \cap E(X) \neq \emptyset$ and $L \cap E(X) \neq \emptyset$.

Let $v \in L$ and $w \in J$ be such that $L \cap E(X) = \{v\}$ and $J \cap E(X) = \{w\}$. By Lemma 6, $\langle J, L\rangle \cap \mathcal{N}\partial F_2(X) = \{\{v, x\} \in F_2(X) : x \in J\} \cup \{\{w, x\} \in F_2(X) : x \in L\}$. Since both sets in this union are arcwise connected and they meet in the element $\{v, w\}$, we obtain that $\mathcal{K} = \{\{v, x\} \in F_2(X) : x \in J\} \cup \{\{w, x\} \in F_2(X) : x \in L\}$ is as in (4).

This completes the proof of the lemma. □

Lemma 8. *Let X be an indecomposable arc continuum and let Y be a continuum such that $F_2(X)$ is homeomorphic to $F_2(Y)$. Then $Y \in \mathfrak{A}$.*

Proof: Let d_Y be a metric for Y. Let $h : F_2(Y) \to F_2(X)$ be a homeomorphism. Let $\delta = \text{diameter}(X)$. Then there exists $\varepsilon > 0$ such that if $A, B \in F_2(Y)$ and $H_Y(A,B) < \varepsilon$, then $H_X(h(A), h(B)) < \frac{\delta}{5}$.

Let $Z \in C(Y)$ be such that $0 < \text{diameter}(Z) < \varepsilon$.

Fix an element $A_0 \in F_2(Z)$. Suppose that $h(A_0) = \{r, s\}$, where $r = s$ in the case that $h(A_0)$ is a one-point set. Given $B \in F_2(Z)$. For each $a \in A_0$ and each $b \in B$, we have that $a, b \in Z$ and $d_Y(a,b) < \varepsilon$. This implies that $H_Y(A_0, B) < \varepsilon$. Thus, $H_X(h(A_0), h(B)) < \frac{\delta}{5}$ and $h(B) \subset N(h(A_0), \frac{\delta}{5}) = B(r, \frac{\delta}{5}) \cup B(s, \frac{\delta}{5})$.

Let $D = \bigcup\{h(B) \in F_2(X) : B \in F_2(Z)\}$. By the previous paragraph, $D \subset B(r, \frac{\delta}{5}) \cup B(s, \frac{\delta}{5})$. By [5, Lemma 1], D has at most two components. Let D_1, D_2 be the components of D, where $D_1 = D_2$ in the case that D is connected. In the case that $D_1 \neq D_2$, each element of $h(F_2(Z))$ intersects both sets D_1 and D_2 ([5, Lemma 1]). So, in this case, $h(F_2(Z)) \subset \langle D_1, D_2 \rangle$. In the case that $D_1 = D_2$, $h(F_2(Z)) \subset F_2(D)$.

We claim that $D \neq X$. Suppose to the contrary that $D = X$. Then $X = B(r, \frac{\delta}{5}) \cup B(s, \frac{\delta}{5})$. By the connectedness of X, $B(r, \frac{\delta}{5}) \cap B(s, \frac{\delta}{5}) \neq \emptyset$. This implies that $\text{diameter}(X) \leq \frac{4\delta}{5} < \delta$, which is a contradiction. We have shown that $D \neq X$. Since X is an arc continuum, each set D_1 and D_2 is either an arc or a one-point set.

In the case that $D_1 = D_2$, $F_2(D)$ is a 2-cell ([11, Section 13]) and $F_2(Z)$ can be embedded in $F_2(D)$. In the case that $D_1 \neq D_2$, $\langle D_1, D_2 \rangle$ is homeomorphic to $D_1 \times D_2$, and then $\langle D_1, D_2 \rangle$ is either an arc or a 2-cell. In both cases, $F_2(Z)$ can be embedded in a 2-cell. By [2, Theorem 5], Z is an arc.

We have shown that $Y \in \mathfrak{A}$. $\square$

Theorem 9. *Let X be an indecomposable arc continuum. Then X has unique hyperspace $F_2(X)$.*

Proof: Let Y be a continuum and let $h : F_2(Y) \to F_2(X)$ be a homeomorphism.

By Lemma 4, $\mathcal{AC}(F_2(X)) = \{\langle J, L \rangle \subset F_2(X) : J, L \in \mathcal{AC}(X)\}$. By [14, Theorem 11.15] it follows that $\mathcal{AC}(X)$ is uncountable. This implies that $\mathcal{AC}(F_2(X))$ is uncountable. Since h is a homeomorphism, $\mathcal{AC}(F_2(Y) = \{h^{-1}(K) : K \in \mathcal{AC}(F_2(X))\}$. Thus, $\mathcal{AC}(F_2(Y)) = \{\langle J, L \rangle \subset F_2(Y) : J, L \in \mathcal{AC}(Y)\}$ is uncountable. Hence, $\mathcal{AC}(Y)$ is uncountable.

Given $K \in \mathcal{AC}(Y)$, we will see that K is dense in Y. Let U be a nonempty open subset of Y. By Lemma 4, $\langle K \rangle$ is an arc component of $F_2(Y)$. Thus, $h(\langle K \rangle)$ is an arc component of $F_2(X)$. Applying Lemma 4 again, we have that there exist $J, L \in \mathcal{AC}(X)$ such that $h(\langle K \rangle) = \langle J, L \rangle$. Since $\langle U \rangle$ is nonempty and open in $F_2(Y)$, $h(\langle U \rangle)$ is nonempty and open in $F_2(X)$. Since $F_2(X) - F_1(X)$ is dense in $F_2(X)$, there exist $u, x \in X$ such that $\{u, x\} \in h(\langle U \rangle)$ and $u \neq x$. Then there exists $\varepsilon > 0$ such that if $A \in F_2(X)$ and $H_X(A, \{u, x\}) < \varepsilon$, then $A \in h(\langle U \rangle)$ and $\varepsilon < \frac{d(u,x)}{2}$. Since J and L are dense in X, there exist points $r \in B(u, \varepsilon) \cap J$ and $s \in B(x, \varepsilon) \cap L$. Then $H_X(\{r, s\}, \{u, x\}) < \varepsilon$ and $\{r, s\} \in h(\langle U \rangle) \cap \langle J, L \rangle$. Thus, $h^{-1}(\{r, s\}) \in \langle U \rangle \cap \langle K \rangle$ and $h^{-1}(\{r, s\}) \subset U \cap K$. Hence, $U \cap K \neq \emptyset$. Therefore, K is dense in Y.

By Lemma 8, we conclude that $Y \in \mathfrak{W}$.

Since $\mathcal{N}\partial F_2(X)$ is defined in terms of topological properties of $F_2(X)$, we obtain that $h(\mathcal{N}\partial F_2(Y)) = \mathcal{N}\partial F_2(X)$. Hence, for each $\mathcal{K} \in \mathcal{AC}(\mathcal{N}\partial F_2(Y))$, $h(\mathcal{K}) \in \mathcal{AC}(\mathcal{N}\partial F_2(X))$ and $h(\mathrm{cl}_{F_2(Y)}(\mathcal{K})) = \mathrm{cl}_{F_2(X)}(h(\mathcal{K}))$.

Given $L \in \mathcal{AC}(Y)$ and $v \in E(X)$, since L is dense in Y, $\mathrm{cl}_{F_2(Y)}(F_1(L)) = F_1(Y)$ and $\mathrm{cl}_{F_2(Y)}(\{\{v,x\} \in F_2(Y) : x \in L\}) = \{\{v,x\} \in F_2(Y) : x \in Y\}$ is homeomorphic to Y.

Notice that an element Z of $\mathfrak{W}$ does not have cut points since a cut point belongs to each dense arc component and elements in $\mathfrak{W}$ have uncountably many dense arc components.

Given $\mathcal{K} \in \mathcal{AC}(\mathcal{N}\partial F_2(Y))$, we have that $\mathcal{K}$ is of one of the forms described in Lemma 7. In the case that $\mathcal{K}$ is of the form (1), then $\mathrm{cl}_{F_2(Y)}(\mathcal{K})$ is homeomorphic to Y; if $\mathcal{K}$ is of the form (2), then $\mathrm{cl}_{F_2(Y)}(\mathcal{K})$ is homeomorphic to two copies of Y joined by a point (the element $\{v\}$); if $\mathcal{K}$ is of the form (3), then $\mathrm{cl}_{F_2(Y)}(\mathcal{K})$ is homeomorphic to Y; and if $\mathcal{K}$ is of the form (4), then $\mathrm{cl}_{F_2(Y)}(\mathcal{K})$ is homeomorphic to two copies of Y joined by a point (the element $\{v,w\}$).

Fix $\mathcal{K} \in \mathcal{AC}(\mathcal{N}\partial F_2(Y))$. Then $\mathrm{cl}_{F_2(Y)}(\mathcal{K})$ is homeomorphic to $\mathrm{cl}_{F_2(X)}(h(\mathcal{K}))$. By the previous paragraph, $\mathrm{cl}_{F_2(Y)}(\mathcal{K})$ (resp., $\mathrm{cl}_{F_2(X)}(h(\mathcal{K}))$) is either homeomorphic to Y (resp., X) or homeomorphic to two copies of Y (resp., X) joined by a point. Since elements in $\mathfrak{W}$ does not have cut points, it is not possible that $\mathrm{cl}_{F_2(Y)}(\mathcal{K})$ is homeomorphic to Y and $\mathrm{cl}_{F_2(X)}(h(\mathcal{K}))$ is homeomorphic to two copies of X joined by a point; and it is not possible that $\mathrm{cl}_{F_2(X)}(h(\mathcal{K}))$ is homeomorphic to X and $\mathrm{cl}_{F_2(Y)}(\mathcal{K})$ is homeomorphic to two copies of Y joined by a point.

Therefore, we only have two possibilities:

(a) $\mathrm{cl}_{F_2(Y)}(\mathcal{K})$ is homeomorphic to Y and $\mathrm{cl}_{F_2(X)}(h(\mathcal{K}))$ is homeomorphic to X; or

(b) $\mathrm{cl}_{F_2(Y)}(\mathcal{K})$ is homeomorphic to two copies of Y joined by a point and $\mathrm{cl}_{F_2(X)}(h(\mathcal{K}))$ is homeomorphic to two copies of X joined by a point.

Clearly, each of the statements (a) and (b) implies that X is homeomorphic to Y. □

Corollary 10. *The Buckethandle continuum X has unique hyperspace $F_2(X)$.*

Corollary 11. *Each solenoid X has unique hyperspace $F_2(X)$.*

3. Arcwise connected continua

Theorem 12. *Let X be an arcwise connected continuum for which there exists a unique point $v_0 \in X$ such that v_0 is the vertex of a simple triod. Then X has unique hyperspace $F_2(X)$.*

Proof: Let Y be a continuum and let $h : F_2(X) \to F_2(Y)$ be a homeomorphism. Define

$$\mathcal{A}(X) = \{A \in F_2(X) : \text{for each neighborhood } \mathcal{U} \text{ of } A \text{ in } F_2(X) \text{ there exists a locally connected subcontinuum } \mathcal{M} \text{ of } F_2(X) \text{ such that } \mathcal{M} \subset \mathcal{U} \text{ and } \mathcal{M} \text{ is not embeddable in a 2-manifold}\}.$$

Since $\mathcal{A}(X)$ is defined only using topological properties, we conclude that $h(\mathcal{A}(X)) = \mathcal{A}(Y)$.

Given an arcwise connected continuum Z, let $V(Z) = \{z \in Z : z$ is the vertex of a simple triod contained in $Z\}$.

Claim 1. Let Z be an arcwise connected continuum. Then

(a) $\{A \in F_2(Z) : A \cap V(Z) \neq \emptyset\} \subset \mathcal{A}(Z)$,

(b) if $A \in F_2(Z)$ and $A \cap \mathrm{cl}_Z(V(Z)) = \emptyset$, then $A \notin \mathcal{A}(Z)$.

In order to prove (a), take $A \in F_2(Z)$ and $v \in V(Z)$ such that $v \in A$. Let $\mathcal{U}$ be a neighborhood of A in $F_2(Z)$. If $A = \{v\}$, we can take an element $z \in Z - \{v\}$ such that $B = \{v, z\} \in \mathrm{int}_{F_2(Z)}(\mathcal{U})$. In the case that $A \neq \{v\}$, $A = \{v, z\}$ for some $z \in Z - \{v\}$ and put $B = A$. In both cases, there exists $B = \{v, z\} \in \mathrm{int}_{F_2(Z)}(\mathcal{U})$, where $v \neq z$. Let T be a simple triod in Z such that v is the vertex of T and $z \notin T$. Since Z is arcwise connected, there exists an arc J in Z such that $z \in J$ and $T \cap J = \emptyset$. Shortening T and J, if necessary, we may assume that the set $\mathcal{M} = \{\{a, b\} \in F_2(Z) : a \in T$ and $b \in J\}$ is contained in $\mathcal{U}$. Notice that $\mathcal{M}$ is homeomorphic to $T \times J$ and thus $\mathcal{M}$ is not embeddable in a 2-manifold (this follows from the Invariance of Domain Theorem [8, Theorem VI 9]). Hence, $A \in \mathcal{A}(Z)$.

To prove (b), take $A \in F_2(Z)$ such that $A \cap \mathrm{cl}_Z(V(Z)) = \emptyset$. Let $\mathcal{U}$ be a closed neighborhood of A in $F_2(Z)$ such that for each $B \in \mathcal{U}$, $B \cap \mathrm{cl}_Z(V(Z)) = \emptyset$. Let $\mathcal{M}$ be a locally connected subcontinuum of $F_2(Z)$ such that $\mathcal{M} \subset \mathcal{U}$. Let $M = \bigcup\{B : B \in \mathcal{M}\}$. By [5, Lemma 1] and [4, Lemma 2.2], M has at most two components and each one of them is a locally connected continuum. Then $M = M_1 \cup M_2$, where M_1 and M_2 are the components of M and it is possible that $M_1 = M_2$. Notice that no point of M is the vertex of a simple triod in Z. Thus, M does not contain simple triods. Hence, M_1 and M_2 are locally connected continua without simple triods. Therefore, each M_i is either an arc or a simple closed curve. In the case that $M_1 = M_2$, M is an arc or a simple closed curve, so ([11, Section 13]) $F_2(M)$ is a 2-cell or a Moebius strip. Since $\mathcal{M} \subset F_2(M)$, $\mathcal{M}$ is embeddable in the Klein Bottle. In the case that $M_1 \neq M_2$, by [5, Lemma 1], each element $B \in \mathcal{M}$ intersects both sets M_1 and M_2. Thus, $\mathcal{M}$ is contained in the set $\mathcal{N} = \langle M_1, M_2 \rangle$ and $\mathcal{N}$ is homeomorphic to $M_1 \times M_2$ which is homeomorphic to some of the following continua: (a) $[0, 1]^2$, (b) $S^1 \times [0, 1]$ or (c) $S^1 \times S^1$. In any case, $\mathcal{N}$ is embeddable in $S^1 \times S^1$. Since $\mathcal{M} \subset \mathcal{N}$, we conclude that $A \notin \mathcal{A}(Z)$.

As a consequence of Claim 1, we obtain the following.

Claim 2. $\mathcal{A}(X) = \{A \in F_2(X) : v_0 \in A\}$.

Since the map $f : X \to F_2(X)$ given by $f(x) = \{v_0, x\}$ is an embedding and $f(X) = \mathcal{A}(X)$, we obtain that X, $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ are homeomorphic.

Claim 3. $V(Y)$ is a one-point set.

We prove Claim 3. If $V(Y) = \emptyset$, by Claim 1(b), $\mathcal{A}(Y) = \emptyset$. This is a contradiction since $\mathcal{A}(Y)$ is homeomorphic to X. If $V(Y)$ contains two different elements v_1 and v_2, by Claim 1(a), $\mathcal{R} = \{A \in F_2(Y) : v_1 \in A\}$ is contained in $\mathcal{A}(Y)$. Since the map $g : Y \to \mathcal{R}$ given by $g(y) = \{v_1, y\}$ is a homeomorphism and $V(Y)$ has two elements, we obtain that $V(\mathcal{R})$ has two elements. Thus, $\mathcal{R}$ is a subcontinuum of $\mathcal{A}(Y)$ such that $|V(\mathcal{R})| \geq 2$. Since $\mathcal{A}(Y)$ is homeomorphic to X, we obtain that X contains a subcontinuum R such that $|V(R)| \geq 2$. Since $|V(X)| \geq |V(R)| \geq 2$, we obtain a contradiction. This completes the proof of Claim 3.

By Claim 3, $V(Y) = \{w\}$, for some $w \in Y$. By Claim 1, $\mathcal{A}(Y) = \{\{w, y\} : y \in Y\}$ which is homeomorphic to Y. Therefore, X is homeomorphic to Y. □

Corollary 13. *If X belongs to one of the following families of continua, then X has unique hyperspace $F_2(X)$.*

(a) *fans,*
(b) *cones over compact metric spaces containing no arcs,*
(c) *cones over hereditarily indecomposable continua.*

By Corollary 13 and [5, Theorem 5], each fan has unique hyperspace for all $n \neq 3$.

Question 14 ([10, Question 41])**.** *Let X be a fan, does X have unique hyperspace $F_3(X)$?*

Question 15. *Let X be an arcwise connected continuum for which there exists a unique point $v_0 \in X$ such that v_0 is the vertex of a simple triod. Does X have unique hyperspace $F_3(X)$?*

The most important question on the topic of this paper is the following.

Question 16 ([10, Question 43])**.** *Does there exist a finite-dimensional continuum X such that X does not have unique hyperspace $F_n(X)$ for some $n > 1$?*

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