Near-homogeneous spherical Latin bitrades

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Abstract. A planar Eulerian triangulation is a simple plane graph in which each face is a triangle and each vertex has even degree. Such objects are known to be equivalent to spherical Latin bitrades. (A Latin bitrade describes the difference between two Latin squares of the same order.) We give a classification in the near-regular case when each vertex is of degree 4 or 6 (which we call a near-homogeneous spherical Latin bitrade, or NHSLB). The classification demonstrates that any NHSLB is equal to two graphs embedded in hemispheres glued at the equator, where each hemisphere belongs to one of nine possible types, each of which may be described recursively.

Keywords: planar Eulerian triangulation; Latin bitrade; Latin square

Classification: 05B15, 05C45, 05C10

1. Introduction

A planar Eulerian triangulation is a simple plane graph in which each face is a triangle and the degree of each vertex is even. Batagelj introduced an algorithm for generating any such triangulation recursively using only one specific graph as a starting point and applying only two types of generating rules ([2]). This theory was applied using the freely available package *plantri* ([3]) to generate isomorph-free planar Eulerian triangulations of small orders ([12]).

Meanwhile, Wanless enumerated isomorphism classes of Latin bitrades of small orders in [17]. A Latin bitrade is a pair of partial Latin squares $\{T_1, T_2\}$ such that T_1 and T_2 each occupy the same set of non-empty cells, corresponding cells contain distinct symbols and each row and column contains the same set of symbols in T_1 as in T_2, yet in a different order. From such a combinatorial structure we can construct a graph $G(\{T_1, T_2\})$ which is a triangulation of some pseudo-surface. This is achieved by associating each occurrence of a symbol s in row r and column c with a triangular face on these 3 vertices (with triangles from T_1 and T_2 corresponding to faces of different colours in a face 2-colouring of the graph G). (See Figure 1 for an example and [4] for more detail.) When the pseudo-surface is the plane, we refer to the Latin bitrade as being *spherical*.

Wanless' enumeration drew attention to the fact that the sequence giving the number of isomorphism classes of spherical Latin bitrades coincides exactly with the sequence giving the number of isomorphism classes of Eulerian triangulations, as far as could be verified computationally. The general equivalence between these two combinatorial objects was proven in [6].

When we restrict ourselves to regularity or near-regularity, there is some evidence to suggest that Latin bitrades become directly classifiable. For example, Latin bitrades that correspond to 6-regular graphs (and are thus of genus 1 by Euler's formula) can be described directly and up to isomorphism via a direct geometric construction ([5]). It is shown in [10] that this result can also be derived as a corollary of work done by Negami [14] and Altsluher [1]. Other toroidal Latin bitrades which have a near-regular structure are classified in [8].

	c	d
a	e	f
b	f	e

T_1

	c	d
a	f	e
b	e	f

T_2

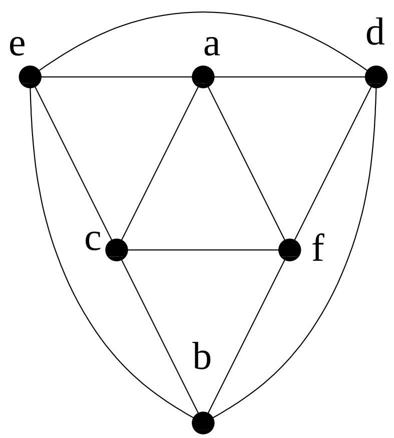

Figure 1. The unique 4-regular planar Eulerian triangulation and its corresponding Latin bitrade $\{T_1, T_2\}$.

It is not hard to show that there is precisely one regular spherical Latin bitrade, the smallest possible one shown above in Figure 1. In this work we focus on spherical Latin bitrades in which every vertex has degree 4 or degree 6; this provides an infinite class which is as close as we can get in some sense to being regular whilst retaining planarity. We henceforth call such structures near-homogeneous spherical Latin bitrades (NHSLB). It is most convenient for our purposes to take the graphical form; i.e. planar Eulerian triangulations in which each vertex has degree 4 or 6. It is well known (see, for example, [15]) that any planar Eulerian triangulation and hence any NHSLB, is tripartite (this follows in fact from the equivalence to spherical Latin bitrade), a fact we make use of throughout this paper.

Euler's formula implies that any NHSLB has precisely six vertices of degree 4. It is given as an "easy" exercise in 13.4 of [11] to show that an NHSLB may have k vertices of degree 6 for any integer $k \geq 0$. A long-standing unsolved conjecture, attributed to Barnette (Conjecture 5 in [16]), purports that the dual graph of any planar Eulerian triangulation has a Hamilton cycle. This conjecture has been shown to be true for graphs with at most 66 vertices in [12]. Using a proof that does not involve a classification, Barnette's conjecture has been proven true in [9] for the special case of dual graphs of NHSLBs.

We begin with a lemma which applies to any planar Eulerian triangulation and is helpful to our classification of NHSLBs.

Lemma 1. *Suppose there exist distinct vertices v_1, v_2 and v_3 and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$ and $\{v_3, v_1\}$ within a planar Eulerian triangulation G. Consider the plane graph G' formed by taking the cycle (v_1, v_2, v_3, v_1) and any vertices and edges from G inside the cycle. Then either $G' = (v_1, v_2, v_3, v_1)$ and thus is a triangular face of G, or the vertices v_1, v_2 and v_3 each have degree at least 4 within G'.*

Proof: Let G' be such a plane graph which is not a triangular face of G and is minimal with respect to the properties described. Then if either v_1, v_2 or v_3 have degree 2 in G', we are forced to have a multiple edge, a contradiction. So we may assume that each of v_1, v_2 and v_3 have degree at least 3 in G'. There are either 2 or 0 vertices from $\{v_1, v_2, v_3\}$ with odd degree in G'. Suppose that v_1 and v_2 have odd degree 3. Then there is an internal vertex w such that $\{v_1, v_2, w\}$ is a triangular face. Thus there must be an edge from w to v_3. Let G_1 and G_2 be the graphs formed by taking the cycles (v_3, v_2, w, v_3) and (v_3, v_1, w, v_3) (respectively) and any vertices and edges inside these cycles. Since v_3 must have degree at least 4, at least one of G_1 and G_2 must have an internal vertex, contradicting the minimality of G'. $\square$

In this paper we use the standard definitions of walk, path, circuit and cycle. Let $\mathcal{H}$ be the infinite 6-regular triangulation of the plane, where each triangle is equilateral and of equal size (unit length). When we say that a plane graph G is some closed polygon from $\mathcal{H}$, it is understood that we include all vertices and edges from the border or inside the polygon, and none from outside.

Our classification method is summarized as follows. We show that any NHSLB can be considered as the glueing of two plane graphs along the equator of a sphere; where the two hemispheres belong to one of nine different types. Each of these types can be constructed by recursive methods. In our classification, in general one NHSLB can arise in many different ways, and consequently isomorphism classes are not fully revealed. We comment more about the advantages and disadvantages of the classification in the conclusion.

2. Two methods for glueing

In this section we describe two methods to "glue" two plane graphs to obtain another plane graph. Let G_1 and G_2 be two plane graphs and let the cycles $(v_1, v_2, \dots, v_\alpha, v_1)$ and $(w_1, w_2, \dots, w_\beta, w_1)$ be their respective external faces. Let k be some integer with $1 \leq k \leq \alpha, \beta$. Then we may form another plane graph by identifying v_i with w_i, for each i, $1 \leq i \leq k$, so that this graph has external face

$$(v_k, v_{k+1}, \dots, v_\alpha, v_1 = w_1, w_\beta, w_{\beta-1}, \dots, w_k = v_k).$$

We say this graph is $G_1 \oplus G_2$ (with respect to paths $P_1 = [v_1, v_2, \dots, v_k]$ and $P_2 = [w_1, w_2, \dots, w_k]$). Note in the case $k = 1$, the resultant graph has a cut vertex.

In the second glueing method, we have G_1 and G_2 as above but we require $\alpha = \beta$. Let $v_{\alpha+1} = v_1$ and $w_{\alpha+1} = w_1$. Then we define $G \circ G'$ to be the plane graph obtained by embedding G and G' on the Northern and Southern hemispheres, respectively, so that the external faces of each lie on the equator, with $v_i = w_i$ and $\{v_i, v_{i+1}\} = \{w_i, w_{i+1}\}$ for each i, $1 \leq i \leq \alpha$. Note that this "glueing" is defined with respect to the labelling of the vertices on the external faces of G and G'.

3. The classification begins

Next, we wish to define certain plane graphs which may occur as subgraphs of a NHSLB. Let x_2, x_4 and x_6 be positive integers such that $0 \leq x_2, x_4, x_6$ and $x_6 \leq 1$. Then we define $\mathcal{G}(x_2, x_4, x_6, z_4, z_6)$ to be the set of plane graphs G such that:

- G has an external face with $x_2 + x_4 + x_6$ vertices (which we call *external*);
- x_i is the number of external vertices of degree i;
- every other (internal) face of G is a triangle and each vertex which is not external (i.e. *internal*) has degree 4 or degree 6;
- z_i is the number of internal vertices of degree i.

Euler's formula gives the following constraint.

Lemma 2. $x_2 + z_4 - x_6 = 3$.

Thus, for convenience, we write $\mathcal{G}(x_2, z_4, x_6)$ instead of $\mathcal{G}(x_2, x_4, x_6, z_4, z_6)$. We also let

$$\mathbb{G} := \bigcup \mathcal{G}(x_2, z_4, x_6).$$

The above lemma means that (x_2, z_4, x_6) must be an element of the following set of size 9:

$$\{(3,0,0), (2,1,0), (1,2,0), (0,3,0), (4,0,1), (3,1,1), (2,2,1), (1,3,1), (0,4,1)\}.$$

Lemma 3. *Any NHSLB includes a subgraph $G \in \mathcal{G}(x_2, z_4, x_6)$, where $x_2 + x_6 \leq 1$.*

Proof: Let G be a NHSLB and let v_0 be a vertex of G. Create a walk $v_0, v_1, \ldots$ within G such that $\{v_i, v_{i+1}\}$ is the third edge adjacent to $\{v_i, v_{i-1}\}$ in a clockwise direction. By this we mean there are two distinct vertices w and w' adjacent to v_i such that $\{v_i, w\}$ and $\{v_i, w'\}$ lie "between" v_{i-1} and v_{i+1} in a clockwise direction. By finiteness such a walk must contain circuits. Consider a minimal circuit (i.e. a cycle) L within the path. Since the graph is on the plane we may assume, without loss of generality, that each vertex on the cycle is adjacent to two vertices strictly inside the loop, with at most one exception (the first/last vertex of the cycle). Thus deleting any points outside of L and any edges adjacent to them, we obtain an element of $G(x_2, z_4, x_6)$ with at most one external vertex not of degree 4. □

Corollary 4. *Any NHSLB is isomorphic to $G \circ G'$ for some $G \in \mathcal{G}(x_2, z_4, x_6)$ and $G' \in \mathcal{G}(x'_2, z'_4, x'_6)$, where $x_2 + x_6 \leq 1$, $x'_6 \leq x_2$, $x_6 \leq x'_2$.*

To obtain a NHSLB any vertex of degree 6 on the equator must be glued to a vertex of degree 2 from the other graph (hence some of the conditions of the above theorem). Thus by classifying the plane graphs in $\mathbb{G}$ for the nine choices of parameters we in effect provide a classification of every possible NHSLB.

Corollary 5. *The number of external vertices of any graph in $\mathbb{G}$ is divisible by 3.*

PROOF: Let $G \in \mathbb{G}$ and consider the graph $G' = G \circ G$. It is possible that G' is not a proper NHSLB (as there may be repeated edges); however it is always an Eulerian *near-triangulation* of the plane; i.e. an Eulerian plane graph with each face a triangle. It is well-known that such structures are tripartite (see, for example, [13]); thus G is also tripartite. Let L be the external cycle of G. Vertices distance 3 apart in L must share the same colour; the result follows. □

With this aim in mind we give some preliminary lemmas.

Lemma 6. *Let $(v_1, v_2, \ldots, v_\alpha, v_{\alpha+1} = v_1)$ be the external face of $G \in \mathcal{G}(x_2, z_4, x_6)$. Suppose that there is an edge $\{v_1, v_j\}$ such that $j \notin \{2, \alpha\}$. Then either:*

- *$j = 3$ and v_2 has degree 2; or*
- *$j = \alpha - 1$ and v_α had degree 2; or*
- *at least one of v_1 and v_j has degree 6.*

PROOF: Suppose first, for the sake of contradiction, that there is an edge $\{v_1, v_j\}$ that none of the above cases arise. Since v_1 and v_j are not adjacent on the external face, they must each have degree 4. If we cut along the edge $\{v_1, v_j\}$ to create two plane graphs G_1 and G_2, so that $G = G_1 \oplus G_2$ along the path $[v_1, v_j]$, we see that for one such graph v_1 and v_j must have degree 2, as otherwise each graph has one vertex of odd degree. It then follows that either G_1 or G_2 is a triangle, contradicting our original assumption. □

Lemma 7. *Let $(v_1, v_2, \ldots, v_\alpha, v_{\alpha+1} = v_1)$ be the external face of $G \in \mathcal{G}(x_2, z_4, x_6)$. Suppose that v_1 and v_j have degree 4 and there is a path $[v_1, w, v_j]$ where w is an internal vertex such that $j \notin \{2, \alpha\}$. Then there is (at least one) internal edge from one of $\{v_\alpha, v_1, v_2\}$ to one of $\{v_{j-1}, v_j, v_{j+1}\}$.*

PROOF: We assume for the sake of contradiction that the lemma is false. Suppose first that w has degree 4. Let v_1, u_1, u_2, u_3 be the vertices adjacent to w in clockwise order. We cannot have v_j adjacent to v_1 so we must have $v_j = u_2$. Since v_1 is external and of degree 4, without loss of generality $\{v_1, u_3\}$ is an external edge. Thus $u_3 \in \{v_\alpha, v_2\}$ and is adjacent to v_j, a contradiction.

Otherwise w has degree 6. Let $v_1, u_1, u_2, u_3, u_4, u_5$ be the vertices adjacent to w in clockwise order. Then $v_j \in \{u_2, u_3, u_4\}$. Since v_1 is external and of degree 4, without loss of generality the edge $\{u_5, v_1\}$ is external. Thus $v_j \neq u_4$. If $v_j = u_3$, cutting the graph along the path $[v_j, w, v_1]$ shows that $\{v_j, u_4\}$ is external (otherwise the components each have one vertex of odd degree). The edge $\{u_4, u_5\}$ must be internal. Thus $u_5 \in \{v_\alpha, v_2\}$ is adjacent to $u_4 \in \{v_{j-1}, v_{j+1}\}$ via an internal edge, a contradiction. Thus $v_j = u_2$. If $\{v_j, u_3\}$ is an external edge this again causes a contradiction cutting the graph along the path $[v_j, w, v_1]$. Thus $\{v_j, u_1\}$ is an external edge, and we get a contradiction as above. □

So if we can classify the structure of each of the types, we will in effect have classified every possible NHSLB. It turns out that the types are related to each other via recursive methods. First, we show that the vertices of degree 2 can be dealt with in many cases in the following manner. Let $G \in \mathcal{G}(x_2, z_4, x_6)$ with $x_2 \geq 1$ and $x_6 \leq 1$. Let the external face of G be $(v_1, v_2, \dots, v_{\alpha+1} = v_1)$, where $\alpha \geq 6$. Let $3 \leq k \leq \alpha - 3$ where v_k has degree 2, v_{k-2}, v_{k-1}, v_{k+1} and v_{k+2} have degree 4 and vertex v_{k+3} has degree at least 4. By observation, v_{k-1} and v_{k+1} are adjacent and there exists a vertex w whose neighbourhood includes $v_{k-2}, v_{k-1}, v_{k+1}$ and v_{k+2}. Since v_{k+3} has degree 4, by Lemma 6 vertices v_{k-2} and v_{k+2} are non-adjacent and the degree of w must be 6. We claim that w must be internal. To see this, suppose that w is external. Remove from G the vertices v_{k-1}, v_k and v_{k+1} and all edges adjacent to them. The resultant plane graph G' has w as a cut-vertex. But cutting G' at w results in two graphs with one vertex of odd degree, a contradiction. Thus w is indeed internal. Let $G' = D(G, v_k)$ be the plane graph created by first deleting v_{k-1}, v_k and v_{k+1} from G and any edges adjacent to these vertices, then adding the (external) edge $\{v_{k-2}, v_{k+2}\}$. Then, $D(G, v_k) \in \mathcal{G}(x_2 - 1, z_4 + 1, x_6)$ and has $\alpha - 3$ external vertices. Note that w is now an internal vertex of degree 4 within $D(G, v_k)$.

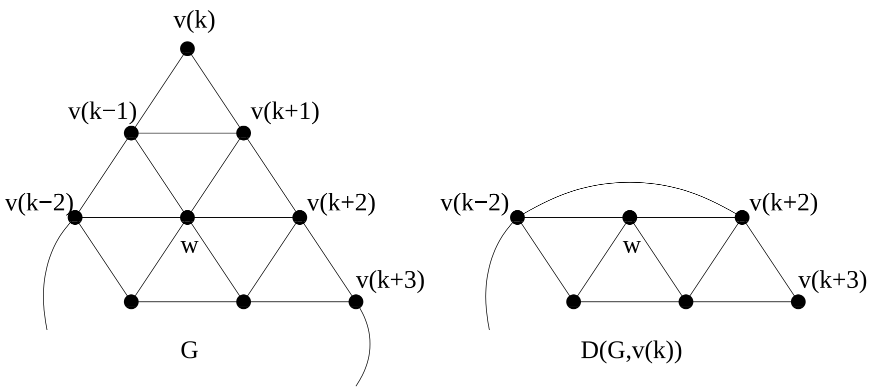

FIGURE 2. Removing external vertices of degree 2 via the operation D.

It is helpful (for the sake of being able to construct our graphs recursively) to observe necessary and sufficient conditions to invert this process. Let $G \in \mathcal{G}(x_2, z_4, x_6)$ where $z_4 \geq 1$. Let v_1 and v_2 be vertices on the external face of degree at least 4 and w an internal vertex of degree 4 such that $\{v_1, v_2, w\}$ is a triangular face and the edge $\{v_1, v_2\}$ is external. Then there exists $G' \in \mathcal{G}(x_2 + 1, z_4 - 1, 0)$ and $v_k \in G'$ such that $D(G', v_k) = G$. In this case we say that $G = D^{-1}(G', w)$.

In fact, $D(D^{-1}(G', w), v_k) = G'$ and $D^{-1}(D(G, v_k), w) = G$ whenever the inner functions are well-defined.

To classify graphs from $\mathbb{G}$ with $x_6 = 0$, we need to identify some small "starter" graphs from which we can recursively build larger ones. To this end, let G_3, G_6 and G_9 be equilateral triangles of widths 1, 2 and 3, respectively, from $\mathcal{H}$. Next, take an equilateral triangle from $\mathcal{H}$ of side 2 units, where A, B and C are the vertices of degree 2. Next add a new vertex D and edges $\{D, B\}$, $\{D, C\}$ and $\{B, C\}$ to create a graph with 6 external vertices so that $\{B, C\}$ is an internal edge. We call this graph G_6'.

In the following lemma we reduce any graph from $G \in \mathbb{G}$ with $x_6 = 0$ and $x_2 \geq 1$ to either a small "starter" graph or a graph from $\mathcal{G}(0, 3, 0)$.

Lemma 8. *Let $G \in \mathcal{G}$ with $x_6 = 0$ and $x_2 \geq 1$. Let v_k be an external vertex of G of degree 2. If $G \in \mathcal{G}(1, 2, 0)$, then $D(G, v_k) \in \mathcal{G}(0, 3, 0)$. Let $G \in \mathcal{G}(2, 1, 0)$. Then either G is isomorphic to G_6' or $D(G, v_k) \in \mathcal{G}(1, 2, 0)$. Let $G \in \mathcal{G}(3, 0, 0)$. Then either G is isomorphic to an element of $\{G_3, G_6, G_9\}$ or $D(G, v_k) \in \mathcal{G}(2, 1, 0)$.*

Proof: From Corollary 5, the number of external vertices of G is divisible by 3. If G has three external vertices, by Lemma 1 G is isomorphic to G_3. Otherwise G has at least 6 external vertices.

If $G \in \mathcal{G}(1, 2, 0)$, then vertices $v_{k-2}, v_{k-1}, v_{k+1}, v_{k+2}$ and v_{k+3} are distinct and each have degree 4; thus we may apply the method described before the proof.

Otherwise, let l be the minimum distance between any two vertices of degree 2 in G. If $l \in \{1, 2\}$, then by observation G is isomorphic to G_3 or G_6. Hence we may assume that $l \geq 3$. If $G \in \mathcal{G}(2, 1, 0)$, the vertices of degree 2 must be separated by a path of at least three edges in either direction within the external face. If they are separated minimally in both directions, we have the graph G_6. Otherwise we can label the external vertices so that $v_{k-2}, v_{k-1}, v_{k+1}, v_{k+2}$ and v_{k+3} are distinct and each have degree 4, so $D(G, v_k)$ is well-defined.

Finally, suppose that $G \in \mathcal{G}(3, 0, 0)$. If $l = 3$, by observation G is isomorphic to G_9. Otherwise $l \geq 4$ and we may again apply the method described before the proof. $\square$

4. Wrapping

Before we classify the structure of the remaining sets of the form $\mathcal{G}(x_2, z_4, x_6)$, we describe various ways that some elements of these sets may be embedded into others. The general idea is to "wrap" a plane graph by adding triangular faces to create an external face with a new set of vertices. This process may either create a plane graph with the same set of parameters $\{x_2, z_4, x_6\}$ or we may change the parameters slightly. The idea is to build plane graphs recursively where possible.

Let $G \in \mathcal{G}(x_2, z_4, x_6)$ and let the external face of G be the cycle

$$(v_1, v_2, \ldots, v_\alpha, v_{\alpha+1} = v_1).$$

Let S be the set of subscripts of vertices of the external face which have degree 2. Let $T \subset S$, with $|T| = y_2$ and $0 \leq y_2 \leq x_2$. Let X be the set of subscripts of vertices of the external face which have degree 4. Let $W \subset X$, where $|W| = y_4 \in \{0,1\}$, $y_4 \leq x_4$ and $y_4 + x_6 \leq 1$. We embed G into a larger plane graph $C_{T,W}(G) \in \mathcal{G}(x_2, x_4, x_6)$ as follows. We add vertices w_i, $1 \leq i \leq \alpha$ and extra edges so that $\{v_i, v_{i-1}, w_{i-1}\}$ is a triangular face, and so that each of these triangles lies outside the original external loop of vertices. Next, for each i:

- if v_i is an external vertex of degree 2 (in G) and $v \notin T$, add three new vertices $w_{i(1)}$, $w_{i(2)}$ and $w_{i(3)}$ and extra edges so that $\{v_i, w_{i-1}, w_{i(1)}\}$, $\{v_i, w_{i(1)}, w_{i(3)}\}$, $\{v_i, w_{i(3)}, w_i\}$ and $\{w_{i(1)}, w_{i(2)}, w_{i(3)}\}$ are new triangular faces;
- if v_i is an external vertex of degree 4 and $v_i \notin W$ *or* $v_i \in T$ (in G), add the edge $\{w_{i-1}, w_i\}$;
- if v_i is an external vertex of degree 6 or $v_i \in W$, first let $w'_{i-2} = w_{(i-1)(3)}$ (if this is defined); otherwise $w'_{i-2} = w_{i-2}$. Similarly, let $w'_{i+1} = w_{(i+1)(1)}$ (if this is defined); otherwise $w'_{i+1} = w_{i+1}$. Next, identify $w_i = v_{i-1}$ and $w_{i-1} = v_{i+1}$ (and appropriate edges), so that $\{v_{i-1}, v_i, v_{i+1}\}$ becomes a face; and identify $w'_{i-2} = w'_{i+1}$ (and appropriate edges) so that $\{v_{i-1}, v_{i+1}, w'_{i-2}\}$ is now a triangular face. In doing so, w'_{i-2} becomes an external vertex.

(In the above, the third case arises for at most one such i; we deal with every other case first, so that w'_{i-2} and w'_{i+1} are well-defined.) We give examples of the wrapping process in Figures 3 and 4.

Apart from where specified in the final case, we specify that the newly created external vertices are pairwise distinct. Observe the following.

Lemma 9. *If $G \in \mathcal{G}(x_2, z_4, x_6)$ and G has α external vertices, then $C_{T,W}(G) \in \mathcal{G}(x_2 - y_2, z_4, x_6 + y_4)$ and has $\alpha + 3(x_2 - y_2 - x_6 - y_4)$ external vertices.*

We next investigate whether $C_{T,W}(G)$ is always well-defined. The above lemma potentially allows the number of external vertices to decrease. From Corollary 5, α is divisible by 3. If $\alpha \geq 6$, then since $x_6 + y_4 \leq 1$ and $y_2 \leq x_2$, $C_{T,W}(G)$ has at least three external vertices. Otherwise consider when $\alpha = 3$. If $x_2 \geq 1$, then from Lemma 1, we must have $x_2 = 3$; thus $x_6 = 0$ and $y_4 \leq x_4 = 0$ and $C_{T,W}(G)$ has at least 3 external vertices.

Otherwise $x_2 = 0$. If $y_4 = 1$, then $C_{T,W}(G)$ contains a doubled edge; thus $C_{T,W}(G)$ is not a subgraph of a NHSLB. If $x_6 = 1$, then our NHSLB is equal to $G \oplus G'$ for some G' with three external vertices, at least one of which has degree 2. Thus, G' is forced to be a triangle by Lemma 1. In summary, the cases when $C_{T,W}(G)$ is not well-defined do not effect our overall method of classification.

5. The classification continues

Lemma 10. *Let $G \in \mathcal{G}(0,3,0)$. Then $G = C_{T,\emptyset}(G')$ for some $G' \in \mathcal{G}(x_2, z_4, 0)$ with $x_2 + z_4 = 3$.*

PROOF: Let $G \in \mathcal{G}(0,3,0)$ with external cycle $(v_1, v_2, \ldots, v_{\alpha+1} = v_1)$. By Lemma 6 there is no edge of the form $\{v_i, v_j\}$ where $j \notin \{i-1, i+1\}$. Furthermore, by Lemma 7, there is no internal vertex w such that $\{v_i, w\}$ and $\{w, v_j\}$ are edges, where $j \notin \{i-1, i, i+1\}$. It follows that there exist pairwise distinct vertices $w_1, w_2, w_3, \ldots, w_\alpha$ such that $\{v_i, v_{i+1}, w_i\}$ and $\{w_i, w_{i+1}, v_{i+1}\}$ are internal faces for each i, $1 \leq i \leq \alpha$. Now, delete the vertices $v_1, v_2, \ldots, v_k$ and any edges adjacent to them. What remains is a plane graph G' with an external face $(w_1, w_2, w_3, \ldots, w_k, w_1)$. Clearly $G' \in \mathcal{G}(x_2, z_4, 0)$ where $x_2 + z_4 = 3$; indeed $G = C_{T,\emptyset}(G')$ for some (possible empty) set T of external vertices in G' of degree 2. □

The following corollary follows from the results we have so far. It suggests a method of generating computationally all graphs in $\mathcal{G}(x_2, z_4, 0)$ of a given size. It is clear there are many slight variations on this approach.

Corollary 11. *Let $G \in \mathcal{G}(x_2, z_4, 0)$. Then there exists a list of graphs $G_0, G_1, \ldots, G_k$ such that:*

- *$G_0 \in \{G_3, G_6, G_9, G_6'\}$;*
- *for each $i \geq 0$, $G_{i+1} = C_{T,\emptyset}(G_i)$ (where T is some (possibly empty) subset of the external vertices of degree 2 in G_i) or $G_{i+1} = D^{-1}(G_i, w)$, where w is an internal vertex of G_i of degree 4 adjacent to two external vertices, each of degree 4;*
- *$G_k = G$.*

Next we consider the cases where $x_6 = 1$. If $x_2 \geq 2$, these can be constructed from the above cases by the removal of a parallelogram from the regular hexagonal triangulation of the plane. To see this, observe the following lemma.

Lemma 12. *Let $G \in \mathcal{G}(x_2, z_4, 1)$ where $x_2 \geq 2$. Let the external cycle of G be $(v_1, v_2, \ldots, v_{\alpha+1} = v_1)$, where v_1 and v_l have degree 2, v_k has degree 6, $1 < k < l$ and if either $1 < j < k$ or $k < j < l$ then v_j has degree 4. Take a parallelogram $ABCD$ from $\mathcal{H}$ with vertices A, B, C and D, $|AB| = |CD| = k-1$ and $|AC| = |BD| = l-k$, with B and D of degree 2. Let P be the path on the external face of $ABCD$ which begins at A, ends at C and includes B. Let G' be the graph $ABCD \oplus G$, where we glue path P to the path $[v_1, v_2, \ldots, v_l]$. Then $G' \in \mathcal{G}(x_2 - 1, z_4, 0)$.*

We have just two remaining classes of graph to classify, namely $\mathcal{G}(1,3,1)$ and $\mathcal{G}(0,4,1)$. Recall that the glueing operation $\oplus$ may glue paths of length 0 (i.e. vertices). The external face of such a planar graph is a circuit rather than a cycle. Let $x_2', x_2'' \geq 1$ and let $G_1 \in \mathcal{G}(x_2', z_4', 0)$ and $G_2 \in \mathcal{G}(x_2'', z_4'', 0)$. Let v' and v'' be external vertices of degree 2 in G_1 and G_2, respectively. Denote $G_1 \oplus G_2$ with respect to the paths $P_1 = [v']$ and $P_2 = [v'']$ by $G(v') \oplus G_2(v'')$. We can define $C_T(G_1(v') \oplus G_2(v'')) \in \mathcal{G}(x_2' + x_2'' - 2, z_4' + z_4'', 1)$ in the natural way, treating $v' = v''$ as a degree of vertex 6 on one side and as a degree of vertex 4 on the other (obtaining possibly two distinct graphs by this choice). We demonstrate this process with an example in Figure 5.

In the following lemma, recall that G_6 is the equilateral triangle from $\mathcal{H}$ with side length 2.

Lemma 13. *Let $G \in \mathcal{G}(0,4,1)$. Then either:*

(a) *there exist $G_1 \in \mathcal{G}(1, z_4', 0)$ and $G_2 \in \mathcal{G}(1, z_4'', 0)$ with $z_4' + z_4'' = z_4$, with v' and v'' being the vertices of degree 2 in G_1 and G_2 respectively, so that G is formed by overlapping the triangles from G_1 and G_2 containing v' and v'' respectively, so that v' and v'' are distinct vertices in G of degree 4;* or

(b) *there exist G_1 and G_2 as in* (a)*, but we form G by overlapping copies of G_6 from G_1 and G_2 containing v' and v'' respectively, so that v' and v'' are distinct vertices in G of degree 4;* or

(c) *$G = C_T(G_1 \oplus G_2)$ for some $G_1 \in \mathcal{G}(x_2', z_4', 0)$ and $G_2 \in \mathcal{G}(x_2'', z_4'', 0)$ such that $x_2' + x_2'' + z_4' + z_4'' = z_4$* and *there exists an external vertex v' of degree 2 in G_1 and an external vertex v'' of degree 2 in G_2 so that the glueing paths are of length 0 and are equal to $[v']$ and $[v'']$;* or

(d) *$G = C_{T,W}(G')$ for some $G' \in \mathcal{G}(x_2', z_4', x_6')$.*

(*See Figure* 6 *for an illustration of these four cases.*)

Proof: Let the external face of G be the sequence of vertices $v_1, v_2, \ldots, v_\alpha, v_{\alpha+1} = v_1$ where v_1 has degree 6. Let the neighbours of v_1 (without loss of generality in clockwise order) be $v_2, u_1, u_2, u_3, u_4, v_\alpha$. Suppose u_1 is external. Then we can cut G into two plane graphs which glue at the edge $\{u_1, v_1\}$; however, as $x_2 = 0$, each graph will have only one vertex of odd degree, which is impossible. Similarly, u_4 is internal.

Consider the case when at least one of u_2 or u_3 is an external vertex. Since u_1 and u_4 are internal, we must have the edge $\{u_2, u_3\}$ external. Let G_1 be the component which includes u_3 when we cut along the edge $\{u_2, v_1\}$. Similarly, let G_2 be the component which includes u_2 when we cut along the edge $\{u_3, v_1\}$. Then u_2 is a vertex of degree 2 in G_1 and u_3 is a vertex of degree 2 in G_2. Moreover, v_1 is a vertex of degree 4 in both G_1 and G_2. Thus, we have case (a).

Otherwise both u_2 and u_3 are internal vertices. Let y be the vertex not equal to v_1 such that $\{y, u_2, u_3\}$ is a triangular face. Since u_2 and u_3 are internal, there must exist w and z such that $\{w, y, u_3\}$ and $\{y, z, u_2\}$ are triangular faces. (It is possible that $w = u_4$ or $z = u_1$ but this does not interfere with our line of argument.) Suppose y is an external vertex. Then y must have degree 4 and w and z are each external. Cut along the path v_1, u_2, z to create a graph G_1 (the component including w) and again along the path v, u_3, w to create a graph G_2 (the component including z). It follows that we have case (b). Otherwise y is internal.

For each i, $2 \leq i \leq \alpha - 1$, there exists an internal vertex w_i such that $\{v_i, v_{i+1}, w_i\}$ is a face. By Lemmas 6 and 7 (and from above), the vertices $v_1, v_2, \ldots, v_\alpha, w_2, w_3, \ldots, w_{\alpha-1}, u_1, u_2, u_3, u_4$ are pairwise distinct. So we obtain an internal circuit $C = [y, u_2, u_1, w_2, w_3, \ldots, w_{\alpha-1}, u_4, u_3, y]$. Suppose that $y = w_j$ where $2 \leq j \leq \alpha - 1$. Then we have case (c).

Otherwise the circuit C is a cycle. Delete any vertices and edges external to this cycle to obtain a new plane graph G. Then at most one external vertex of G has degree 6 (i.e. the vertex y) and the remaining external vertices have degree 2 or 4. Thus we have case (d). □

Finally, we deal with the class of graphs $\mathcal{G}(1,3,1)$, by showing how each graph in this class can be related to one from $\mathcal{G}(0,3,0)$. To this end, let $G \in \mathcal{G}(1,3,1)$ and suppose that $(v_1, v_2, \ldots, v_\alpha, v_{\alpha+1} = v_1)$ is the external face of G, where v_1 has degree 6. Let v_k be the vertex of degree 2. Suppose that $k \notin \{2, 3, \alpha-1, \alpha\}$. Then $D(G, v_k) \in \mathcal{G}(0,4,1)$. Otherwise, by Lemma 1 and Corollary 5, $\alpha \geq 6$ and without loss of generality the vertex of degree 2 is either v_2 or v_3.

First consider the case that v_2 is the vertex of degree 2. Let G' be the graph created by adding the edges $\{v_\alpha, v_{\alpha-2}\}$, $\{v_2, v_{\alpha-2}\}$ and $\{v_2, v_\alpha\}$, so that vertices v_1, $v_{\alpha-1}$ and v_α are internal in G'. Then $G' \in \mathcal{G}(0,4,1)$ with vertex $v_{\alpha-2}$ of degree 6 and the internal vertex $v_{\alpha-1}$ of degree 4 in G'. Otherwise v_3 has degree 2. Let $T = \{v_3\}$, then $C_{T,\emptyset}(G) \in \mathcal{G}(0,4,1)$.

6. Conclusion

Now that the classification is fully presented, it is worth discussing its strengths and weaknesses. It is quite clear that a NHSLB can, in general, arise from our constructions in more than one way. Indeed, a disadvantage of the classification presented is that it does not directly reveal the structure of isomorphism classes. It could be argued, then, that the method given by Batagelj is also a classification, in that it provides a way of generating every possible example, with isomorphic graphs being constructed in many different ways.

However, although the method outlined does not provide clearly stated isomorphism classes, it provides considerably more direct information than the Batagelj method about isomorphisms. It tells us a great deal about substructures that can and must occur within a NHSLB.

At some point I thought that it would be nice to classify each set of graphs $\mathcal{G}(x_2, z_4, x_6)$ in terms of glueing together regions cut out from $\mathcal{H}$. Such an approach could potentially make isomorphisms more obvious. For example, it is not hard to show that any graph from $\mathcal{G}(3,0,0)$ is an equilateral triangle from $\mathcal{H}$. Any graph from $\mathcal{G}(2,1,0)$ can be constructed by glueing a trapezium from $\mathcal{H}$ with a triangle from $\mathcal{H}$, so that two sides of the triangle are glued along one side of the trapezium. Next, any element of $\mathcal{G}(4,0,1)$ can be obtained by taking a triangle T of length k from $\mathcal{H}$ and "deleting" a parallelogram P (within T) with side lengths strictly less than k and overlapping some vertex of T. So in these cases we can classify types directly without using recursion. However, for other cases this approach appears to be complicated and lengthy to articulate. The classification we have presented has the advantage of being succinct enough to present in one paper.

In terms of computation, although like Batagelj we have used recursive methods, it seems likely that in general the number of recursive steps will be much smaller, since in the Batagelj construction each step creates at most one or two new vertices. Whether the methods in this paper do indeed lead to more efficient enumeration of NHSLBs is an open question. Should alternate classifications of NHSLBs be conjectured, the result in this paper might be used to test their validity.

We make one final important point which is that there does exist an infinite family of NHSLB's which may not be formed by glueing two elements of $\mathbb{G}$, each of which have $x_6 = 0$. Let H be any convex hexagon taken from $\mathcal{H}$. Next, glue two copies of H together via the operation $\circ$, so that corresponding external vertices and edges are equal. Let the resultant NHSLB be H'. Then it can be shown that any path taken as in the proof of Lemma 3 results in at least one of G or G' containing an external vertex of degree 6, where $H' = G \circ G'$ as in Corollary 4. It follows that the method described by Corollary 11 does not give rise to every possible NHSLB.

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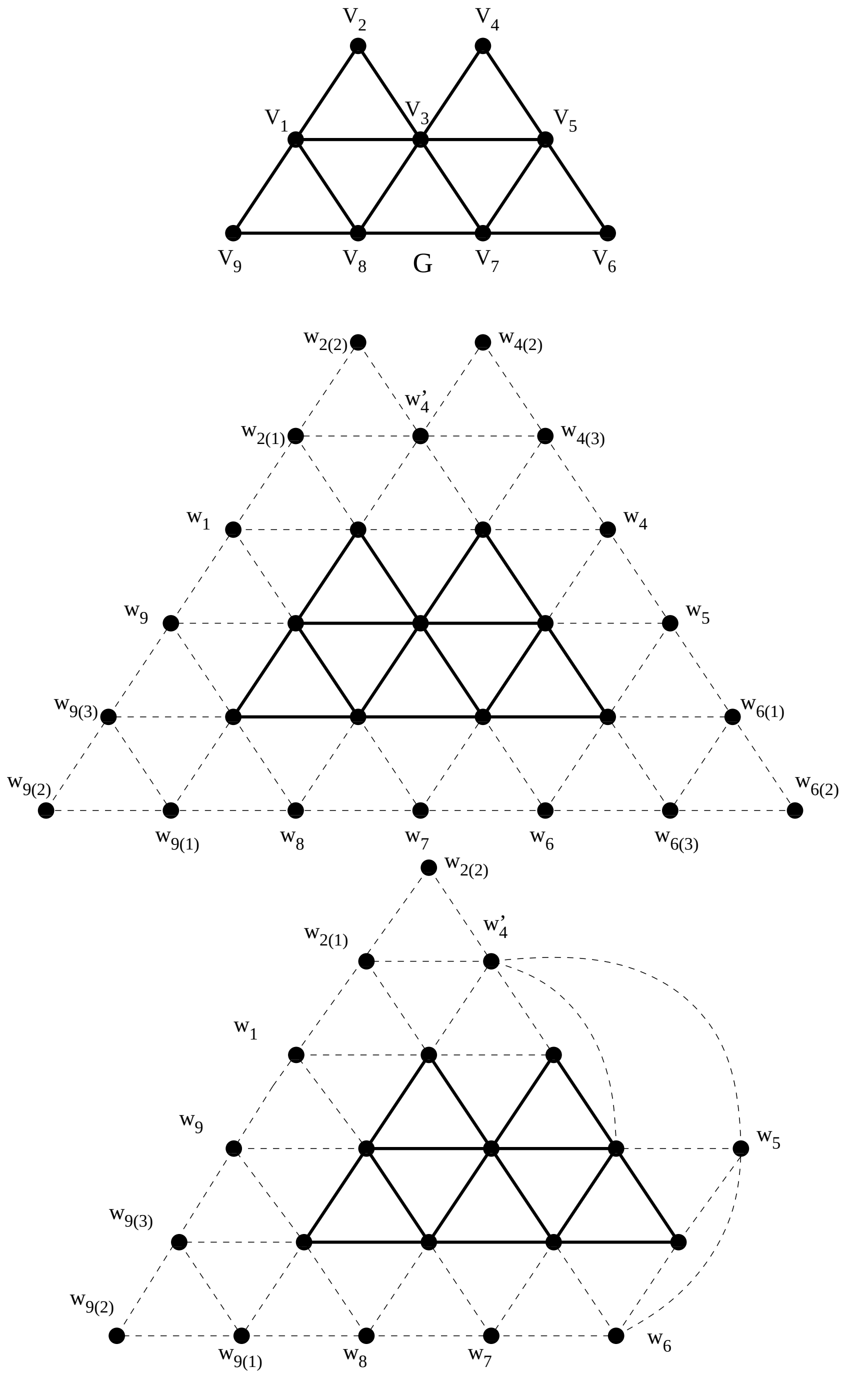

Figure 3. The graph $G \in \mathcal{G}(4,0,1)$ is "wrapped" to the graphs $C(G)$ and $C_T(G)$ (where $T = \{v_4, v_6\}$).

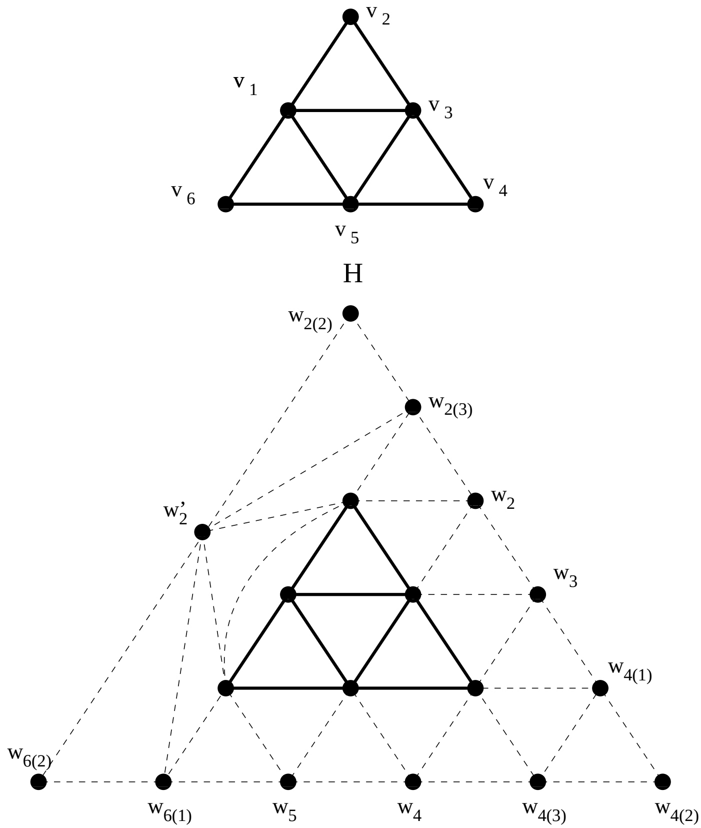

FIGURE 4. The graph $H \in \mathcal{G}(3,0,0)$ is "wrapped" to the graph $C_W(H)$ (where $W = \{v_1\}$.)

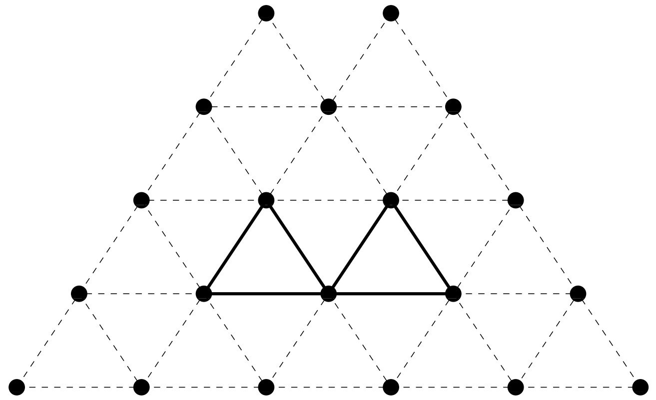

FIGURE 5. The graph $C_T(G_3(v') \oplus G_3(v''))$.

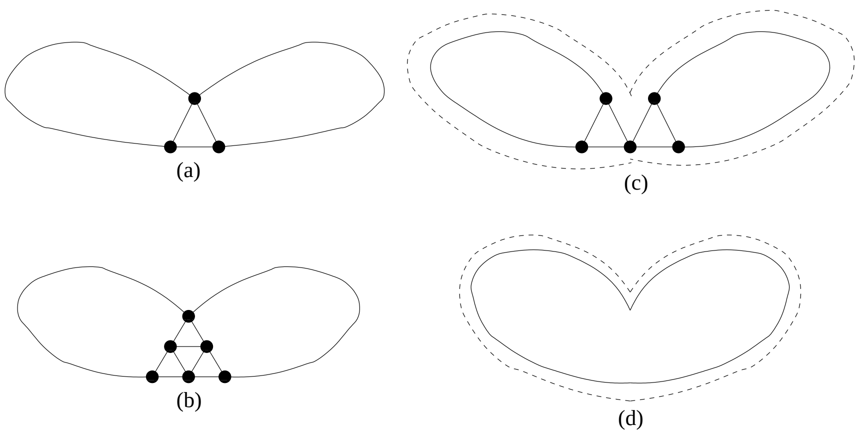

FIGURE 6. The four cases of Lemma 13.