Images of some functions and functional spaces under the Dunkl-Hermite semigroup

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Abstract. We propose the study of some questions related to the Dunkl-Hermite semigroup. Essentially, we characterize the images of the Dunkl-Hermite-Sobolev space, $\mathcal{S}(\mathbb{R})$ and $L^p_\alpha(\mathbb{R})$, $1 < p < \infty$, under the Dunkl-Hermite semigroup. Also, we consider the image of the space of tempered distributions and we give Paley-Wiener type theorems for the transforms given by the Dunkl-Hermite semigroup.

Keywords: Dunkl-Hermite functions; Dunkl-Hermite semigroup; Dunkl-Hermite-Sobolev space

Classification: 42B25, 46E35, 47B38, 47D03

1. Introduction and statement of the results

Let D_α, $\alpha \geq -\frac{1}{2}$, be the Dunkl operator on the real line defined by

$$D_\alpha f(x) = f'(x) + \frac{2\alpha+1}{x}\Big[\frac{f(x)-f(-x)}{2}\Big], \ f \in C^1(\mathbb{R}).$$

To this operator is associated the Dunkl-Hermite operator

$$\mathcal{H}_\alpha = -D_\alpha^2 + x^2.$$

Its spectral decomposition is given by the Dunkl-Hermite functions h_n^α defined by

$$h_n^\alpha(x) = e^{-\frac{x^2}{2}} H_n^\alpha(x), \ n \in \mathbb{N},$$

namely we have (see [11])

$$\mathcal{H}_\alpha h_n^\alpha(x) = (2n+2\alpha+2)h_n^\alpha(x).$$

Here H_n^α is the Dunkl-Hermite polynomial given by

$$H_n^\alpha(x) = 2^{-\frac{n}{2}}\sqrt{\frac{b_n(\alpha)}{\Gamma(\alpha+1)}}\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k}{k!b_{n-2k}(\alpha)}(2x)^{n-2k},$$

where $b_n(\alpha)$ is the generalized factorial defined by Rosenblum in [10],

$$b_n(\alpha) = \frac{2^n([\frac{n}{2}])!}{\Gamma(\alpha+1)}\Gamma\Big([\frac{n+1}{2}]+\alpha+1\Big),$$

$[n/2]$ denotes the integral part of $n/2$. More precisely, these polynomials are expressed in terms of the Laguerre polynomials,

$$H_n^\alpha(x) = \frac{(-1)^{[\frac{n}{2}]}}{\sqrt{\Gamma(\alpha+1)}}\,\frac{2^{\frac{n}{2}}([\frac{n}{2}])!}{\sqrt{b_n(\alpha)}}\; x^{\theta_n} L_{[\frac{n}{2}]}^{\alpha+\theta_n}(x^2),$$

where θ_n is defined to be 0 if n is even and 1 if n is odd.

Hereafter, $L^p_\alpha(\mathbb{R}) = L^p(\mathbb{R}, |x|^{2\alpha+1}dx)$, $1 \le p < +\infty$, denotes the space of measurable functions on $\mathbb{R}$ satisfying

$$\|f\|_{\alpha,p} := \Big(\int_{\mathbb{R}} |f(x)|^p |x|^{2\alpha+1}\,dx\Big)^{\frac{1}{p}} < +\infty.$$

It is known that $\{h_n^\alpha, n \in \mathbb{N}\}$ forms an orthonormal basis of $L^2_\alpha(\mathbb{R})$. So for $f \in L^2_\alpha(\mathbb{R})$

$$\mathcal{H}_\alpha f = \sum_{n=0}^{\infty}(2n+2\alpha+2)a_n^\alpha(f)h_n^\alpha$$

with $a_n^\alpha(f) = \int_{\mathbb{R}} f(x)h_n^\alpha(x)|x|^{2\alpha+1}\,dx$.

Then, for a non-negative integer m, the Dunkl-Hermite-Sobolev space $\mathcal{W}^{m,2}_{\mathcal{H}_\alpha}(\mathbb{R})$ is defined to be the image of $L^2_\alpha(\mathbb{R})$ under $(\mathcal{H}_\alpha)^{-m}$. We remark that $\mathcal{W}^{m,2}_{\mathcal{H}_\alpha}(\mathbb{R})$ is a Hilbert space under the inner product

$$\langle f, g\rangle_{\mathcal{W}^{m,2}_{\mathcal{H}_\alpha}} = \sum_{n=0}^{\infty}(2n+2\alpha+2)^{2m}a_n^\alpha(f)\overline{a_n^\alpha(g)}.$$

The Dunkl-Hermite semigroup denoted by $e^{-t\mathcal{H}_\alpha}$, $t > 0$, is defined by

$$e^{-t\mathcal{H}_\alpha}f = \sum_{n=0}^{\infty} e^{-(2n+2\alpha+2)t}a_n^\alpha(f)h_n^\alpha$$

for $f \in L^2_\alpha(\mathbb{R})$ and $f = \sum_{n=0}^{\infty} a_n^\alpha(f)h_n^\alpha$.

Using the Mehler formula for the Dunkl-Hermite polynomials H_n^α (see [10]), we can write $e^{-t\mathcal{H}_\alpha}$, on a dense subspace of $L^2_\alpha(\mathbb{R})$, as an integral operator with kernel $\mathcal{M}_t^\alpha(x,y)$

$$[e^{-t\mathcal{H}_\alpha}f](x) = \int_{\mathbb{R}} f(y)\mathcal{M}_t^\alpha(x,y)|y|^{2\alpha+1}dy. \tag{1}$$

The kernel $\mathcal{M}_t^\alpha(x,y)$ can be explicitly written as

$$\mathcal{M}_t^\alpha(x,y) = \frac{1}{\Gamma(\alpha+1)(2\sinh(2t))^{\alpha+1}}e^{-\frac{1}{2}\coth(2t)(x^2+y^2)}E_\alpha\Big(\frac{x}{\sinh(2t)}, y\Big),$$

where $E_\alpha(\xi, x)$ is the Dunkl kernel given by

$$E_\alpha(\xi, x) = j_\alpha(\xi x) + \frac{\xi x}{2(\alpha+1)} j_{\alpha+1}(\xi x),$$

j_β being the spherical Bessel function of order β given by

$$j_\beta(t) = \Gamma(\beta+1) \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\beta+1)} (\frac{t}{2})^{2n} .$$

We define the holomorphic Dunkl-Sobolev space $\mathcal{W}^{m,2}_{t,\alpha}(\mathbb{C})$ as the image of $\mathcal{W}^{m,2}_{\mathcal{H}_\alpha}(\mathbb{R})$ under $e^{-t\mathcal{H}_\alpha}$. It can be viewed as a Hilbert space simply by transfering the Hilbert space structure of $\mathcal{W}^{m,2}_{\mathcal{H}_\alpha}(\mathbb{R})$. In what follows, we give a characterization of the space $\mathcal{W}^{m,2}_{t,\alpha}(\mathbb{C})$.

Using the reproducing kernel property, we show that if F is a holomorphic function on $\mathbb{C}$, then there exists a function $f \in \mathcal{S}(\mathbb{R})$ (the Schwartz space) such that $F = e^{-t\mathcal{H}_\alpha} f$ if and only if F satisfies

$$|F(z)|^2 \leq C_{t,\alpha,m} \frac{e^{-\tanh(2t)x^2 + \coth(2t)y^2}}{(1+x^2+y^2)^{2m}}, \quad z = x+iy,$$

for some constant $C_{t,\alpha,m}$ $m = 1, 2, 3, \ldots$

The formula (1) permits to extend $e^{-t\mathcal{H}_\alpha}$ on the spaces $L^p_\alpha(\mathbb{R})$. We establish that if $f \in L^p_\alpha(\mathbb{R})$ for $1 < p < \infty$ then $e^{-t\mathcal{H}_\alpha}(f)$ is holomorphic and $e^{-t\mathcal{H}_\alpha}(f) \in L^s_\alpha(\mathbb{C}, V^{\frac{s+\epsilon}{2}}_{t,\frac{p}{2}})$ for every $\epsilon > 0$ and any $1 \leq s < \infty$, where

$$V^r_{t,\frac{p}{2}}(x+iy) = \exp\Big(-2r\big(\frac{p}{(p-1)\sinh 4t} x^2 + \frac{\coth 2t}{2} y^2\big)\Big).$$

Next, we consider the space of tempered distributions. For $S \in \mathcal{S}'(\mathbb{R})$, we show that $e^{-t\mathcal{H}_\alpha}$ is given by a function defined by

$$e^{-t\mathcal{H}_\alpha} S(x) = e^{-\frac{1}{2}(\frac{\cosh 2t - 1}{\sinh 2t})x^2} \big(e^{-\frac{1}{2}(\frac{\cosh 2t-1}{\sinh 2t})y^2} S *_\alpha q_{\frac{\sinh 2t}{2}}\big)(x),$$

where q_t, $t > 0$, denotes the heat kernel associated with the Dunkl operator D_α, given by

$$q_t(x) = \frac{1}{\Gamma(\alpha+1)} \ (4t)^{-(\alpha+1)} e^{-\frac{x^2}{4t}} ,$$

and $*_\alpha$ is the generalized convolution product associated with the Dunkl operator D_α (see [13]). Moreover, $e^{-t\mathcal{H}_\alpha} S$ is a $\mathcal{C}^\infty$ function on $\mathbb{R}$.

These results permit us to characterize the image of tempered distributions on $\mathbb{R}$ under the Dunkl-Hermite semigroup. We establish that if F is a holomorphic function on $\mathbb{C}$, then there exists a distribution $f \in \mathcal{S}'(\mathbb{R})$ with $F = e^{-t\mathcal{H}_\alpha} f$ if and only if F satisfies

$$|F(z)|^2 \leq C_{t,\alpha}(1+|z|^2)^{2m} \exp\big(-\tanh(2t)x^2 + \coth(2t)y^2\big),$$

for some non-negative integer m.

Next, we define the transform $\mathcal{T}_a^\alpha$, for $a > 0$, by

$$\mathcal{T}_a^\alpha(S)(x) = \langle S, e^{-\frac{1}{2}a(\cdot)^2} E_\alpha(-ix,\cdot)\rangle, \ S \in \mathcal{S}'(\mathbb{R}).$$

We prove that this transform is related to the Dunkl-Hermite semigroup and we establish a Paley-Wiener theorem for $\mathcal{T}_a^\alpha f$. For any $a > 0$ the transform $\mathcal{T}_a^\alpha$ of a tempered distribution f on $\mathbb{R}$ extends to $\mathbb{C}$ as an entire function which satisfies the estimate

$$|\mathcal{T}_a^\alpha f(z)| \leq C_\alpha(1+x^2+y^2)^m e^{\frac{1}{2}a^{-1}y^2}$$

for some non-negative integer m. Conversely, if an entire function F satisfies such an estimate, then $F = \mathcal{T}_a^\alpha f$ for some tempered distribution f.

Again relating the Dunkl-Hermite semigroup and the Dunkl transform, we obtain a characterization of the image of compactly supported distributions under the Dunkl-Hermite semigroup. If f is a distribution supported in a ball of radius R centered at the origin then for any $t > 0$ the function $e^{-t\mathcal{H}_\alpha} f$ extends to $\mathbb{C}$ as an entire function which satisfies

$$|e^{-t\mathcal{H}_\alpha} f(z)| \leq C e^{-\frac{1}{2}\coth 2t(x^2-y^2)} e^{\frac{R|x|}{\sinh 2t}}.$$

Conversely, any entire function F satisfying the above condition is of the form $e^{-t\mathcal{H}_\alpha} f$, where f is supported inside a ball of radius R centered at the origin.

We point out that the results of this paper extend naturally those established in [8] by R. Radha and S. Thangavelu.

We conclude this introduction by giving the organization of this paper. In the next section, we define the Dunkl-Hermite-Sobolev space and we characterize its images under the Dunkl-Hermite semigroup. The third section deals with a characterization of the image of $\mathcal{S}(\mathbb{R})$ and $L^p_\alpha(\mathbb{R})$ under the Dunkl-Hermite semigroup. In the last section we establish Paley-Wiener type theorems for the tempered distributions and the compactly supported distributions under the Dunkl-Hermite semigroup.

2. Holomorphic Dunkl-Sobolev spaces

We have established in [1] that every element in the range of the operator $e^{-t\mathcal{H}_\alpha}$ defined on L^2_α can be analytically extended to the complex plane $\mathbb{C}$, hence we shall consider the operator $e^{-t\mathcal{H}_\alpha}$ as a linear operator from L^2_α into an entire function space and the entire extension will be simply denoted by $e^{-t\mathcal{H}_\alpha} f(z)$, $z = x + iy$.

In this section, we introduce the Dunkl-Hermite-Sobolev space and we give a characterization of its images under the Dunkl-Hermite semigroup.

Notation 1. Let $U^{\alpha}_{t,e}(z)=\frac{2}{\pi\sinh(4t)}K_{\alpha}(\frac{|z|^2}{\sinh(4t)})\exp\{\coth(4t)(x^2-y^2)\}|z|^{2\alpha+2}$ and $U^{\alpha}_{t,o}(z)=\frac{2}{\pi\sinh(4t)}K_{\alpha+1}(\frac{|z|^2}{\sinh(4t)})\exp\{\coth(4t)(x^2-y^2)\}|z|^{2\alpha+2}$. We have

$$U^{\alpha}_{t,o}(z)=\frac{U^{\alpha+1}_{t,e}(z)}{|z|^2}\,.$$

Here K_ν is the Macdonald function defined in [4] by:

$$K_\nu(z)=\frac{\pi}{2}\frac{I_{-\nu}(z)-I_\nu(z)}{\sin(\nu\pi)}\,,\ \nu\in\mathbb{C}\backslash\mathbb{Z},\ |\arg(z)|<\pi$$

where

$$I_\nu(z)=\frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^{\nu}j_\nu(z)$$

and for an integer n,

$$K_n(z)=\lim_{\nu\to n}K_\nu(z).$$

Let $\mathcal{H}^{\alpha}_{t,e}(\mathbb{C})$ denote the Hilbert space of all even entire functions on $\mathbb{C}$ which are square integrable with respect to the weight function $U^{\alpha}_{t,e}$, equipped with the inner product defined by

$$\langle f,g\rangle_{\alpha,e}=\int_{\mathbb{C}}f(z)\overline{g(z)}U^{\alpha}_{t,e}(z)\,dz.$$

Let $\mathcal{H}^{\alpha}_{t,o}(\mathbb{C})$ denote the Hilbert space of all odd entire functions on $\mathbb{C}$ which are square integrable with respect to the weight function $U^{\alpha}_{t,o}$, equipped with the inner product defined by

$$\langle f,g\rangle_{\alpha,o}=\int_{\mathbb{C}}f(z)\overline{g(z)}U^{\alpha}_{t,o}(z)\,dz.$$

Let $\mathcal{H}^{\alpha}_{t}$ denote the direct sum of $\mathcal{H}^{\alpha}_{t,e}$ and $\mathcal{H}^{\alpha}_{t,o}$ admitting the inner product

$$\langle f,g\rangle_{\alpha,t}=\langle f_e,g_e\rangle_{\alpha,e}+\langle f_o,g_o\rangle_{\alpha,o}\,,$$

where $f_e(z)=\frac{f(z)+f(-z)}{2}$ and $f_o(z)=\frac{f(z)-f(-z)}{2}$.

We recall the following results proved in [1].

Theorem 1. *The image of $L^2_\alpha(\mathbb{R})$ under the Dunkl-Hermite semigroup is the Fock type space $\mathcal{H}^{\alpha}_{t}$. The Dunkl-Hermite semigroup $e^{-t\mathcal{H}_\alpha}$ is an isometric isomorphism from $L^2_\alpha(\mathbb{R})$ into $\mathcal{H}^{\alpha}_{t}(\mathbb{C})$.*

Also we have the orthogonality property

$$\begin{aligned}\langle h^{\alpha}_{n},h^{\alpha}_{m}\rangle_{\alpha,t}&=\int_{\mathbb{C}}h^{\alpha}_{n,e}(z)\overline{h^{\alpha}_{m,e}(z)}U^{\alpha}_{t,e}(z)\,dz+\int_{\mathbb{C}}h^{\alpha}_{n,o}(z)\overline{h^{\alpha}_{m,o}(z)}U^{\alpha}_{t,o}(z)\,dz\\&=e^{2(2n+2\alpha+2)t}\delta_{n,m}\,,\end{aligned}\tag{2}$$

where $h_n^\alpha(z)$ is the extension of the Dunkl-Hermite function $h_n^\alpha(x)$ to $\mathbb{C}$ as an entire function.

Let $\widetilde{h_n^\alpha}(z) = e^{-(2n+2\alpha+2)t} h_n^\alpha(z)$, then $\{\widetilde{h_n^\alpha}, n \in \mathbb{N}\}$ forms an orthonormal basis for $\mathcal{H}_t^\alpha(\mathbb{C})$. Thus any $F \in \mathcal{H}_t^\alpha(\mathbb{C})$ can be written as

$$F = \sum_{n=0}^{\infty} \langle F, \widetilde{h_n^\alpha} \rangle_{\alpha,t} \widetilde{h_n^\alpha}.$$

Definition 1. Let m be a non-negative integer. The Dunkl-Hermite-Sobolev space $\mathcal{W}_{\mathcal{H}_\alpha}^{m,2}(\mathbb{R})$ is defined to be the image of $L_\alpha^2(\mathbb{R})$ under $(\mathcal{H}_\alpha)^{-m}$.

Remark 1. We remark that $f \in \mathcal{W}_{\mathcal{H}_\alpha}^{m,2}(\mathbb{R})$ if and only if $\sum_{n=0}^{\infty}(2n + 2\alpha + 2)^{2m} |a_n^\alpha(f)|^2 < \infty$. The Sobolev space $\mathcal{W}_{\mathcal{H}_\alpha}^{m,2}(\mathbb{R})$ is an Hilbert space under the inner product

$$\langle f, g \rangle_{\mathcal{W}_{\mathcal{H}_\alpha}^{m,2}} = \sum_{n=0}^{\infty} (2n + 2\alpha + 2)^{2m} a_n^\alpha(f) \overline{a_n^\alpha(g)}.$$

As $(\mathcal{H}_\alpha)^m f = \sum_{n=0}^{\infty}(2n + 2\alpha + 2)^m a_n^\alpha(f) h_n^\alpha$ then

$$\langle f, g \rangle_{\mathcal{W}_{\mathcal{H}_\alpha}^{m,2}} = \langle (\mathcal{H}_\alpha)^m f, (\mathcal{H}_\alpha)^m g \rangle_{L_\alpha^2}.$$

Definition 2. We define the holomorphic Dunkl-Sobolev space $\mathcal{W}_{t,\alpha}^{m,2}(\mathbb{C})$ to be the image of $\mathcal{W}_{\mathcal{H}_\alpha}^{m,2}(\mathbb{R})$ under $e^{-t\mathcal{H}_\alpha}$.

Remark 2. It is clear that by transferring the Hilbert space structure of $\mathcal{W}_{\mathcal{H}_\alpha}^{m,2}(\mathbb{R})$ to $\mathcal{W}_{t,\alpha}^{m,2}(\mathbb{C})$, the space $\mathcal{W}_{t,\alpha}^{m,2}(\mathbb{C})$ becomes a Hilbert space. The Dunkl-Hermite semigroup $e^{-t\mathcal{H}_\alpha}$ is an isometric isomorphism from $\mathcal{W}_{\mathcal{H}_\alpha}^{m,2}(\mathbb{R})$ onto $\mathcal{W}_{t,\alpha}^{m,2}(\mathbb{C})$. Then we can write

$$\langle F, G \rangle_{\mathcal{W}_{t,\alpha}^{m,2}} = \sum_{n=0}^{\infty} (2n + 2\alpha + 2)^{2m} a_n^\alpha(f) \overline{a_n^\alpha(g)}$$

whenever $F = e^{-t\mathcal{H}_\alpha} f$ and $G = e^{-t\mathcal{H}_\alpha} g$.

Notation 2. We denote by $\mathcal{O}(\mathbb{C})$ the set of all holomorphic functions on $\mathbb{C}$. Let $\mathcal{F}_{t,e}^{m,\alpha}(\mathbb{C})$ be the space of all even functions in $\mathcal{O}(\mathbb{C})$ which are square integrable with respect to the measure $|\frac{d^{2m}}{dt^{2m}} U_{t,e}^\alpha(z)| dz$. We equip $\mathcal{F}_{t,e}^{m,\alpha}(\mathbb{C})$ with the sesquilinear form

$$\langle F, G \rangle_{m,e} = \int_{\mathbb{C}} F(z) \overline{G(z)} \frac{d^{2m}}{dt^{2m}} U_{t,e}^\alpha(z)\, dz.$$

Let $\mathcal{F}_{t,o}^{m,\alpha}(\mathbb{C})$ be the space of all odd functions in $\mathcal{O}(\mathbb{C})$ which are square integrable with respect to the measure $|\frac{d^{2m}}{dt^{2m}} U_{t,o}^\alpha(z)| dz$. We equip $\mathcal{F}_{t,o}^{m,\alpha}(\mathbb{C})$ with the sesquilinear form

$$\langle F, G\rangle_{m,o} = \int_{\mathbb{C}} F(z)\overline{G(z)}\frac{d^{2m}}{dt^{2m}}U^{\alpha}_{t,o}(z)\,dz.$$

Let $\mathcal{F}^{m,\alpha}_t(\mathbb{C})$ be the direct sum of $\mathcal{F}^{m,\alpha}_{t,e}(\mathbb{C})$ and $\mathcal{F}^{m,\alpha}_{t,o}(\mathbb{C})$ admitting the sesquilinear form

$$\langle F, G\rangle_{m,\alpha} = \langle F_e, G_e\rangle_{m,e} + \langle F_o, G_o\rangle_{m,o}.$$

We shall show below that this defines a pre-Hilbert space structure on $\mathcal{F}^{m,\alpha}_t(\mathbb{C}) \cap \mathcal{H}^{\alpha}_t(\mathbb{C})$.

Let $\mathcal{B}^{m,\alpha}_t(\mathbb{C})$ denote the completion of $\mathcal{F}^{m,\alpha}_t(\mathbb{C}) \cap \mathcal{H}^{\alpha}_t(\mathbb{C})$ with respect to the norm induced by the above inner product. In the following proposition, we also show that $\|F\|_{m,\alpha}$ and $\|F\|_{\mathcal{W}^{m,2}_{t,\alpha}}$ coincide up to a constant multiple.

Proposition 1. *The sesquilinear form $\langle F, G\rangle_{m,\alpha}$, for a non-negative integer m, is an inner product on $\mathcal{F}^{m,\alpha}_t(\mathbb{C}) \cap \mathcal{H}^{\alpha}_t(\mathbb{C})$ and hence induces a norm $\|F\|^2_{m,\alpha} = \langle F, F\rangle_{m,\alpha}$. We also have*

$$\|F\|^2_{m,\alpha} = 2^{2m}\|F\|^2_{\mathcal{W}^{m,2}_{t,\alpha}}$$

for all functions $F = e^{-t\mathcal{H}_\alpha} f$ with $f \in \mathcal{S}(\mathbb{R})$.

Proof: Let F be in $\mathcal{F}^{m,\alpha}_t(\mathbb{C}) \cap \mathcal{H}^{\alpha}_t(\mathbb{C})$. We expand the restriction of F to $\mathbb{R}$ into an orthogonal expansion in terms of h^{α}_n (see [1]), and we can write

$$F(x+iy) = \sum_n \langle F, h^{\alpha}_n\rangle_{2,\alpha} h^{\alpha}_n(x+iy),$$

so we have that

$$\begin{aligned} I^{\alpha}_t &:= \int_{\mathbb{C}} |F_e(x+iy)|^2 U^{\alpha}_{t,e}(z)\,dz + \int_{\mathbb{C}} |F_o(x+iy)|^2 U^{\alpha}_{t,o}(z)\,dz \\ &= \Big\langle \sum_n \langle F, h^{\alpha}_n\rangle_{2,\alpha} h^{\alpha}_n, \sum_q \langle F, h^{\alpha}_q\rangle_{2,\alpha} h^{\alpha}_q \Big\rangle_{\alpha,t}. \end{aligned}$$

Using the orthogonality relation (2), we can show that

$$I^{\alpha}_t = \sum_n |\langle F, h^{\alpha}_n\rangle_{2,\alpha}|^2 e^{2(2n+2\alpha+2)t}.$$

By definition, for a nonnegative integer m, we have

$$\begin{aligned}\langle F, F\rangle_{m,\alpha} &= \int_{\mathbb{C}} |F_e(z)|^2 \frac{d^{2m}}{dt^{2m}} U^\alpha_{t,e}(z)\, dz + \int_{\mathbb{C}} |F_o(z)|^2 \frac{d^{2m}}{dt^{2m}} U^\alpha_{t,o}(z)\, dz \\ &= \frac{d^{2m}}{dt^{2m}} I^\alpha_t \\ &= 2^{2m} \sum_n (2n+2\alpha+2)^{2m} |\langle F, h^\alpha_n\rangle_{2,\alpha}|^2 e^{2(2n+2\alpha+2)t}.\end{aligned}$$

Thus it follows that the sesquilinear form defined above is positive definite and induces the norm $\|F\|_{m,\alpha}$.

On the other hand, we have the expansion

$$F(z) = \sum_{m=0}^\infty \langle F, \widetilde{h^\alpha_m}\rangle_{\alpha,t} \widetilde{h^\alpha_m}(z)$$

and

$$F = e^{-t\mathcal{H}_\alpha} f \quad \text{with} \quad f \in L^2_\alpha(\mathbb{R}).$$

Thus we have

$$\begin{aligned}\langle F, h^\alpha_n\rangle_{2,\alpha} &= \int_{\mathbb{R}} \sum_{m=0}^\infty \langle F, \widetilde{h^\alpha_m}\rangle_{\alpha,t} \widetilde{h^\alpha_m}(x) h^\alpha_n(x) |x|^{2\alpha+1}\, dx \\ &= \int_{\mathbb{R}} \sum_{m=0}^\infty \langle f, h^\alpha_m\rangle_{2,\alpha} e^{-(2m+2\alpha+2)t} h^\alpha_m(x) h^\alpha_n(x) |x|^{2\alpha+1}\, dx \\ &= \sum_{m=0}^\infty \langle f, h^\alpha_m\rangle_{2,\alpha} e^{-(2m+2\alpha+2)t} \int_{\mathbb{R}} h^\alpha_m(x) h^\alpha_n(x) |x|^{2\alpha+1}\, dx \\ &= \langle f, h^\alpha_n\rangle_{2,\alpha} e^{-(2n+2\alpha+2)t}.\end{aligned}$$

Interchanging the order of summation and integration is justified by Lebesgue's dominated convergence theorem and limiting behavior of $\|h^\alpha_n\|_{\alpha,p}$ given in [2]. Again using the orthogonality relation (2), we get

$$\begin{aligned}\|F\|^2_{m,\alpha} &= 2^{2m} \sum_n (2n+2\alpha+2)^{2m} |\langle F, h^\alpha_n\rangle_{2,\alpha}|^2 e^{2(2n+2\alpha+2)t} \\ &= 2^{2m} \sum_n (2n+2\alpha+2)^{2m} |\langle f, h^\alpha_n\rangle_{2,\alpha}|^2 \\ &= 2^{2m} \sum_n (2n+2\alpha+2)^{2m} |\langle F, \widetilde{h^\alpha_n}\rangle_{\alpha,t}|^2 \\ &= 2^{2m} \|F\|^2_{\mathcal{W}^{m,2}_{t,\alpha}}.\end{aligned}$$

□

Using this proposition we can easily prove the following result on the range of the Dunkl-Hermite-Sobolev spaces under the Dunkl-Hermite semigroup.

Theorem 2. *For every nonnegative integer m, $\mathcal{W}^{m,2}_{t,\alpha}(\mathbb{C})$ coincides with $\mathcal{B}^{m,\alpha}_t(\mathbb{C})$ and the Dunkl-Hermite semigroup $e^{-t\mathcal{H}_\alpha}$ is an isometric isomorphism from $\mathcal{W}^{m,2}_{\mathcal{H}_\alpha}(\mathbb{R})$ onto $\mathcal{B}^{m,\alpha}_t(\mathbb{C})$ up to a constant multiple.*

Proof: Let $F \in \mathcal{F}^{m,\alpha}_t(\mathbb{C}) \cap \mathcal{H}^\alpha_t(\mathbb{C})$, hence F is of the form $e^{-t\mathcal{H}_\alpha} f$ with $f \in L^2_\alpha(\mathbb{R})$. Further, it follows from the above proposition, as the norms $\|F\|_{m,\alpha}$ and $\|F\|_{\mathcal{W}^{m,2}_{t,\alpha}}$ coincide, that $f \in \mathcal{W}^{m,2}_{\mathcal{H}_\alpha}(\mathbb{R})$. Consequently, $\mathcal{F}^{m,\alpha}_t(\mathbb{C}) \cap \mathcal{H}^\alpha_t(\mathbb{C})$ is contained in $\mathcal{W}^{m,2}_{t,\alpha}(\mathbb{C})$. We have $\widetilde{h^\alpha_n} = e^{-t\mathcal{H}_\alpha} h^\alpha_n$, and

$$\begin{aligned}
\|\widetilde{h^\alpha_n}\|^2_{m,\alpha} &= 2^{2m} \|\widetilde{h^\alpha_n}\|_{\mathcal{W}^{m,2}_{t,\alpha}} \\
&= 2^{2m} \|h^\alpha_n\|_{\mathcal{W}^{m,2}_{\mathcal{H}_\alpha}} \\
&= 2^{2m} (2n + 2\alpha + 2)^{2m} < \infty.
\end{aligned}$$

So *for all* $n \in \mathbb{N}$, $\widetilde{h^\alpha_n} \in \mathcal{B}^{m,\alpha}_t(\mathbb{C})$. We have

$$\begin{aligned}
\langle F, \widetilde{h^\alpha_n} \rangle_{\mathcal{W}^{m,2}_{t,\alpha}} &= \sum_{p=0}^{\infty} (2p + 2\alpha + 2)^{2m} \langle F, \widetilde{h^\alpha_p} \rangle_{\alpha,t} \langle \widetilde{h^\alpha_n}, \widetilde{h^\alpha_p} \rangle_{\alpha,t} \\
&= (2n + 2\alpha + 2)^{2m} \langle F, \widetilde{h^\alpha_n} \rangle_{\alpha,t}.
\end{aligned}$$

Then it can be easily seen that if $\langle F, \widetilde{h^\alpha_n} \rangle_{\mathcal{W}^{m,2}_{t,\alpha}} = 0$ then $\langle F, \widetilde{h^\alpha_n} \rangle_{\alpha,t} = 0$. This gives that $F = 0$ because $\{\widetilde{h^\alpha_n},\ n \in \mathbb{N}\}$ form an orthonormal basis for $\mathcal{H}^\alpha_t(\mathbb{C})$, so we have

$$\{\widetilde{h^\alpha_n},\ n \in \mathbb{N}\} \subset \mathcal{B}^{m,\alpha}_t(\mathbb{C}) \subset \mathcal{W}^{m,2}_{t,\alpha}(\mathbb{C})$$

and

$$\overline{\{\widetilde{h^\alpha_n},\ n \in \mathbb{N}\}}^{\mathcal{W}^{m,2}_{t,\alpha}(\mathbb{C})} = \mathcal{W}^{m,2}_{t,\alpha}(\mathbb{C}).$$

Hence $\mathcal{F}^{m,\alpha}_t(\mathbb{C}) \cap \mathcal{H}^\alpha_t(\mathbb{C})$ is dense in $\mathcal{W}^{m,2}_{t,\alpha}(\mathbb{C})$. □

3. The image of $\mathcal{S}(\mathbb{R})$ and $L^p_\alpha(\mathbb{R})$ under the Dunkl-Hermite semigroup

3.1 The image of $\mathcal{S}(\mathbb{R})$ under the Dunkl-Hermite semigroup. We begin by establishing that $\mathcal{S}(\mathbb{R})$ is stable under the Dunkl-Hermite semigroup.

First we recall that the heat kernel q_t, $t > 0$, associated with the Dunkl operators, see [12], is given by

$$q_t(x) = \frac{1}{\Gamma(\alpha+1)}\, (4t)^{-(\alpha+1)} e^{-\frac{x^2}{4t}}\,.$$

This function belongs to $\mathcal{S}(\mathbb{R})$ and satisfies the following property

$$\tau^\alpha_{-y} q_t(x) = \frac{1}{\Gamma(\alpha+1)}\ (4t)^{-(\alpha+1)} e^{-\frac{(x^2+y^2)}{4t}}\ E_\alpha\Big(\frac{x}{2t}, y\Big),$$

where τ^α_y is the generalized translation associated with the Dunkl operator D_α (see [13]).

Using the Mehler formula for the Dunkl-Hermite polynomials H^α_n (see [10]), we can write $e^{-t\mathcal{H}_\alpha}$ on $\mathcal{S}(\mathbb{R})$ as an integral operator with kernel $\mathcal{M}^\alpha_t(x, y)$

$$[e^{-t\mathcal{H}_\alpha} f](x) = \int_{\mathbb{R}} f(y)\mathcal{M}^\alpha_t(x, y)|y|^{2\alpha+1}\, dy.$$

The kernel $\mathcal{M}^\alpha_t(x, y)$ can be explicitly written as

$$\mathcal{M}^\alpha_t(x, y) = \frac{1}{\Gamma(\alpha+1)(2\sinh(2t))^{\alpha+1}}\ e^{-\frac{1}{2}\coth(2t)(x^2+y^2)} E_\alpha\Big(\frac{x}{\sinh(2t)}, y\Big),$$

where $E_\alpha(\xi, x)$ is the Dunkl kernel. We can see that the kernel $\mathcal{M}^\alpha_t(x, y)$ satisfies the following relation

$$\mathcal{M}^\alpha_t(x, y) = e^{-\frac{1}{2}(\frac{\cosh 2t-1}{\sinh 2t})(x^2+y^2)}\ \tau^\alpha_{-y} q_{\frac{\sinh 2t}{2}}(x).$$

So for $\varphi \in \mathcal{S}(\mathbb{R})$, we have

$$e^{-t\mathcal{H}_\alpha}\varphi(y) = e^{-\frac{1}{2}(\frac{\cosh 2t-1}{\sinh 2t})y^2}\big(e^{-\frac{1}{2}(\frac{\cosh 2t-1}{\sinh 2t})x^2}\varphi *_\alpha q_{\frac{\sinh 2t}{2}}\big)(y),$$

where $*_\alpha$ is the generalized convolution product associated with the Dunkl operator D_α (see [13]).

As a consequence we have the following result.

Proposition 2. *The Dunkl-Hermite semigroup $e^{-t\mathcal{H}_\alpha}$ is a continuous transform from $\mathcal{S}(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$.*

In the following, we shall give a characterization of the image of the Schwartz space under the Dunkl-Hermite semigroup.

Let $F \in \mathcal{H}^\alpha_t(\mathbb{C})$ and for $z \in \mathbb{C}$, $F(z)$ be its entire extension. Since $F \to F(z)$ is a continuous linear functional on $\mathcal{H}^\alpha_t(\mathbb{C})$ for each $z \in \mathbb{C}$, Riesz representation theorem ensures that there exists a unique $\mathcal{N}^\alpha_t(z, \cdot) \in \mathcal{H}^\alpha_t(\mathbb{C})$ such that

$$F(z) = \langle F, \mathcal{N}^\alpha_t(z, \cdot)\rangle_{\alpha,t} = \langle F_e, \mathcal{N}^\alpha_{t,e}(z, \cdot)\rangle_{\alpha,e} + \langle F_o, \mathcal{N}^\alpha_{t,o}(z, \cdot)\rangle_{\alpha,o}.$$

The function $\mathcal{N}^\alpha_t(z, w)$ is called the reproducing kernel for $\mathcal{H}^\alpha_t(\mathbb{C})$. By expanding F in terms of $\widetilde{h^\alpha_n}$, we can write

$$F(z) = \sum_{n=0}^{\infty}\langle F, \widetilde{h^\alpha_n}\rangle_{\alpha,t}\widetilde{h^\alpha_n}(z) = \langle F, \sum_{n=0}^{\infty}\widetilde{h^\alpha_n}(\cdot)\overline{\widetilde{h^\alpha_n}(z)}\rangle_{\alpha,t}\,.$$

So, we deduce that

$$\mathcal{N}_t^\alpha(z,w) = \sum_n e^{-(2n+2\alpha+2)2t} h_n^\alpha(w) h_n^\alpha(\overline{z}).$$

Cauchy-Schwartz inequality gives us

$$|F(z)|^2 = |\langle F, \mathcal{N}_t^\alpha(z,\cdot)\rangle_{\alpha,t}|^2 \leq \|F\|_{\alpha,t}^2 \|\mathcal{N}_t^\alpha(z,\cdot)\|_{\alpha,t}^2 = \|F\|_{\alpha,t}^2 \mathcal{N}_t^\alpha(z,z).$$

Using Mehler's formula, we can explicitly calculate $\mathcal{N}_t^\alpha(z,z)$, in fact, we get

$$\begin{aligned}\mathcal{N}_t^\alpha(z,z) &= \sum_n e^{-(2n+2\alpha+2)2t} h_n^\alpha(z) h_n^\alpha(\overline{z}) = e^{-(2\alpha+2)2t} \sum_n (e^{-4t})^n h_n^\alpha(z) h_n^\alpha(\overline{z}) \\ &= \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} (\sinh(4t))^{-(\alpha+1)} \exp\Big(-\frac{1}{2}\coth(4t)(z^2+\overline{z}^2)\Big) E_\alpha\Big(\frac{1}{\sinh(4t)}, z\overline{z}\Big).\end{aligned}$$

If $z = x+iy$ we have that

$$\begin{aligned}|F(z)|^2 \leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} &(\sinh(4t))^{-(\alpha+1)} \exp\big(-\coth(4t)(x^2-y^2)\big) \\ &\times E_\alpha\Big(\frac{1}{\sinh(4t)}, x^2+y^2\Big) \|F\|_{\alpha,t}^2.\end{aligned}$$

It is known that the kernel E_α satisfies the inequality below for all $x, y \in \mathbb{R}$ (see [3])

$$E_\alpha\Big(\frac{1}{\sinh(4t)}, x^2+y^2\Big) \leq \exp\Big(\frac{1}{\sinh(4t)}(x^2+y^2)\Big). \tag{3}$$

As

$$-\coth(4t)(x^2-y^2) + \frac{1}{\sinh(4t)}(x^2+y^2) = -\tanh(2t)x^2 + \coth(2t)y^2,$$

we deduce

$$|F(z)|^2 \leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} (\sinh(4t))^{-(\alpha+1)} \exp\big(-\tanh(2t)x^2 + \coth(2t)y^2\big) \|F\|_{\alpha,t}^2,$$

which gives a pointwise estimate for functions $F \in \mathcal{H}_t^\alpha(\mathbb{C})$.

Notation 3. We denote by $\mathcal{N}_t^{\alpha,2m}(z,w)$ the kernel defined by

$$\mathcal{N}_t^{\alpha,2m}(z,w) = \sum_n (2n+2\alpha+2)^{-2m} \widetilde{h_n^\alpha}(\overline{z}) \widetilde{h_n^\alpha}(w).$$

In order to obtain pointwise estimates for $F \in \mathcal{W}_{t,\alpha}^{m,2}(\mathbb{C})$, we have to show the following result.

Proposition 3. *$\mathcal{N}_t^{\alpha,2m}(z,w)$ is a reproducing kernel for $\mathcal{W}_{t,\alpha}^{m,2}(\mathbb{C})$.*

Proof: For $z \in \mathbb{C}$, the function $w \to \mathcal{N}_t^{\alpha,2m}(z,w)$ belongs to $\mathcal{W}_{t,\alpha}^{m,2}(\mathbb{C})$ because $\widetilde{h_n^\alpha}(w) \in \mathcal{W}_{t,\alpha}^{m,2}(\mathbb{C})$ for all $w \in \mathbb{C}$. We show now the reproducing property. For $z \in \mathbb{C}$ and $F \in \mathcal{W}_{t,\alpha}^{m,2}(\mathbb{C})$, we have

$$\begin{aligned}\langle F, \mathcal{N}_t^{\alpha,2m}(z,\cdot)\rangle_{\mathcal{W}_{t,\alpha}^{m,2}} &= \sum_{n=0}^{\infty}(2n+2\alpha+2)^{2m}\langle F, \widetilde{h_n^\alpha}\rangle_{\alpha,t}\overline{\langle \mathcal{N}_t^{\alpha,2m}(z,\cdot), \widetilde{h_n^\alpha}\rangle_{\alpha,t}}\\ &= \sum_{n=0}^{\infty}(2n+2\alpha+2)^{2m}\langle F, \widetilde{h_n^\alpha}\rangle_{\alpha,t}(2n+2\alpha+2)^{-2m}\widetilde{h_n^\alpha}(z)\\ &= \sum_{n=0}^{\infty}\langle F, \widetilde{h_n^\alpha}\rangle_{\alpha,t}\widetilde{h_n^\alpha}(z) = F(z). \qquad \square\end{aligned}$$

The last kernel can be written as

$$\mathcal{N}_t^{\alpha,2m}(z,w) = \frac{2^{2m}}{(2m-1)!}\int_0^{+\infty} s^{2m-1}\mathcal{N}_{s+t}^{\alpha}(z,w)\,ds.$$

Using the explicit formula for $\mathcal{N}_s^\alpha(z,z)$, we have

$$\begin{aligned}\mathcal{N}_t^{\alpha,2m}(z,z) = \frac{2^{2m}}{(2m-1)!2^{\alpha+1}\Gamma(\alpha+1)}&\int_0^{+\infty} s^{2m-1}(\sinh 4(t+s))^{-(\alpha+1)}\\ &\times \exp\big(-\coth 4(t+s)(x^2-y^2)\big)\times E_\alpha\Big(\frac{1}{\sinh 4(t+s)}, x^2+y^2\Big)\,ds.\end{aligned}$$

Theorem 3 (Dunkl-Sobolev-embedding theorem)**.** *Let m be a nonnegative integer. Then every $F \in \mathcal{W}_{t,\alpha}^{m,2}(\mathbb{C})$ satisfies the estimate*

$$|F(z)|^2 \le C_{t,\alpha}(1+x^2+y^2)^{-2m}\exp\big(-\tanh(2t)x^2+\coth(2t)y^2\big),$$

where $C_{t,\alpha}$ is a constant depending on t and α.

Proof: We begin by estimating the integral appearing in the representation of the reproducing kernel $\mathcal{N}_t^{\alpha,2m}(z,z)$, using the inequality (3) we obtain

$$\begin{aligned}\mathcal{N}_t^{\alpha,2m}(z,z) \le \frac{2^{2m}}{(2m-1)!2^{\alpha+1}\Gamma(\alpha+1)}&\int_0^{+\infty} s^{2m-1}(\sinh 4(t+s))^{-(\alpha+1)}\\ &\times e^{-\tanh 2(t+s)x^2+\coth 2(t+s)y^2}\,ds.\end{aligned}$$

We rewrite this in the following form

$$\mathcal{N}_t^{\alpha,2m}(z,z) \le \frac{2^{2m}}{(2m-1)!2^{\alpha+1}\Gamma(\alpha+1)}\, e^{-\tanh(2t)x^2+\coth(2t)y^2} J_t^\alpha,$$

where

$$J_t^\alpha = \int_0^{+\infty} s^{2m-1}(\sinh 4(t+s))^{-(\alpha+1)} \times e^{-x^2(\tanh 2(t+s)-\tanh(2t))} \times e^{y^2(\coth 2(t+s)-\coth(2t))}\, ds,$$

which after some simplification yields

$$J_t^\alpha = \int_0^{+\infty} s^{2m-1}(\sinh 4(t+s))^{-(\alpha+1)} \times \exp\Big(-x^2(\frac{\sinh 2s}{\cosh 2(t+s)\cosh 2t}) - y^2(\frac{\sinh 2s}{\sinh 2(t+s)\sinh 2t})\Big)\, ds.$$

Thus we only need to show that the above integral is bounded by $C_{t,\alpha}(1+x^2+y^2)^{-2m}$.

To prove this estimate we break up the above integral into two parts. Using the elementary properties of the functions sinh and cosh, we see that

$$\int_0^{t} s^{2m-1}(\sinh 4(t+s))^{-(\alpha+1)} \times \exp\Big(-x^2(\frac{\sinh 2s}{\cosh 2(t+s)\cosh 2t}) - y^2(\frac{\sinh 2s}{\sinh 2(t+s)\sinh 2t})\Big)\, ds$$

is bounded by

$$\begin{aligned}&\int_0^{+\infty} s^{2m-1}e^{-4(\alpha+1)s}\exp\big(-2(\frac{x^2}{\cosh^2 4t}+\frac{y^2}{\sinh^2 4t})s\big)\, ds\\ &= (2m-1)![2(2(\alpha+1)+\frac{x^2}{\cosh^2 4t}+\frac{y^2}{\sinh^2 4t})]^{-2m}\\ &\le C_{t,\alpha,m}(1+x^2+y^2)^{-2m}.\end{aligned}$$

On the other hand the integral

$$\int_t^{\infty} s^{2m-1}(\sinh 4(t+s))^{-(\alpha+1)} \times \exp\Big(-x^2(\frac{\sinh 2s}{\cosh 2(t+s)\cosh 2t}) - y^2(\frac{\sinh 2s}{\sinh 2(t+s)\sinh 2t})\Big)\, ds,$$

is bounded by

$$\frac{(2m-1)!}{(4(\alpha+1))^{2m}}\exp\Big(-(\frac{\tanh 2t}{\cosh 4t}x^2+\frac{1}{\sinh 4t}y^2)\Big).$$

The above clearly gives the required estimate. □

Now we are in a position to prove the following result which characterizes the image of $\mathcal{S}(\mathbb{R})$ under $e^{-t\mathcal{H}_\alpha}$.

Theorem 4. *Let $t > 0$ be fixed, and F be a holomorphic function on $\mathbb{C}$. Then there exists a function $f \in \mathcal{S}(\mathbb{R})$ such that $F = e^{-t\mathcal{H}_\alpha} f$ if and only if F satisfies*

$$|F(z)|^2 \leq C_{t,\alpha,m} \frac{e^{-\tanh(2t)x^2+\coth(2t)y^2}}{(1+x^2+y^2)^{2m}}$$

for some constants $C_{t,\alpha,m}$, $m = 1, 2, 3, \ldots$

Proof: If $f \in \mathcal{S}(\mathbb{R})$, then $(\mathcal{H}_\alpha)^m f \in L^2_\alpha(\mathbb{R})$ for all integer m, so $f \in \mathcal{W}^{m,2}_{\mathcal{H}_\alpha}(\mathbb{R})$ for all m, which implies that

$$F = e^{-t\mathcal{H}_\alpha} f \in \mathcal{W}^{m,2}_{t,\alpha}(\mathbb{C}) \quad \text{for all } m.$$

From Theorem 3, we have $|F(z)|^2$ is bounded by $C_{t,\alpha,m} \frac{e^{-\tanh(2t)x^2+\coth(2t)y^2}}{(1+x^2+y^2)^{2m}}$ for all m.

Conversely, suppose F satisfies the necessity condition. Using [6, p. 140],

$$K_\alpha(z) = \Big(\frac{\pi}{2z}\Big)^{\frac{1}{2}} \frac{e^{-z}}{\Gamma(\alpha+\frac{1}{2})} \int_0^{+\infty} e^{-s} s^{\alpha-\frac{1}{2}} \Big(1+\frac{s}{2z}\Big)^{\alpha-\frac{1}{2}}\, ds \qquad \text{for } |\arg z| < \pi,\ \alpha > -\frac{1}{2}, \tag{4}$$

then by choosing m large enough, we see that

$$\int_{\mathbb{C}} |F_e(z)|^2 U^\alpha_{t,e}(z)\, dz + \int_{\mathbb{C}} |F_o(z)|^2 U^\alpha_{t,o}(z)\, dz < +\infty,$$

from which it follows that $F \in \mathcal{H}^\alpha_t(\mathbb{C})$, thus there exists a function $f \in L^2_\alpha(\mathbb{R})$ such that $F = e^{-t\mathcal{H}_\alpha} f$.

We have

$$\begin{aligned} K_\alpha\Big(\frac{|z|^2}{\sinh 4t}\Big) \times |z|^{2\alpha+2} &= \Big(\frac{\pi \sinh 4t}{2}\Big)^{\frac{1}{2}} \frac{|z|^2}{\Gamma(\alpha+\frac{1}{2})} \\ &\times e^{-\frac{|z|^2}{\sinh 4t}} \int_0^{+\infty} e^{-s} s^{\alpha-\frac{1}{2}} \Big(|z|^2 + \frac{s(\sinh 4t)}{2}\Big)^{\alpha-\frac{1}{2}}\, ds, \end{aligned}$$

so it is an easy matter to see that $\frac{d^{2m}}{dt^{2m}} U^\alpha_{t,e}(z)$ and $\frac{d^{2m}}{dt^{2m}} U^\alpha_{t,o}(z)$ are a sum of $(2m+1)$ terms times $e^{\tanh(2t)x^2-\coth(2t)y^2}$, where each term is of the form

$$\big(p(t,\alpha)x^2 + q(t,\alpha)y^2 + c(t,\alpha)\big)^k \leq C_{t,\alpha}\big(1+x^2+y^2\big)^{2m} \quad \text{with} \quad k \leq 2m,$$

where $p(t,\alpha)$, $q(t,\alpha)$ and $c(t,\alpha)$ are real constants. In view of Theorem 2, it follows that $F \in \mathcal{B}^{m,\alpha}_t(\mathbb{C}) = \mathcal{W}^{m,2}_{t,\alpha}(\mathbb{C})$. This leads to the fact that $F \in \mathcal{W}^{m,2}_{t,\alpha}(\mathbb{C})$

for all m. Consequently $f \in \mathcal{W}^{m,2}_{\mathcal{H}_\alpha}(\mathbb{R})$ for all m. Since

$$\bigcap_m \mathcal{W}^{m,2}_{\mathcal{H}_\alpha}(\mathbb{R}) = \mathcal{S}(\mathbb{R}),$$

the result follows. $\square$

3.2 The image of $L^p_\alpha(\mathbb{R})$ under the Dunkl-Hermite semigroup. We begin this subsection by recalling that in [2] the authors have proved that the Dunkl-Hermite semigroup initially defined on $L^2_\alpha \cap L^p_\alpha(\mathbb{R})$ extends to the whole of L^p_α and we have

$$\|e^{-t\mathcal{H}_\alpha} f\|_{\alpha,p} \leq (\cosh(2t))^{-(\alpha+1)} \|f\|_{\alpha,p}.$$

In the following, we give a characterization of the image of L^p_α under the Dunkl-Hermite semigroup.

Theorem 5. *Fix $t > 0$ and let $1 < p < \infty$. Then for all $f \in L^p_\alpha(\mathbb{R})$, we have*

$$|e^{-t\mathcal{H}_\alpha} f(x+iy)| \leq C_{t,p,\alpha} \|f\|_{p,\alpha} \exp\Big(\big(\frac{p}{(p-1)\sinh 4t} - \frac{\coth 2t}{2}\big)x^2 + \frac{\coth 2t}{2} y^2\Big).$$

Proof: As we have shown previously, we have

$$e^{-t\mathcal{H}_\alpha} f(z) = e^{-\frac{1}{2}(\frac{\cosh 2t-1}{\sinh 2t})z^2} \Big(e^{-\frac{1}{2}(\frac{\cosh 2t-1}{\sinh 2t})x^2} f *_\alpha q_{\frac{\sinh 2t}{2}}\Big)(z),$$

so

$$|e^{-t\mathcal{H}_\alpha} f(x+iy)| \leq \frac{1}{\Gamma(\alpha+1)} (2\sinh 2t)^{-(\alpha+1)} e^{-\frac{\coth 2t}{2}(x^2-y^2)} I_{t,\alpha},$$

where

$$I_{t,\alpha} = \int_{\mathbb{R}} |f(s)| \Big| e^{-\frac{\coth 2t}{2}s^2} E_\alpha\Big(\frac{s}{\sinh 2t}, z\Big)\Big| |s|^{2\alpha+1}\, ds.$$

So by Hölder's inequality, we have

$$I_{t,\alpha} \leq \|f\|_{p,\alpha} \Big\| e^{-\frac{\coth 2t}{2}s^2} E_\alpha\Big(\frac{s}{\sinh 2t}, z\Big)\Big\|_{p',\alpha},$$

where p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$.

We know that

$$\Big| E_\alpha\Big(\frac{s}{\sinh 2t}, z\Big)\Big|^{p'} \leq e^{\frac{p' sx}{\sinh 2t}},$$

so

$$\Big\| e^{-\frac{\coth 2t}{2}s^2} E_\alpha\Big(\frac{s}{\sinh 2t}, z\Big)\Big\|^{p'}_{p',\alpha} \leq \int_{\mathbb{R}} e^{-\frac{\coth 2t}{2}p' s^2} e^{\frac{p' sx}{\sinh 2t}} |s|^{2\alpha+1}\, ds.$$

We can easily verify that

$$e^{-\frac{\coth 2t}{2}p' s^2} \; e^{\frac{p' sx}{\sinh 2t}} = e^{\frac{p' x^2}{\sinh 4t}} \; e^{-\frac{p'}{2}(\sqrt{\coth 2t}s - \sqrt{\frac{2}{\sinh 4t}}x)^2}$$

which completes the proof. $\square$

Notation 4. We denote by $V_{t,\frac{p}{2}}(z)$ the function defined by

$$V_{t,\frac{p}{2}}(x+iy) = \exp\Big(-2\big(\frac{p}{(p-1)\sinh 4t}x^2 + \frac{\coth 2t}{2}y^2\big)\Big)$$

and by $V^s_{t,\frac{p}{2}}$, the s-th power of $V_{t,\frac{p}{2}}$.

We write $\mathcal{H}L^p_\alpha(\mathbb{C}, V_{t,\frac{p}{2}}(z))$ for the class of holomorphic functions in $L^p_\alpha(\mathbb{C}, V_{t,\frac{p}{2}}(z))$.

The next corollary follows from Theorem 5, by a straightforward computation.

Corollary 1. *Let $f \in L^p_\alpha(\mathbb{R})$, $1 < p < \infty$ and fix $t > 0$, then*

(i) $e^{-t\mathcal{H}_\alpha}(f) \in \mathcal{H}L^p_\alpha(\mathbb{C}, V^{\frac{p+\epsilon}{2}}_{t,\frac{p}{2}})$, *for $\epsilon > 0$.*
So

$$e^{-t\mathcal{H}_\alpha}(f) \in \bigcap_{\epsilon>0} \mathcal{H}L^p_\alpha\big(\mathbb{C}, V^{\frac{p+\epsilon}{2}}_{t,\frac{p}{2}}\big).$$

(ii) $e^{-t\mathcal{H}_\alpha}(f) \in \mathcal{H}L^{p'}_\alpha(\mathbb{C}, V^{\frac{p+\epsilon}{2}}_{t,\frac{p}{2}})$, *for $\epsilon > 0$, where $2 \le p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.*
So

$$e^{-t\mathcal{H}_\alpha}(f) \in \bigcap_{\epsilon>0} \mathcal{H}L^{p'}_\alpha(\mathbb{C}, V^{\frac{p+\epsilon}{2}}_{t,\frac{p}{2}}).$$

(iii) $e^{-t\mathcal{H}_\alpha}(f) \in \mathcal{H}L^s_\alpha(\mathbb{C}, V^{\frac{s+\epsilon}{2}}_{t,\frac{p}{2}})$, *for $\epsilon > 0$, where $1 \le s < \infty$.*

4. Paley Wiener type Theorems

In this section we establish Paley-Wiener type theorems for the tempered distributions and the compactly supported distributions under the Dunkl-Hermite semigroup.

Theorem 6. *Let m be a positive integer. Then every $F \in \mathcal{W}^{-m,2}_{t,\alpha}(\mathbb{C})$ satisfies the estimate*

$$|F(z)|^2 \le C_{t,\alpha}(1+|z|^2)^{2m} \exp\big(-\tanh(2t)x^2 + \coth(2t)y^2\big).$$

Conversely, if an entire function F satisfies the above estimate, then F belongs to $\mathcal{W}^{-m-1,2}_{t,\alpha}(\mathbb{C})$.

Proof: It is easy to see that the reproducing kernel for $\mathcal{W}^{-m,2}_{t,\alpha}(\mathbb{C})$ is given by

$$\mathcal{N}^{\alpha,-2m}_t(z,w) = \sum_n (2n+2\alpha+2)^{2m}\widetilde{h^\alpha_n}(\overline{z})\widetilde{h^\alpha_n}(w).$$

So we only need to estimate the $(2m)$-th derivate of $\mathcal{N}^\alpha_t(z,z)$ with respect to t.

Thanks to inequality (3), we have

$$\frac{d^{2m}}{dt^{2m}}\mathcal{N}^\alpha_t(z,z) \le C_{t,\alpha}(1+|z|^2)^{2m} e^{-\tanh(2t)x^2+\coth(2t)y^2}.$$

Then if $F \in \mathcal{W}_{t,\alpha}^{-m,2}(\mathbb{C})$

$$|F(z)|^2 \leq C_{t,\alpha}(1+|z|^2)^{2m} e^{-\tanh(2t)x^2+\coth(2t)y^2}.$$

To prove the converse, we need to make use of duality between $\mathcal{W}_{\mathcal{H}_\alpha}^{m,2}(\mathbb{R})$ and $\mathcal{W}_{\mathcal{H}_\alpha}^{-m,2}(\mathbb{R})$.

The duality bracket is given by

$$\langle F, G\rangle = \int_{\mathbb{C}} F_e(z)\overline{G_e(z)} U_{t,e}^\alpha(z)\, dz + \int_{\mathbb{C}} F_o(z)\overline{G_o(z)} U_{t,o}^\alpha(z)\, dz.$$

If F satisfies the given estimates then F_e and F_o satisfy them too, and for any $G \in \mathcal{W}_{t,\alpha}^{m+1,2}(\mathbb{C})$ the integral defining $\langle F, G\rangle$ converges and hence F defines a continuous linear functional on $\mathcal{W}_{t,\alpha}^{m+1,2}(\mathbb{C})$.

Consequently, F belongs to $\mathcal{W}_{t,\alpha}^{-m-1,2}(\mathbb{C})$ which proves the converse. □

We recall the following definition given in [14].

Definition 3. Let S be in $\mathcal{S}'(\mathbb{R})$ and φ in $\mathcal{S}(\mathbb{R})$, the Dunkl convolution product of S and φ is the function $S *_\alpha \varphi$ defined by

$$\forall x \in \mathbb{R},\ S *_\alpha \varphi(x) = \langle S_y, \tau_{-y}^\alpha \varphi(x)\rangle,$$

where τ_y^α is the generalized translation associated with the Dunkl operator D_α (see [13]).

It was shown in [14] that $S *_\alpha \varphi$ is a $\mathcal{C}^\infty$ function on $\mathbb{R}$ and for all $n \in \mathbb{N}$, we have

$$D_\alpha^n(S *_\alpha \varphi) = S *_\alpha (D_\alpha^n \varphi) = (D_\alpha^n S) *_\alpha \varphi.$$

It can be obviously seen that for fixed $x \in \mathbb{R}$ and $t > 0$, the function

$$y \longrightarrow \mathcal{M}_t^\alpha(x, y) \in \mathcal{S}(\mathbb{R}).$$

Definition 4. The Dunkl-Hermite semigroup of a distribution S in $\mathcal{S}'(\mathbb{R})$ is defined by

$$e^{-t\mathcal{H}_\alpha}(S)(x) = \langle S_y, \mathcal{M}_t^\alpha(x, y)\rangle.$$

Remark 3. For $S \in \mathcal{S}'(\mathbb{R})$, we have

$$e^{-t\mathcal{H}_\alpha} S(x) = e^{-\frac{1}{2}(\frac{\cosh 2t-1}{\sinh 2t})x^2} \left(e^{-\frac{1}{2}(\frac{\cosh 2t-1}{\sinh 2t})y^2} S *_\alpha q_{\frac{\sinh 2t}{2}}\right)(x),$$

so $e^{-t\mathcal{H}_\alpha} S$ is a $\mathcal{C}^\infty$ function on $\mathbb{R}$.

Theorem 7. *Suppose F is a holomorphic function on $\mathbb{C}$. Then there exists a distribution $f \in \mathcal{S}'(\mathbb{R})$ with $F = e^{-t\mathcal{H}_\alpha} f$ if and only if F satisfies*

$$|F(z)|^2 \leq C_{t,\alpha}(1+|z|^2)^{2m} \exp\left(-\tanh(2t)x^2 + \coth(2t)y^2\right),$$

for some nonnegative integer m.

Proof: Let $f \in \mathcal{S}'(\mathbb{R})$. Since the union of all $\mathcal{W}_{\mathcal{H}_\alpha}^{-m,2}(\mathbb{R})$ is $\mathcal{S}'(\mathbb{R})$, then there exists m such that $f \in \mathcal{W}_{\mathcal{H}_\alpha}^{-m,2}(\mathbb{R})$. Thus

$$e^{-t\mathcal{H}_\alpha} f \in \mathcal{W}_{t,\alpha}^{-m,2}(\mathbb{C}),$$

and from Theorem 6 we have the result.

Conversely, suppose that F satisfies the hypothesis, then F belongs to $\mathcal{W}_{t,\alpha}^{-m-1,2}(\mathbb{C})$ and $F = e^{-t\mathcal{H}_\alpha} f$ with $f \in \mathcal{W}_{\mathcal{H}_\alpha}^{-m-1,2}(\mathbb{R})$. Then $f \in \mathcal{S}'(\mathbb{R})$. □

In [7], the authors introduced the generalized windowed transform associated with D_α as follows. Given a function g in the Schwartz space, the windowed Dunkl transform of a regular function f, with window g, is defined by

$$\mathcal{V}_g^\alpha(f)(x,y) = \int_{\mathbb{R}} f(u)\tau_{-y}^\alpha g(u) E_\alpha(-ix,u)|u|^{2\alpha+1}\,du.$$

Here we extend this definition to the tempered distribution.

Definition 5. The windowed Dunkl transform of a tempered distribution S with window $g \in \mathcal{S}(\mathbb{R})$ is defined by

$$\mathcal{V}_g^\alpha(S)(x,y) = \langle S, \tau_{-y}^\alpha g E_\alpha(-ix,\cdot)\rangle.$$

When S is given by the function $f|u|^{2\alpha+1}$, $S = S_{f|u|^{2\alpha+1}}$, then

$$\mathcal{V}_g^\alpha(S_{f|u|^{2\alpha+1}})(x,y) = \int_{\mathbb{R}} f(u)\tau_{-y}^\alpha g(u) E_\alpha(-ix,u)|u|^{2\alpha+1}\,du,$$

which we write simply $\mathcal{V}_g^\alpha(f)(x,y)$.

In the case where $g(x) = \varphi_a(x) = e^{-\frac{1}{2}ax^2}$, for $a > 0$, $\mathcal{V}_{\varphi_a}^\alpha f$ is called gaussian Dunkl windowed transform. In our context, we are interested in the case $y = 0$ and we denote

$$\mathcal{T}_a^\alpha f(x) = \mathcal{V}_{\varphi_a}^\alpha(f)(x,0).$$

Hence, for $a > 0$, the transform $\mathcal{T}_a^\alpha$ is defined by

$$\mathcal{T}_a^\alpha(S)(x) = \langle S, e^{-\frac{1}{2}a(\cdot)^2} E_\alpha(-ix,\cdot)\rangle,\ S \in \mathcal{S}'(\mathbb{R}).$$

If $f \in \mathcal{S}(\mathbb{R})$ we have

$$\mathcal{T}_a^\alpha(f)(x) = \int_{\mathbb{R}} f(u) e^{-\frac{1}{2}au^2} E_\alpha(-ix,u)|u|^{2\alpha+1}\,du.$$

We see that $\mathcal{T}_a^\alpha f$ extends to $\mathbb{C}$ as an entire function even when f is in $\mathcal{S}'(\mathbb{R})$. This property of $\mathcal{T}_a^\alpha$ allows us to prove the following analogue of Paley-Wiener theorem given by Trimèche in [13].

Theorem 8. *For any $a > 0$ the transform $\mathcal{T}_a^\alpha$ of a tempered distribution f on $\mathbb{R}$ extends to $\mathbb{C}$ as an entire function which satisfies the estimate*

$$|\mathcal{T}_a^\alpha f(z)| \leq C_\alpha(1 + x^2 + y^2)^m e^{\frac{1}{2}a^{-1}y^2}$$

for some non-negative integer m.

Conversely, if an entire function F satisfies such an estimate, then $F = \mathcal{T}_a^\alpha f$ for some tempered distribution f.

Proof: We relate the transform $\mathcal{T}_a^\alpha f$ to $e^{-t\mathcal{H}_\alpha} f$. Indeed, considering the case $a > 1$ first and writing $a = \coth 2t$ for some $t > 0$, we can easily verify that

$$e^{-t\mathcal{H}_\alpha} f(z) = \frac{1}{\Gamma(\alpha+1)(2\sinh 2t)^{\alpha+1}} e^{-\frac{1}{2}\coth 2t z^2} \mathcal{T}_a^\alpha f\Big(\frac{iz}{\sinh 2t}\Big) \quad \forall z \in \mathbb{C}.$$

We obtain the required estimate on $\mathcal{T}_a^\alpha f(z)$ by applying Theorem 7.

Conversely, if F satisfies the given estimates then again by Theorem 7 the function

$$G(z) = \frac{1}{\Gamma(\alpha+1)(2\sinh 2t)^{\alpha+1}} e^{-\frac{1}{2}\coth 2t z^2} F\Big(\frac{iz}{\sinh 2t}\Big)$$

should be of the form $e^{-t\mathcal{H}_\alpha} f(z)$ with a tempered distribution f.

When $a < 1$ we take $t > 0$ so that $a = \tanh 2t$ and the proof requires an analogue of Theorem 7 for functions of the form $e^{-(t+i\frac{\pi}{4})\mathcal{H}_\alpha} f$ (see [1]).

The image of tempered distributions under $e^{-(t+i\frac{\pi}{4})\mathcal{H}_\alpha}$ can be characterized in a similar way. The final estimates do not depend on the factor $e^{-i\frac{\pi}{4}\mathcal{H}_\alpha}$ which is just the Dunkl transform $\mathcal{F}_D$.

Here the Dunkl transform of a distribution f in $\mathcal{S}'(\mathbb{R})$ is defined by

$$\langle \mathcal{F}_D(f), \psi\rangle = \langle f, \mathcal{F}_D(\psi)\rangle, \ \psi \in \mathcal{S}(\mathbb{R})$$

and for $f \in \mathcal{S}(\mathbb{R})$

$$\mathcal{F}_D(f)(x) = \int_{\mathbb{R}} f(y) E_\alpha(-ix, y)|y|^{2\alpha+1}\, dy.$$

We have

$$e^{-(t+i\frac{\pi}{4})\mathcal{H}_\alpha} f = e^{-t\mathcal{H}_\alpha}\big(e^{-i\frac{\pi}{4}\mathcal{H}_\alpha} f\big)$$

and

$$e^{-i\frac{\pi}{4}\mathcal{H}_\alpha} f = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\ e^{(\alpha+1)i\frac{\pi}{2}} \mathcal{F}_D f.$$

We know that $\mathcal{F}_D$ is an isomorphism from $\mathcal{S}'(\mathbb{R})$ onto $\mathcal{S}'(\mathbb{R})$ (see [13]), so we have the analogue of Theorem 7. □

Finally, we remark that we also have the following result which characterizes the image of compactly supported distributions under the Dunkl-Hermite semigroup.

Theorem 9. *Let f be a distribution supported in a ball of radius R centered at the origin. Then for any $t > 0$ the function $e^{-t\mathcal{H}_\alpha} f$ extends to $\mathbb{C}$ as an entire function which satisfies*

$$|e^{-t\mathcal{H}_\alpha} f(z)| \leq Ce^{-\frac{1}{2}\coth 2t(x^2-y^2)} e^{\frac{R|x|}{\sinh 2t}},$$

with C being a positive constant.

Conversely, any entire function F satisfying the above estimate is of the form $e^{-t\mathcal{H}_\alpha} f$ where f is supported inside a ball of radius R centered at the origin.

Proof: We have to relate the Dunkl-Hermite semigroup and the Dunkl transform in $\mathcal{E}'(\mathbb{R})$

$$e^{-t\mathcal{H}_\alpha} S(z) = \frac{1}{\Gamma(\alpha+1)(2\sinh(2t))^{\alpha+1}}\; e^{-\frac{1}{2}\coth 2tz^2} \mathcal{F}_D\big[S_y e^{-\frac{1}{2}\coth 2ty^2}\big]\Big(\frac{iz}{\sinh 2t}\Big).$$

Here the Dunkl transform of a distribution S in $\mathcal{E}'(\mathbb{R})$ is defined by

$$\forall y \in \mathbb{R},\; \mathcal{F}_D(S)(y) = \langle S_x, E_\alpha(-iy, x)\rangle.$$

We obtain the necessity condition by appealing Theorem 5.3 given in [13], i.e., Paley-Wiener theorem for compactly supported distributions and the Dunkl transform.

Conversely, if F satisfies the given estimates then again by the same Theorem 5.3, the function

$$G(z) = \Gamma(\alpha+1)(2\sinh(2t))^{\alpha+1} e^{-\frac{1}{4}\sinh 4tz^2} F(-iz\sinh 2t)$$

should be of the form $\mathcal{F}_D(f)$ for a distribution f supported inside a ball of radius R centered at the origin and

$$F(z) = e^{-t\mathcal{H}_\alpha}\big(f(y)e^{\frac{1}{2}\coth 2ty^2}\big)(z),$$

where $f(y)e^{\frac{1}{2}\coth 2ty^2}$ is also a distribution supported inside a ball of radius R centered at the origin. This completes the proof of the theorem. □

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